Chapter Four

Discrete Distributions

Discrete Uniform Distribution

If the random variable X is equally likely to be each of the n integers $1, 2, \ldots, n$. Then the pdf of X is:

$$f_X(x) = \begin{cases} \frac{1}{n} & x = 1,2,..,n \\ 0 & elsewhere \end{cases}$$

Example:

Throwing a die once yields the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{6} & x = 1,2,...,6 \\ 0 & elsewhere \end{cases}$$

Then X has a discrete uniform distribution.

Example:

One ball is selected at random from 5 balls numbered from 1 to 5. Find the pdf of the number of the selected ball.

Let X be the random variable of the selected ball, then

$$f_X(x) = \begin{cases} \frac{1}{5} & x = 1,2,3,4,5 \\ 0 & elsewhere \end{cases}$$

Lemma:

If X is has discrete uniform distribution, then

$E(X) = \frac{n+1}{2}$ and, $Var(X) = \frac{n^2-1}{12}$

Proof:

$$E(X) = \sum_{x \in S} x\, f_X(x) = \sum_{i=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} x$$

$$= \frac{1}{n}\left(\frac{n(n+1)}{2}\right) = \frac{n+1}{2}$$

$$E(X^2) = \sum_{x\in S} x^2 f_X(x) = \sum_{i=1}^{n} x^2 \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} x^2$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6}$$

$$Var(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n+1}{2}\left[\frac{4n-2-3n-3}{6}\right] = \frac{n+1}{2}\left[\frac{n-1}{6}\right] = \frac{n^2-1}{12}$$

Bernoulli distribution:

If an experiment has two possible outcomes, "success" and "failure" and their probabilities are, respectively, p and $(1-p)$, then the number of successes $Y = 1$ or $Y = 0$ has a Bernoulli distribution.

The *pdf* is given by:

$$p(y) = p^y (1-p)^{1-y} \qquad y = 0, 1$$

$$\mu = E(Y) = \sum_{x\in S} y p(y) = 0\,p(Y=0) + 1\,p(Y=1)$$

$$= 0(1-p) + 1(p) = p$$

$$Var(Y) = E(Y^2) - [E(Y)]^2 = \sum_{x\in S} y^2 p(y) - p^2$$

$$= 0(1-p) + 1(p) - p^2 = p(1-p)$$

The mgf is:

$$M_Y(t) = E(e^{tY}) = \sum_{y=0}^{1} e^{ty} p^y (1-p)^{1-y} = [(1-p) + pe^t]$$

$M'_Y(t) = pe^t$

$M'_Y(0) = pe^0 = p = E(Y) = \mu$

$M''_Y(t) = pe^t$

$M''_Y(0) = pe^0 = p$

$Var(Y) = p - p^2 = p(1-p)$

The characteristic function is:

$\phi_X(t) = E\, e^{itx} = M_X(it) = (1-p)+pe^{it} = 1-p(1-e^{it})$

Various examples of Bernoulli trials are: tossing a coin (head or tail), firing a target (hit or miss), fighting an election (win or not win), playing a game (win or lose), etc.

<u>Binomial distribution</u>:

The inspection of a single item taken from an assembly line is a Bernoulli random variable. Suppose we independently inspect n items and record values for $Y_1, Y_2, \ldots, Y_n$ where $Y_i = 1$ if the inspected item is defective and $Y_i = 0$ otherwise. The random variable $X = \sum_{i=1}^{n} Y_i$ denotes the number of defectives among n items possesses a binomial distribution with *pdf*

$$p(X = x) = C_x^n p^x (1-p)^{n-x} \qquad x = 0,1,2,\ldots,n$$

In general, a random variable X has a binomial distribution if:

1. The experiment consists of a fixed number n of identical trials.
2. Each trial can result in one of two possible outcomes, “success” or “failure”.
3. X is the number of successes among the n trials.
4. The probability of “success” is constant from trial to trial = p.
5. The trials are independent.

Remark:

The binomial expansion if n is a positive integer is

$$(a+b)^n = \sum_{x=0}^{n} C_x^n a^x b^{n-x}$$

The mean and the variance of the binomial distribution are:

$$\mu = E(X) = \sum_{x \in S} xp(X = x) = \sum_{x=0}^{n} xC_x^n p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!\,(n-x)!} p\, p^{x-1} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$$

Let $n\text{-}1 = m$, $x\text{-}1 = y$

$$E(X) = np \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y (1-p)^{m-y} = np[p + (1-p)]^m$$

$$= np$$

$$Var(X) = E(X^2) - [E(X)]^2 = E(X^2) - E(X) + E(X) - [E(X)]^2$$

$$= E(X^2 - X) + E(X) - [E(X)]^2$$

$$= E(X(X-1)) + E(X) - [E(X)]^2$$

$$E\big(X(X-1)\big) = \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!} p^2 p^{x-2} (1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} (1-p)^{n-x}$$

Let *n-2=m, x-2=y,* then

$$E(X(X-1)) = n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y (1-p)^{m-y}$$

$$= n(n-1)p^2[p+1-p]^m = n(n-1)p^2$$

$$Var(X) = E(X(X-1))+E(X)-[E(X)]^2$$

$$= n(n-1)p^2+np-n^2p^2 = n^2p^2-np^2+np-n^2p^2$$

$$= np(1-p)$$

Or, since the random variable X is the sum of n Bernoulli random variables, then

$$E(X) = E(\sum_{i=1}^{n} y_i) = \sum_{i=1}^{n} p = np$$

$$\mathrm{var}(X) = \sum_{i=1}^{n} \mathrm{var}(Y_i) = \sum_{i=1}^{n} p(1-p) = np(1-p)$$

The moment generating function of X is

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} C_x^n p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} C_x^n (pe^t)^x (1-p)^{n-x}$$

Using the formula for the binomial expansion with $a = 1-p$ and $b = pe^t$ We see that

$$M_X(t) = [(1-p)+pe^t]^n$$

for all real values of t. The first two derivatives of $M_X(t)$ are:

$$M'_X(t) = n[(1-p)+pe^t]^{n-1}(pe^t)$$

$$M''_X(t) = n(n-1)[(1-p)+pe^t]^{n-2}(pe^t)^2 + n[(1-p)+pe^t]^{n-1}(pe^t)$$

Thus

$$\mu = M'_X(0) = n\left(1-p+pe^0\right)pe^0 = np$$

$$\sigma^2 = M''_X(0)-[M'_X(0)]^2 = n(n-1)p^2+np-(np)^2 = np(1-p)$$

The characteristic function for the binomial distribution is:

$\phi_X(t) = E\,e^{itx} = M_X(it) = [(1-p)+pe^{it}]^n = [1-p(1-e^{it})]^n$

Example 1:

A large lot of fuses contain *10*% defectives. If *4* fuses are randomly sampled from the lot.

a. Find the probability that exactly one fuse is defective.

b. Find the probability that at least one fuse is defective.

$p = 0.10, \; n = 4$

$$a.\, p(X = 1) = C_1^4 (0.1)^1 (0.9)^{4-1} = \frac{4!}{3!1!}(0.1)(0.9)^3$$

$$= 4(0.1)(0.729) = 0.2916$$

$$b.\, p(X \geq 1) = 1 - p(X = 0) = 1 - C_0^4 (0.1)^0 (0.9)^4 = 0.3439$$

$E(X) = np = 0.4$

$Var(X) = np(1-p) = 0.36$

Example 2:

Find the probability that *7* of *10* persons will recover from a tropical disease, if we can assume independence with probability of recovery *0.80*.

$x = 7 \qquad n = 10 \qquad p = 0.8$

$$p(X = 7) = C_7^{10} (0.8)^7 (0.20)^3 = 120(0.8)^7 (0.20)^3 \approx 0.20$$

In actual practice, binomial probabilities are tabulated extensively for various values of *p* and *n*, and there exists an abundance of computer software yielding binomial probabilities as well as cumulative probabilities upon simple command, where

$$B(X; n, p) = \sum_{k=0}^{x} b(k; n, p)$$

Example:

For *n = 10, p = 0.4;* find the probability of *3* or more successes.

$$B(X \geq 3; 10, 0.4) = \sum_{k=3}^{10} b(k; n, p) = 1 - \sum_{k=0}^{2} b(k; n, p) = 0.833$$

H.W.

1. Three coins are tossed once. Let the r.v. X denote the number of heads that appears. What is the probability distribution of X? Find the probability of no heads, one head, two heads and three heads.

Also find: $P(X \leq 2)$, $E(X)$, and $Var(X)$.

2. The probability that a patient recover from a blood disease is 0.6. If it is known that 10 people caught the disease. What is the probability that:

a) exactly 3 people survive. b) at least 8 people survive.

Also find: $P(2 \leq X \leq 5)$, $E(X)$ and $Var(X)$.

3. Team A has 2/3 chance of winning a game whenever it plays. If A plays 4 games, find the probability of winning:

1) Exactly two games. 2) At least one game. 3) More than two games.