Chapter Three

Mathematical Expectation

Def.: If X is a discrete random variable and $f_X(x)$ is the value of its probability distribution at x, the expected value of X is:

$$E(X) = \sum_{x \in S} x\, f_X(x)$$

Correspondingly, if X is a continuous random variable, and $f_X(x)$ is the value of its probability density at x, the expected value of X is:

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) \quad dx$$

That is; $E(X)$ is the weighted average of the possible values of X, each value is weighted by its probability

Example 1:
A pair of fair dice is thrown once, let X be the random variable whose value is the sum of the two numbers on the dice. Then the probability function of X is:

x	2	3	4	5	6	7	8	9	10	11	12
$f_X(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

And

$$p(3 \le X \le 9) = \sum_{x=3}^{9} f_X(x) = \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} = \frac{29}{36}$$

$$p(0 \le X \le 4) = \sum_{x=2}^{4} f_X(x) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$$

$$E(X) = \sum_{x=2}^{12} x f_X(x) = 2(\frac{1}{36}) + 3(\frac{2}{36}) + 4(\frac{3}{36}) + 5(\frac{4}{36}) + \dots + 12(\frac{1}{36}) = \frac{252}{36} = 7$$

Example 2:

Let X be a random variable with probability distribution:

$$f(x) = \frac{\binom{2}{x}\binom{10}{3-x}}{\binom{12}{3}} \qquad x = 0,1,2$$

x	*0*	*1*	*2*
$f_X(x)$	$\frac{12}{22}$	$\frac{9}{22}$	$\frac{1}{22}$

Find *E(X)*.

$$E(x) = \sum_{x=0}^{2} x f_X(x) = 0(\frac{12}{22}) + 1(\frac{9}{22}) + 2(\frac{1}{22}) = \frac{11}{22} = \frac{1}{2}$$

Example 3:

Suppose that X is a continuous random variable having *pdf*:

$$f(x) = \begin{cases} 3x^2 & 0 \le X \le 1 \\ 0 & elsewhere \end{cases}$$

Find *E(X)*.

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x(3x^2) dx = 3\int_0^1 x^3 dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}[1-0] = \frac{3}{4}$$

Example 4:

If X is a random variable with the following *pdf:*

$$f(x) = \begin{cases} \dfrac{4}{\pi(1+x^2)} & 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

Find *E(X)*.

$$E(X) = \int_0^1 x \frac{4}{\pi(1+x^2)} dx = \frac{4}{\pi}\int_0^1 \frac{x}{1+x^2} dx$$

$$= \frac{1}{2}\frac{4}{\pi}\int_0^1 \frac{2x}{1+x^2} dx \quad = \frac{2}{\pi}\ln(1+x^2) \Big|_0^1$$

$$= \frac{2}{\pi}(\ln 1 + 1 - \ln 1 + 0) \quad = \frac{2}{\pi}\ln 2 = \frac{\ln 4}{\pi} = 0.4413$$

Note: Not every random variable has an expected value. The integral that defines the mean might be infinite.
Example:
Let X be a random variable pdf given by:

$$f_X(x) = \begin{cases} \dfrac{1}{X^2} & 1 < x < \infty \\ 0 & elsewhere \end{cases}$$

Then

$$E(X) = \int_1^\infty \frac{x}{x^2}\,dx = \int_1^\infty \frac{1}{x}\,dx$$

$$= \ln x|_1^\infty = \lim_{a\to\infty} \ln x|_1^a = \lim_{a\to\infty} (\ln a - \ln 1) = \infty$$

So we say that *E(X)* does not exist.

***<u>Def</u>*:** If X is a discrete random variable and $f_X(x)$ is the value of its probability distribution at x, the expected value of $g(x)$ is given by:

$$E[g(x)] = \sum_{x\in S} g(x).f_X(x)$$

Correspondingly, if X is a continuous random variable and $f_X(x)$ is the value of its probability density at x, then the expected value of $g(x)$ is given by:

$$E[g(x)] = \int_{-\infty}^{\infty} g(x).f_X(x)dx$$

Example 1:
If X is the number of points rolled with a balanced die, find the expected value of $g(x) = 2X^2 + 1$

$$f(x) = \frac{1}{6} \qquad x = 1,2,3,4,5,6$$

$$E[g(x)] = \sum_{x\in S} g(x).f(x) = 3.\frac{1}{6} + 9.\frac{1}{6} + \ldots + 73.\frac{1}{6} = \frac{94}{3}$$

Example 2:

If the random variable X has the following probability function,

x	*-1*	*0*	*1*
f(x)	*0.2*	*0.3*	*0.5*

Then

1. $E(X)=\sum_{x=-1}^{1} xf(x) = -1(0.2)+0(0.3)+1(0.5)=0.3$
2. $E(2X)=\sum_{x=-1}^{1} 2xf(x) = -2(0.2)+0(0.3)+2(0.5)=0.6$
3. $E(X+1)=(-1+1)(0.2)+(0+1)(0.3)+(1+1)(0.5)=1.3$
4. $E(2X+1)=(-2+1)(0.2)+(0+1)(0.3)+(2+1)(0.5)=1.6$
5. $E(X^2)=-1^2(0.2)+0^2(0.3)+1^2(0.5)=0.7$

Note that: $E(X^2) \neq [E(X)]^2$

Example 3:

For the following *pdf*:

$$f(x)=\begin{cases} 3x^2 & 0 \le X \le 1 \\ 0 & elsewhere \end{cases}$$

Find $E(X^2)$.

$$E(X^2)=\int_{-\infty}^{\infty} x^2 f(x)dx=\int_0^1 x^2(3x^2)dx=3\int_0^1 x^4 dx = 3\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{5}[1-0]=\frac{3}{5}$$

Example 4:

If X has the probability density:

$$f(x)=\begin{cases} e^{-x} & x>0 \\ 0 & elsewhere \end{cases}$$

Find the expected value of $g(X) = e^{3x/4}$

$$E(e^{3x/4})=\int_0^{\infty} e^{3x/4}e^{-x}dx=\int_0^{\infty} e^{-x/4}dx=-4\int_0^{\infty}-\frac{1}{4}e^{-x/4}dx$$

$$=-4e^{-x/4}\Big|_0^{\infty}=-4[0-1]=4$$

Theorem: If a and b are constants, then

$E(aX + b) = aE(X) + b$

Proof:

$$E(aX+b)=\int_{-\infty}^{\infty}(aX+b)f(x)dx = a\int_{-\infty}^{\infty}xf(x)dx + b\int_{-\infty}^{\infty}f(x)dx$$

$$= aE(X) + b$$

If we set $b = 0$, then: $E(aX) = aE(X)$

If we set $a = 0$, then: $E(b) = b$.

Theorem:

If a_1, a_2 are constants, then

$E[a_1g_1(X) + a_2g_2(X)] = a_1E[g_1(X)] + a_2E[g_2(X)]$.

Proof:

Assume X is continuous

$$E[a_1g_1(X) + a_2g_2(X)] = \int_{-\infty}^{\infty}[a_1g_1(x) + a_2g_2(x)]\, f_X(x)dx$$

$$= \int_{-\infty}^{\infty}a_1g_1(x)f_X(x)\,dx + \int_{-\infty}^{\infty}a_2g_2(x)f_X(x)dx$$

$$= a_1\int_{-\infty}^{\infty}g_1(x)f_X(x)\,dx + a_2\int_{-\infty}^{\infty}g_2(x)f_X(x)dx$$

$= a_1E[g_1(X)] + a_2E[g_2(X)]$.

In general,

If $a_1, a_2, \ldots$ and a_n, are constants, then:

$E[a_1g_1(X) + a_2g_2(X) + \ldots + a_ng_n(X)] = a_1E[g_1(X)] + a_2E[g_2(X)] + \ldots + a_nE[g_n(X)]$.

Example:

If the probability density of X is given by:

$$f(x)=\begin{cases}2(1-x) & 0<x<1\\ 0 & elsewhere\end{cases}$$

Show that: $E(X^r) = \dfrac{2}{(r+1)(r+2)}$

$$E(X^r) = \int_0^1 x^r\, 2(1-x)dx = 2\int_0^1 (x^r - x^{r+1})dx = 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+2}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right] = \frac{2(r+2-r-1)}{(r+1)(r+2)} = \frac{2}{(r+1)(r+2)}$$

2. Use this result to evaluate *E [(2X+1)^2]*.

E [(2X+1)^2] =E [4X^2 + 4X + 1] = 4E(X^2) + 4E(X) + 1

$E(X^r) = \dfrac{2}{(r+1)(r+2)}$ When *r = 1*, $E(X) = \dfrac{2}{(2)(3)} = \dfrac{1}{3}$

E [(2X+1)^2] = 4(1/6) + 4(1/3) +1 =3