Marginal Distributions:

Theorem:

1. $\forall\, x \in \boldsymbol{R}$, the function given by $F_X(x) = \lim\limits_{y\to\infty} F_{X,Y}(x,y)$ is called the marginal distribution function of the random variable *X*.
2. $\forall\, x \in \boldsymbol{R}$, the function given by $F_Y(y) = \lim\limits_{x\to\infty} F_{X,Y}(x,y)$ is called the marginal distribution function of the random variable *Y*.

Def: If *X* and *Y* are discrete random variables and $f_{X,Y}(x, y)$ is the value of their joint probability distribution at (x, y), the marginal probability function of *X* can be found as follows:

$$f_X(x) = P(X = x) = \sum_y f(x,y)$$

Correspondingly, the marginal probability function of *Y* can be found as follows:

$$f_Y(y) = P(Y = y) = \sum_x f(x,y)$$

Example:

For the following joint distribution:

$$f_{X,Y}(x,y) = \frac{x+y}{21} \qquad x = 1,2,3 \quad y = 1,2$$

$$f_X(x) = \sum_{y=1}^{2} \frac{x+y}{21} = \frac{x+1}{21} + \frac{x+2}{21}$$

$$f_X(x) = \frac{2x+3}{21} \qquad x = 1,2,3$$

$$f_Y(y) = \sum_{x=1}^{3} \frac{x+y}{21} = \frac{1+y}{21} + \frac{2+y}{21} + \frac{3+y}{21}$$

$$f_Y(y) = \frac{3y+6}{21} \qquad y = 1,2$$

y	1	2

x	1	2	3
$f(x)$	5/21	7/21	9/21

$f(y)$	9/21	12/21

Def: If X and Y are continuous random variables and $f_{X,Y}(x, y)$ is the value of their joint probability density at (x, y), then the marginal pdf of X is given by:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy \qquad for \quad -\infty < x < \infty$$

Correspondingly, the marginal pdf of Y is given by:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx \qquad for \quad -\infty < y < \infty$$

Example:

For the joint distribution

$f(x, y)=x + y \qquad 0 < x < 1 \text{ , } 0 < y < 1$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^1 (x+y)dy = xy + \frac{y^2}{2} \Big|_0^1 = x + \frac{1}{2} \qquad 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx = \int_0^1 (x+y)dx = \frac{x^2}{2} + xy \Big|_0^1 = \frac{1}{2} + y \quad 0 < y < 1$$

$$p(\frac{1}{4} < y < 1) = \int_{\frac{1}{4}}^{1} (\frac{1}{2} + y)dy = \frac{y}{2} + \frac{y^2}{2} \Big|_{1/4}^{1} = \frac{1}{2} + \frac{1}{2} - \frac{1}{8} - \frac{1}{32}$$

$$= 1 - \frac{4+1}{32} = \frac{27}{32}$$

Example:

Given the joint pdf:

$$f(x, y) = \begin{cases} \frac{2}{3}(x+2y) & 0 < x < 1 \quad 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

Find the marginal of X and Y.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{2}{3}(x+2y) \quad dy$$

$$= \frac{2}{3}(xy + \frac{2y^2}{2} \Big|_0^1 = \frac{2}{3}[X+1] \qquad 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^1 \frac{2}{3}(x+2y)dx$$

$$= \quad \frac{2}{3}[\frac{x^2}{2} + 2xy \Big|_0^1 = \frac{2}{3}[\frac{1}{2} + 2y] \qquad 0 < y < 1$$

Independent Random Variables:

Def: Discrete random variables X and Y are said to be independent if and only if:

$p(X = x, Y = y) = p(X = x) \cdot p(X = y) \quad \forall (x, y) \in \boldsymbol{R}^2$

Continuous random variables X and Y are said to be independent if and only if:

$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \quad \forall (x, y) \in \boldsymbol{R}^2$

The concept of joint *pdf* and independence extend immediately to n random variables, where n is any finite positive integer.

Example:

Let X and Y be two random variables with the following joint *pdf:*

x \ y	0	1	2	p(y)
0	1/9	2/9	1/9	4/9
1	2/9	2/9	0	4/9

2	*1/9*	*0*	*0*	*1/9*
p(x)	*4/9*	*4/9*	*1/9*	*1*

Are X and Y independent?

Since for one entry in the table
$p(X = 0, Y = 0) = 1/9$
$p(X = 0) \,.\, P(Y = 0) = (4/9).(4/9)$
$\therefore$ X and Y are not independent.

Example: Given the joint *pdf*

$$f_{X,Y}(x,y) = \begin{cases} 4xy & 0 < x < 1, \quad 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

Find the marginal densities of X and Y. Are X and Y indep.?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^1 4xydy = 4x\frac{y^2}{2}\Big|_0^1 = 2x \qquad 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 4xydx = 4y\frac{x^2}{2}\Big|_0^1 = 2y \qquad 0 < y < 1$$

$\Theta\ f_{X,Y}(x,y) = f_X(x).f_Y(y)$

$\therefore$ X and Y are independent random variables.

H.W

1. If the joint probability distribution of X and Y is given by:
$f(x, y) = c\,(x + y) \quad for \quad x = 0,1,2,3 \;\; ; \;\; y = 0,1,2$
a. Find the value of c.
b. Construct a table showing the values of the joint distribution function of X and Y.
c. Find the marginal distribution of X and Y.
d. Find $f_{X,Y}(x|\,y)$, $f_{X,Y}(x|y=1)$, $w_{X,Y}(y|x)$, $w_{X,Y}(y|x=2)$.

2. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{4}(2x+y) & 0<x<1, \quad 0<y<1\\ 0 & elsewhere\end{cases}$$

1. Find the marginal density of X and Y.
2. The conditional density of Y given $X=1/4$.
3. The conditional density of X given $Y=1$.

3. Find the joint pdf of the two random variables X and Y whose joint distribution function is given by:

$$F_{X,Y}(x,y)=\begin{cases}1-e^{-x}-e^{-y}+e^{-(x+y)} & x>0, \quad y>0\\ 0 & elsewhere\end{cases}$$

Also find

1. The marginal density of X and Y, $p(X>2, Y>1)$.
2. The conditional density of X given $Y=y$.
3. The conditional density of Y given $X=x$.
4. $P(X>2 | Y>1)$