Probability Density Functions:

A function with values $f_X(x)$, defined over the set of all real numbers, is called a probability density function of the continuous random variable X if and only if:

$$p(a < x \le b) = \int_a^b f(x)dx \quad \text{for any real constants } a < b$$

Note: When X is a discrete random variable $p(X = a) = f_X(a)$, but when X is a continuous random variable $p(X = a) = 0$. So we can look for probabilities of X taking values in intervals.

The probability that X takes on a value in the interval $[a, b]$ is the area under the graph of the density function.

A function can serve as a probability density of a continuous random variable X if its values, $f_X(x)$, satisfy the following conditions:

1. $f_X(x)$ is a piece-wise continuous, with space R that is an interval or union of intervals.

2. $f_X(x) \ge 0 \quad \forall x \in R$

3. $\int_{-\infty}^{\infty} f(x)dx = 1$

Example 1:

If X has the probability density

$$f_X(x) = \begin{cases} \dfrac{2}{x^3} & x \ge 1 \\ 0 & Otherwise \end{cases}$$

Is $f_X(x)$ a p.d.f.?

1. $f(x) \ge 0$

2. $\int_1^{\infty} \frac{2}{x^3}dx \quad = 2\left(-\frac{1}{2}\right)\left[\frac{1}{x^2} -\right]_1^{\infty} = 0 + 1 = 1$

Example 2:

If X has the probability density:

$$f_X(x) = \begin{cases} ke^{-3x} & x > 0 \\ 0 & elsewhere \end{cases}$$

Find *k* and *P(0.5≤ X ≤ 1).*

$$1)\int_{-\infty}^{\infty} f_X(x)dx \quad = \int_{0}^{\infty} ke^{-3x}dx \quad = k\left(-\frac{1}{3}\right)\left[e^{-3x}\right]_0^{\infty} \quad = -\frac{k}{3}[0-1] = \frac{k}{3} = 1$$

$$\therefore k = 3$$

$$2)P(0.5 \le X \le 1) = \int_{0.5}^{1} 3e^{-3x}dx \quad = -e^{-3x} \Big]_{0.5}^{1}$$

$$= -e^{-3} + e^{-1.5} = 0.173$$

***<u>Distribution Function</u>*:**

If *X* is a continuous random variable and the value of its probability density at *t* is $f_X(t)$,then the function given by:

$$F_X(x) = p(X \le x) = \int_{-\infty}^{x} f_X(t)dt \qquad for \quad -\infty \le x \le \infty$$

is called the distribution function, or the cumulative distribution of *X*.

Suppose that $f_X(x)$ and $F_X(x)$ are the values of the probability density and the distribution function of *X* at *x*, then:

$p(a < X \le b) = F_X(b) - F_X(a)$ for any real constants *a* and *b* with $a \le b$.

Remark:

If *X* is a continuous random variable and *a*, *b* are real constants with $a \le b$, then:

$$p(a \le x \le b) = p(a \le x < b) = p(a < x \le b) = p(a < x < b)$$

$$= F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(t)dt - \int_{-\infty}^{a} f_X(t)dt = \int_a^b f_X(t)\, dt$$

Also, for every $a \in \mathbb{R}$,

$$P(X > x) = 1 - P(X \le x) = 1 - F_X(x)$$

Finally, for continuous random variables,

$$f_X(x) = \frac{dF_X(x)}{dx}$$, where the derivative exists.

Remark:

We will use the term probability density function to represent both discrete and continuous random variables.

Example 1:

Find the distribution function of the random variable X of the last example.

$$F_X(x) = \int_{-\infty}^{x} f(t)dt \quad = \int_0^x 3e^{-3t}dt \quad = -e^{-3t}\Big]_0^x \quad = -e^{-3x} + 1$$

$$\therefore F_X(x) = \begin{cases} 0 & x \le o \\ 1 - e^{-3x} & x > 0 \end{cases}$$

To determine $p(0.5 \le X \le 1) = F(1) - F(0.5)$

$$= \left(1 - e^{-3}\right) - \left(1 - e^{-1.5}\right) = 0.173$$

Example 2:

Find a pdf for the random variable whose distribution function is:

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

Or $f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & elsewhere \end{cases}$

Example 3:

Let X be a random variable with:

$$f_X(x) = \begin{cases} \frac{1}{4} & -2 < x < 2 \\ 0 & elsewhere \end{cases}$$

$$F_X(x) = \int_{-2}^{x} f(t)dt \quad = \int_{-2}^{x} \frac{1}{4}dt \quad = \frac{t}{4} \Big]_{-2}^{x} \quad = \frac{x+2}{4}$$

$$\therefore F_X(x) = \begin{cases} 0 & x \le 2 \\ \frac{x+2}{4} & -2 < x < 2 \\ 1 & x \ge 2 \end{cases}$$

$$p(a \le X \le b) = F_X(b) - F_X(a)$$

$$p(-\frac{1}{4} < X < \frac{3}{4}) = F_X(\frac{3}{4}) - F_X(-\frac{1}{4}) = \frac{11}{16} - \frac{7}{16} = \frac{4}{16} = \frac{1}{4}$$

$$p(-5 < X < 1) = F_X(1) - F_X(-5) = \frac{3}{4} - 0 = \frac{3}{4}$$

$$p(-\frac{1}{2} < X < 10) = F_X(10) - F_X(-\frac{1}{2}) = 1 - \frac{3}{8} = \frac{5}{8}$$

Example 4:

Let the random variable *X* be the length of life of an electron tube. Suppose that the probability model foe *X* is:

$$f_X(x) = \frac{1}{100} e^{\frac{-x}{100}} \qquad 0 \le x < \infty$$

1. Is *f(x)* a pdf?

2. Find the probability that this electron tube lasts more than 100 hours.

$i.\ f_X(x) \ge 0$

$$ii. \int_0^\infty f_X(x)dx \quad = \int_0^\infty \frac{1}{100} e^{\frac{-x}{100}} dx \quad = -e^{\frac{-x}{100}} \Big]_0^\infty = 0 + 1 = 1$$

$\therefore\ f_X(x)$ is a pdf.

$$p(X > 100) = \int_{100}^\infty \frac{1}{100} e^{\frac{-x}{100}} dx \quad = -e^{\frac{-x}{100}} \Big]_{100}^\infty = 0 + e^{-1} = 0.368$$

Or $P(X > 100) = 1 - P(X \le 100) = 1 - F_X(100) = 1 - [1 - e^{-1}] = e^{-1}$

<u>H.W.</u>:

1. Show that the following function is a pdf

$$f_X(x) = \begin{cases} \frac{1}{5} & 2 < x < 7 \\ 0 & elsewhere \end{cases}$$

Find the distribution function and $p(3 < X < 5)$.

2. The pdf of the random variable *X* is:

$$f_X(x) = \begin{cases} kxe^{-x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find k, and then find the distribution function of *X*.

3. Let X be a continuous random variable with distribution function:

$$F_X(x) = \begin{cases} ax^2 & 0 < x < 2 \\ 0 & x \leq 0 \\ 1 & 2 \leq x \end{cases}$$

Find $f_X(x)$ of X , $P(1<X<1.5)$, and $P(X \geq 0.8)$.

4. Let X be a random variable with pdf

$$f_X(x) = \begin{cases} kx^2 & -1 < x < 1 \\ \\ 0 & o.w \end{cases}$$

- Find the value of k and $F_X(x)$.
- Find $P(0 < X < 1)$, $P(0 < X \leq 3)$, $P(X = \frac{1}{2})$ and $P(-1 < X < \frac{1}{2})$.