Expectation of a Sum of Random variables:

Theorm:

If X and Y are random variables, then

$E(X+Y) = E(X) + E(Y)$

Proof:

$$E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\,f(x+y)dxdy$$

$$=\int_{-\infty}^{\infty}\int_{=\infty}^{\infty}xf(x,y)dxdy \quad +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}yf(x,y)dxdy$$

$$=\int_{-\infty}^{\infty}x(\int_{-\infty}^{\infty}f(x,y)dy)dx+\int_{-\infty}^{\infty}y(\int_{=\infty}^{\infty}f(x,y)dx)dy$$

$$=\int_{-\infty}^{\infty}xf(x)dx \quad +\int_{-\infty}^{\infty}yf(y)dy \qquad =E(X)+E(Y)$$

A similar result can be shown in the discrete case.

In general, for any n,

$$E(X_1+X_2+...+X_n)=E(X_1)+E(X_2)+...+E(X_n)$$

Expectation of a Product of Independent Random Variables:

Theorem:

If X and Y are two independent random variables,then

$E(XY) = E(X)E(Y)$

Proof:

$$E(XY) = \sum_x \sum_y xyf(x,y) = \sum_x \sum_y xyf(x)\, f(y)$$
$$= \sum_x xf(x) \quad \sum_y yf(y)$$
$$= E(X)E(Y)$$

Covariance and Correlation Coefficient:

Def: If X and Y are two random variables with respective means $\overline{X}$ and $\overline{Y}$, then the covariance between X and Y is

$$\mathbf{cov}(X,Y) = E(X - \overline{X})(Y - \overline{Y})$$

Note: If X and Y are independent random variables, then
$\text{cov}(X,Y) = E(X - \overline{X})E(Y - \overline{Y}) = 0$

Since

$$E(X - \overline{X}) = E(X) - E(\overline{X}) = \overline{X} - \overline{X} = 0$$

A useful expression for *cov(X,Y)* can be obtained by expanding the right side of the definition. This yields:

$$\text{cov}(X,Y) = E(X - \overline{X})(Y - \overline{Y}) = E(X - \mu_x)(Y - \mu_y)$$
$$= E(XY - \mu_x Y - \mu_y X + \mu_x \mu_y)$$
$$= E(XY) - \mu_x E(Y) - \mu_y E(X) + \mu_x \mu_y$$
$$= E(XY) - 2\mu_x \mu_y + \mu_x \mu_y = E(XY) - E(X)E(Y)$$

From its definition, we see that:

cov(X, Y) = cov(Y, X) and

cov(X, X) = var(X)

Def : The ratio

$$\rho_{XY} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} = \frac{E(X-\bar{X})(Y-\bar{Y})}{\sqrt{E(X-\bar{X})^2}\ \sqrt{E(Y-\bar{Y})^2}}$$

is defined as the correlation coefficient between X and Y.