Chapter One

Random Variables and Probability Distributions

Definition:

Let $(\Omega, \mathcal{A}, P)$ be a probability measure space and let $X: \Omega \to \mathbb{R}$ be a function from Ω into the set $\mathbb{R}$ of real numbers such that the pre-image of every interval of $\mathbb{R}$ is an event of Ω. Then X is a random variable.

That is $X: \Omega \to \mathbb{R}$ is a random variable if:

$$X^{-1}\big((-\infty, x]\big) = \{\omega \in \Omega : X(\omega) \in (-\infty, x]\} \in \mathcal{A}, \forall\, x \in \mathbb{R}.$$

i.e., the inverse X^{-1}maps all of the sets$(-\infty, x] \subset \mathbb{R}$ to $\mathcal{A}$.

Remark: In general, if $\mathcal{A} = \mathcal{P}(\Omega)$, any function $X: \Omega \to \mathbb{R}$ is a random variable.

Example:

Consider the random experiment of flipping a fair coin twice. The sample space is $\Omega = \{HH, HT, TH, TT\}$. Since Ω is finite we take $\mathcal{A} = \mathcal{P}(\Omega)$. Suppose that we are primarily interested in the number of tails that have occurred, i.e, suppose that our interest is on the function $X: \Omega \to \mathbb{R}$ defined by:

$$X(\omega) = \begin{cases} 0 & if\ \omega = \{HH\} \\ 1 & if \omega = \{HT, HT\} \\ 2 & if\ \omega = \{TT\} \end{cases}$$

Verify that X is a random variable.

$\forall\, x \in \mathbb{R}$

$$X^{-1}\big((-\infty, x]\big) = \begin{cases} \emptyset \in \mathcal{A} & x < 0 \\ \{HH\} \in \mathcal{A} & 0 \le x < 1 \\ \{HH, HT, TH\} \in \mathcal{A} & 1 \le x < 2 \\ \Omega \in \mathcal{A} & 2 \le x \end{cases}$$

It is clear that for each outcome ω of Ω, there corresponds a real number $X(\omega)$.Hence, X is a random variable.
Sometimes it is easier to work with the following equivalent condition:

$\forall x \in \mathbb{R} \Longrightarrow \{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{A}$

This means that once we know the value $X(\omega)$, we know which of the events in $\mathcal{A}$ have happened.

Note:

- If X is a random variable then $aX+b$ will also be a random variable where a, b are constants. Also X^2 and $\frac{1}{X}$ will be random variables.
- Sums and products of random variables are also random variables.

<u>Discrete and Continuous Random Variables</u>:

<u>Def.</u> A random variable is called discrete if its range includes finite number of values or countable infinite number of values $x_1, x_2, \ldots$.

Example: The number of members of a family, number of passengers on a plane, number of road accidents occurs in a day in a city …etc.

<u>Def.</u> A random variable is called continuous if it can assume any value from a specified interval of the form $[a, b]$. That is; if for some $a < b$ any number x between a and b is possible.

Example: Heights of students in a class, temperature and barometric pressures of different cities, age of people or objects … etc.

Distribution Function:

With every random variable is associated another function known as its distribution function.

Def. The cumulative distribution function of a random variable X is a function F_x: $\mathbb{R} \to [0, 1]$ which satisfies

$$F_X(x) = P[-\infty < X \leq x] = P[X \leq x] = P[\{\omega : X(\omega) \leq x\}], -\infty < x < \infty$$

Example:

Find the distribution function of the random variable X in the previous example.

$$F_X(x) = P(X^{-1}(-\infty, x]) = P[-\infty < X \leq x] = P[X \leq x]$$

$$= \begin{cases} P(\emptyset) = 0 & x < 0 \\ P(HH) = \frac{1}{4} & 0 \leq x < 1 \\ P(\{HH, HT, TH\}) = \frac{3}{4} & 1 \leq x < 2 \\ P(\Omega) = 1 & 2 \leq x \end{cases}$$

Lemma:

If F_x is the distribution function of the random variable X and if a < b, then

1. $P(a < X \leq b) = F_X(b) - F_X(a)$

Proof:

$(-\infty, b] = (-\infty, a] \cup (a, b]$

$P(-\infty < X \leq b) = P(-\infty < X \leq a) + P(a < X \leq b)$

$P(a < X \leq b) = P(-\infty < X \leq b) - P(-\infty < X \leq a)$

$= F_X(b) - F_X(a)$

2. $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$

Proof:

$(-\infty, b] = (-\infty, a] \cup (a, b) \cup \{b\}$

$P(-\infty < X \leq b) = P(-\infty < X \leq a) + P(a < X < b) + P(X = b)$

$$P(a < X < b) = P(-\infty < X \leq b) - P(-\infty < X \leq a) - P(X = b)$$

$P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$

3. $P(a \leq X \leq b) = F_X(b) - F_X(a) + P(X = a)$

Proof:

$(-\infty, b] = (-\infty, a] \cup [a, b]$

$P(-\infty < X \leq b) = P(-\infty < X \leq a) + P(a \leq X \leq b) - P(X = a)$

$$P(a \leq X \leq b) = P(-\infty < X \leq b) - P(-\infty < X \leq a) + P(X = a)$$

$= F_X(b) - F_X(a) + P(X = a)$

4. $P(a \leq X < b) = F_X(b) - F_X(a) - P(X = b) + p(X = a)$

Proof:

$(-\infty, b] = (-\infty, a\ ] \cup [\mathrm{a, b}) \cup \{b\}$

$P(-\infty < X \leq b)$

$= P(-\infty < X \leq a) + P(a \leq X < b) - P(X = a) + P(X = b)$

$P(a \leq X < b)$

$$= P(-\infty < X \leq b) - P(-\infty < X \leq a) + P(X = a) - P(X = b)$$

$= F_X(b) - F_X(a) + P(X = a) - P(X = b)$