Properties of a Cumulative Distribution Function:

1. $F_X(-\infty) = \lim_{x\to-\infty} F_X(x) = 0$, and $F_X(\infty) = \lim_{x\to\infty} F_X(x) = 1$

Proof:

$$\lim_{x\to-\infty} F_X(x) = \lim_{x\to-\infty} P(-\infty < X \le x)$$

$$= \lim_{x\to-\infty} P(X^{-1}(-\infty, x]) = P\left(x^{-1}(-\infty, -\infty)\right) = P(\emptyset) = 0$$

Also,

$$\lim_{x\to\infty} F_X(x) = \lim_{x\to\infty} P(-\infty < X \le x)$$

$$\lim_{x\to\infty} P(X^{-1}(-\infty, x]) = P(X^{-1}(-\infty, \infty]) = P(\Omega) = 1$$

2. The function $F_X(x)$ is a monotone non-decreasing function; that is, if $a < b$, then $F_X(a) \le F_X(b)$ for any real numbers a and b.

Proof:

$$F_X(b) = P(X \le b) = P(X \le a) + P(a < X \le b)$$

$$F_X(b) = F_X(a) + P(a < X \le b)$$

Hence, $F_X(a) \le F_X(b)$

3. F_x is continuous from the right at each point x; that is

$$\lim_{h\to0} F_X(x+h) = F_X(x), \qquad \text{with } h > 0.$$

Remark: If we want to use $F_X(x)$ to compute the $P(X=b)$.

$$P(X = b) = \lim_{h\to0} P(b - h < X \le b)$$

$$= \lim_{h\to0}\left(F_X(b) - F_X(b-h)\right) = F_X(b) - \lim_{h\to0} F_X(b-h)$$

That is, $P(X=b)$ is the height of the step that $F_X(X)$ has at $x=b$.

If $F_X(x)$ is continuous at $X=b$, then there is no jump in $F_X(x)$ at b. Hence, $P(X=b)=0$.

Example:

Let $F_X(X) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & otherwise \end{cases}$

Then, $P(X=1) = F_X(1) - \lim\limits_{h \to 0} F_X(1-h)$

$$= 1 - e^{-1} - \left(1 - \mathrm{e}^{-(1-0)}\right) = 0$$

<u>Probability Distributions</u>:

If X is a discrete random variable with distinct values $X_1, X_2, \ldots, X_n$, …,and S is a countable set of real numbers, then the function $f_x: \mathbb{R} \to \mathbb{R}$ given by

$$f_X(x) = \begin{cases} P(X = x) & if\ x \in S \\ 0 & if\ x \notin S \end{cases}$$

is called the probability distribution or the probability mass function of X, and the set S is called the support of X.

A function can serve as the probability distribution of a discrete random variable X if its values, $f_X(x)$, satisfy the following conditions:

1. $f_X(x) > 0 \ \forall\, x \in S$
2. $f_X(x) = 0 \ \forall\, x \notin S$
3. $\sum_{x \in S} f_X(x) = \sum_{x \in S} P[(X = x)] = P(\{X \in S\}) = 1$

Example 1:

Let $f(x) = \begin{cases} \frac{x}{10} & x = 1, 2, 3, 4 \\ 0 & elsewhere \end{cases}$

Is *f(x)* a probability mass function?

Note that f_X is of a discrete type r.v.

1. $p(X = 1) = \dfrac{1}{10}$, $p(X = 2) = \dfrac{2}{10}$, $p(X = 3) = \dfrac{3}{10}$, $p(X = 4) = \dfrac{4}{10}$

2. $\sum_{x \in S} f(x) = \dfrac{1}{10} + \dfrac{2}{10} + \dfrac{3}{10} + \dfrac{4}{10} = 1$

$\therefore$ *f(x)* is a probability mass function.

Example 2:

Check whether the function:

$f(x) = \dfrac{x+2}{25}$, for $x = 1, 2, 3, 4, 5$

can serve as the probability distribution of a discrete random variable.

1) $P(X = 1) = \dfrac{3}{25}$, $P(X = 2) = \dfrac{4}{25}$, $P(X = 3) = \dfrac{5}{25}$, $P(X = 4) = \dfrac{6}{25}$,

$P(X = 5) = \dfrac{7}{25}$

2) $\sum_{x \in S} f_X(x) = 1$

<u>Distribution Function</u>:

If X is a discrete random variable, the function given by:

$$F_X(x) = P(X \leq x) = \sum_{\substack{t \in S \\ t \leq x}} f_X(t)$$

Where $f_X(t)$ is the value of the probability distribution of X at t, is called the distribution function, or the cumulative distribution of X.

Example:

Three coins are tossed. How many fall "heads"?

Let X be the number of heads which appear in the three tosses. Thus number may be 0, 1, 2, or 3.

A sample space for the experiment and the number of heads for each sample point and the probability of each sample point are shown in the following table:

Sample point	*HHH*	*HHT*	*HTH*	*THH*	*HTT*	*THT*	*TTH*	*TTT*
Number of heads	3	2	2	2	1	1	1	0
Probability	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

Then

$S = \{x \in \mathbb{R}: x = 0, 1, 2, 3\}$

$$f_X(x) = \begin{cases} \frac{1}{8} & x = 0 \\ \frac{3}{8} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{1}{8} & x = 3 \\ 0 & o.w. \end{cases}$$

It is usually preferable to give a formula to express the probabilities by means of a function such that its values, $f_X(x)$, equal $p(X=x)$ for each x.

Observing that the numerators of these probabilities, 1, 3, 3, 1, are the binomial coefficients C_0^3, C_1^3, C_2^3, C_3^3, we find that the formula for the probability distribution can be written as:

$$f_X(x) = \frac{C_x^3}{8} \qquad \text{for } x = 0, 1, 2, 3$$

The distribution function of the random variable X is given by:

$$F_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ \frac{1}{8} & 0 \le x < 1 \\ \frac{4}{8} & 1 \le x < 2 \\ \frac{7}{8} & 2 \le x < 3 \\ 1 & 3 \le x \end{cases}$$

Example:

Find the distribution function of the total number of heads obtained in four tosses of a coin.

Ω= {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}.

$$f(x) = \frac{C_x^4}{16} \qquad x = 0, 1, 2, 3, 4$$

$$F_X(0) = f_X(0) = \frac{1}{16}$$

$$F_X(1) = f_X(0) + f_X(1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

$$F_X(2) = f_X(0) + f_X(1) + f_X(2) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} = \frac{11}{16}$$

$$F_X(3) = f_X(0) + f_X(1) + f_X(2) + f_X(3) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{15}{16}$$

$$F_X(4) = \sum_{x \in S} f_X(x) = 1$$

Hence, the distribution function is given by:

$$F(x) = \begin{cases} 0 & for \quad x < 0 \\ \frac{1}{16} & for \quad 0 \le x < 1 \\ \frac{5}{16} & for \quad 1 \le x < 2 \\ \frac{11}{16} & for \quad 2 \le x < 3 \\ \frac{15}{16} & for \quad 3 \le x < 4 \\ 1 & for \quad x \ge 1 \end{cases}$$

Observe that this distribution function is defined for all real numbers.
For instance $F_X(1.7) = 5/16$, $f_X(100) = 1$.
The distribution function will be a step function since it is defined at discrete points only.

Example:

In the case of rolling a fair die. Let $X = \{1, 2, 3, 4, 5, 6\}$ be the random variable representing the face of the die that turns up. Then

$$f_X(x) = \frac{1}{6} \qquad\qquad x = 1, 2, 3, 4, 5, 6$$

$$F_X(x) = p(X \le x) = \frac{x}{6} \qquad\qquad x = 1, 2, 3, 4, 5, 6$$

Theorem:
Let X be a discrete random variable, then $F_X(x)$ can be obtained from $f_X(x)$, and vice versa.
Proof:
Suppose that $F_X(x)$ is given; then

$$f_X(x_j) = F_X(x_j) - \lim_{h\to 0} F_X(x_j - h)$$

And $f_X(x) = 0$ for $x \neq x_j$, $j = 1, 2, \ldots$
So, $f_X(x)$ is determined for all real numbers.

Example:

If X has the distribution function:

$$F(x)=\begin{cases}0 & for \quad x<0\\ \frac{3}{28} & for \quad 0\le x<1\\ \frac{18}{28} & for \quad 1\le x<2\\ 1 & for \quad x\ge 2\end{cases}$$

Find the probability distribution for this random variable.

$$P(X=0)=F_x(0)-\lim_{h\to 0}F_x(0-h)=\frac{3}{28}-0=\frac{3}{28}$$
$$P(X=1)=F_x(1)-\lim_{h\to 0}F_x(1-h)=\frac{18}{28}-\frac{3}{28}=\frac{15}{28}$$
$$P(X=2)=F_x(2)-\lim_{h\to 0}F_x(2-h)=1-\frac{18}{28}=\frac{10}{28}$$

***H.W.*:**

1. Determine *c* so that the following functions can serve as the probability distribution of a random variable with the given range:

(a) $f_X(x) = cx$ $\quad x = 1, 2, 3, 4, 5$

(b) $f_X(x) = c\ C_x^5$ $\quad x = 0, 1, 2, 3, 4, 5$

2. Check whether the following function can serve as a probability distribution of a random variable, find the distribution function.

$$f_X(x)=\frac{C_x^2 C_{3-x}^4}{C_3^6} \qquad for \quad x=0,1,2$$

2. Find the distribution function of the random variable which has the probability distribution:

$$f_X(x) = \frac{x}{15} \qquad \textit{for} \qquad x = 1,2,3,4,5$$