$\rightarrow A = -\frac{1}{2}\ ,\ B = \frac{3}{2}$

$$\therefore f(z) = \frac{-1}{2}(z-1)^{-1} + \frac{3/2}{z-3}$$

$$= \frac{-1}{2}(z-1)^{-1} - \frac{3/2}{2-(z-1)}$$

$$= \frac{-1}{2}(z-1)^{-1} - \frac{3/2}{2\left[1-\frac{(z-1)}{2}\right]}$$

$$= \frac{-1}{2}(z-1)^{-1} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

$$= \frac{-1}{2}(z-1)^{-1} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

Example: Represent the function

$$f(z) = \frac{1}{1+z}$$

into a series of negative power of z.

Solution:

$$\frac{1}{1+z} = \frac{1}{z\left(1+\frac{1}{z}\right)}$$

$$= \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n$$

$$= \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\ ,\ \left(\left|\frac{1}{z}\right| < 1 \rightarrow |z| > 1\right)$$

Example: Evaluate Taylor series of $f(z) = \log(1+z)$ about zero.

Solution: note that $f(z) = \log(1+z)$ is not analytic when

$$Im(1+z) = 0 \text{ and } Re(1+z) < 0$$

$\rightarrow y = 0$ and $x + 1 < 0 \rightarrow x < -1$

$f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z} \to f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2} \to f''(0) = -1$

$f^{(3)}(z) = \frac{2}{(1+z)^3} \to f^{(3)}(0) = 2$

$\vdots$

$\therefore f(z) = \log(1+z) = z - \frac{1}{2!}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots, (|z| < 1)$

[2] Laurent Series

If a function f fails to be analytic at z_0, then we can apply Taylor's theorem at z_0. It is possible however, to find a series representation for $f(z)$ involving both positive and negative powers of $(z - z_0)$. Now, we represent the theory of such representation and begin with Laurent theorem.

<u>Laurent's Theorem</u>:

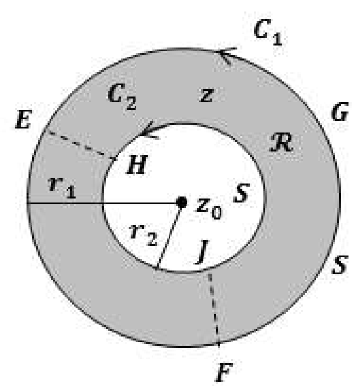
If $f(z)$ is analytic inside and on the boundary of the ring $\mathcal{R}$ bounded by two concentric C_1 and C_2 with center z_0 and respective radii r_1 and r_2, then for all z in $\mathcal{R}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

Such that:

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)dz}{(z-z_0)^{n+1}}, \quad n = 0,1,2, \ldots$$

$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)dz}{(z-z_0)^{-n+1}}, \quad n = 1,2, \ldots$$



Note:

1. If f is analytic on and inside C_2, then $b_n = 0$, i.e.:

$$b_n = \int f(z)(z - z_0)^{n-1}\, dz \,, n = 1,2, \dots$$

2. Laurent's theorem can be reform as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Such that: $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)dz}{(z-z_0)^{n+1}}$, $n = 0,1,2, \dots$, where C is any simple closed curve lies between C_1 and C_2.

3. If f is analytic on and inside C_1 then the Laurent series turns to Taylor series.
4. The Laurent series expansion contains negative powers and usually begins from $-\infty$.
5. The Taylor series expansion about z_0 is a special case of Laurent expansion, that is when calculating Laurent coefficients in this case all the negative power coefficients appear as zeros and the Taylor series remains.

   For example, if $n = -1$ then $(z - z_0)^0 = 1$ and the function is analytic i.e. ($\oint_C = 0$), and it is the same when $n = -2, -3, \dots$

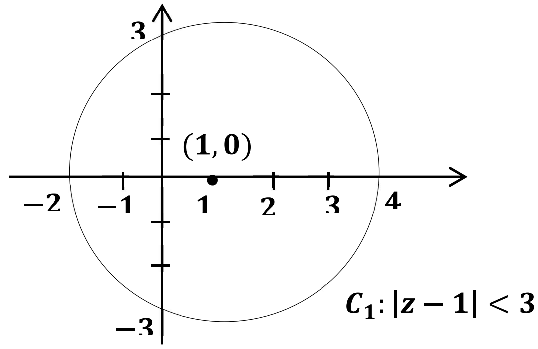
6. z_0 might be the only singular point of f on C_1 and in this case $0 < r_2 < r_1$, so the series will be at the region $0 < |z - z_0| < r_1$.
7. Taylor series can be written for a point inside the circle.
8. Laurent series can be written for a point outside the circle.

Example: Represent $f(z) = \frac{1}{(z-1)(z+2)}$ into a Laurent series about $0 < |z - 1| < 3$.

Solution:

$$f(z) = \frac{1}{(z-1)(z+2)}$$

$$= \frac{1}{3}\,\frac{1}{(z-1)} - \frac{1}{3}\,\frac{1}{(z+2)}$$



We don't need to make it $(z-1)$, the singular points are $1, -2$. Note that the singular point $0 < |z-1| < 3 \not\ni -2$.

$$f(z) = \frac{1}{3(z-1)} - \frac{1}{3}\left(\frac{1}{3+z-1}\right)$$

$$= \frac{1}{3(z-1)} - \frac{1}{3}.\frac{1}{3}\left[\frac{1}{1+\frac{z-1}{3}}\right]$$

$\rightarrow \frac{1}{1+\frac{z-1}{3}}$ is a geometric series and $a = 1\,, r = \frac{z-1}{3}$ with alternative sign $(-1)^n$.

$$f(z) = \frac{1}{3(z-1)} - \frac{1}{3^2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{3^n}$$

$$\left|\frac{z-1}{3}\right| < 1 \rightarrow |z-1| < 3$$

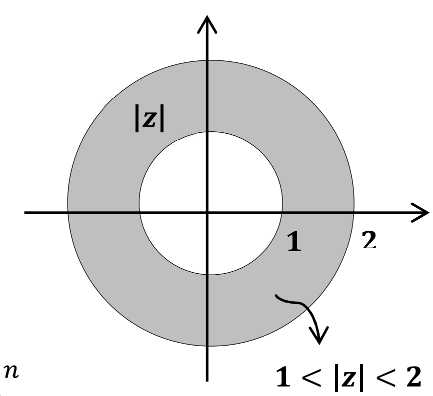
The other part of the region $|z-1| > 0$ we avoid that $|z-1| \neq 0$ in the term $\frac{1}{3(z-1)}$.

In the same example about $1 < |z| < 2$

$$f(z) = \frac{1}{3}\,\frac{1}{(z-1)} - \frac{1}{3}\,\frac{1}{(z+2)}$$

$$= \frac{1}{3}.\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{3}.\frac{1}{2}\left(\frac{1}{1+\frac{z}{2}}\right)$$

$\|1/z\| < 1$	$\|z/2\| < 1$
$\rightarrow 1 < \|z\|$	$\rightarrow \|z\| < 2$



$$\therefore f(z) = \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

Example: Expand $f(z) = \frac{1}{z^2(z-1)(z-2)}$ in a Laurent series about $z_0 = 0$.

Solution: the possibilities depend on the singular points $0, 1, 2$:

Case 1: $0 < |z| < 1$

Case 2: $1 < |z| < 2$

Case 3: $2 < |z| < \infty$

♦ Case 1: if $0 < |z| < 1$

$$f(z) = \frac{1}{z^2}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}\right]$$

$$= \frac{1}{z^2}\left[\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right]$$

$|z| < 1$ & $\left|\frac{z}{2}\right| < 1 \rightarrow |z| < 1$ & $|z| < 2$, note the connection between the intervals and the solution, then $0 < |z| < 1$.

♦ Case 2: if $1 < |z| < 2$

$$f(z) = \frac{1}{z^2}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z^2}\left[\frac{-1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right]$$

$\left|\frac{z}{2}\right| < 1 \rightarrow |z| < 2$ & $\left|\frac{1}{z}\right| < 1 \rightarrow |z| > 1$, note the connection between the intervals and the solution, then $1 < |z| < 2$.

$$\therefore f(z) = \frac{1}{z^2}\left[\frac{-1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$

$$= -\left[\sum_{n=0}^{\infty}\frac{z^{n-2}}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{z^{n+3}}\right]$$

♦ Case 3: if $2 < |z| < \infty$ ($|2/z| < 1$)

$$f(z) = \frac{1}{z^2}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right]$$

We need $|z| > 2$, so

$\left|\frac{2}{z}\right| < 1 \rightarrow |z| > 2$ & $\left|\frac{1}{z}\right| < 1 \rightarrow |z| > 1$, note the connection between the two intervals and the solution, then $2 < |z|$.

$$\therefore f(z) = \frac{1}{z^3}\left[\sum_{n=0}^{\infty}\frac{2^n}{z^n} - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$

<u>Example</u>: Expand $f(z) = \frac{e^z}{z^2}$ in a Laurent series.

<u>Solution</u>: $f(z)$ is analytic everywhere except the origin. We take C_1 big and C_2 a little smaller.

$$f(z) = \frac{e^z}{z^2} = \frac{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots}{z^2}$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots, \quad 0 < |z| < \infty$$

<u>Example</u>: Write $f(z) = \frac{\sin z}{z^2-1}$ into a Laurent series in powers of $z-1$.

<u>Solution</u>:

1. Locate the singular points which are $-1,1$.
2. Leave every factor of the form $z-1$ in the denominator and otherwise is considered a part of the numerator.

$$f(z) = \frac{\sin z}{z^2-1} = \frac{\sin z}{(z-1)(z+1)}$$

$$= \frac{\sin z/(z+1)}{z-1}$$

3. Write Taylor expansion for the new numerator about $z_0 = 1$ and then simplify to get Laurent series,

$$\frac{\sin z}{(z+1)} = \frac{\sin 1}{1+1} + \frac{(\sin z/(z+1))'|_{z=1}}{1!}(z-1) + \frac{(\sin z/(z+1))''|_{z=1}}{2!}(z-1)^2 + \cdots$$

$$f(z) = \frac{\sin z/(z+1)}{z-1}$$