Chapter Two

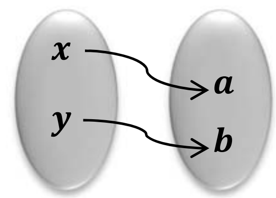
Analytic Functions

[1] Functions of a Complex Variable

Definition:

A function f defined on a set A to a set B is a rule assigns a unique element of B to each element of A; in this case we call f a single function. i.e.: $f: A \to B$, $A, B \subseteq \mathbb{C}$

$$\forall z \in A, \exists!\, w \in B \text{ s.t } w = f(z) \in B$$



Definition:

The domain of f in the above def. is A and the range is the set R of elements of B which f associate with elements of A.

Note: The elements in the domain of f are called independent variables and those in the range of f are called dependent variables.

Definition:

A f rule which assigns more than one number of B to any number of A is called a multiple valued function.

Example:

1. $f(z) = (z)^{1/2}$

Has two roots therefore $f(z)$ is a multiple function.

2. $f(z) = (z)^{3/5} = (z^3)^{1/5}$

Has five roots therefore $f(z)$ is a multiple function. In general, if $f(z) = \arg z$ then f is a multiple function.

3. If $f(z) = \operatorname{Arg} z$ then f is a single function.

Note:

1. Let $f: Z \to W$, if Z and W are complex, then f is called complex variables function (a complex function) or a complex valued function of a complex variable.

2. If A is a set of complex numbers and B is a set of real numbers then f is called real−valued function of a complex variable, conversely f is a complex−valued function of real variables.

Example: Find the domain of the following functions

1. $f(z) = \frac{1}{z}$

Ans.: $D_f = \mathbb{C} \setminus \{0\}$

2. $f(z) = \frac{1}{z^2+1}$

Ans.: $D_f = \mathbb{C} \setminus \{-i, i\}$

3. $f(z) = \frac{z+\bar{z}}{2}$

Ans.: $D_f = \mathbb{C}$, f is real−valued.

4. $f(z) = y \underbrace{\int_0^\infty e^{-xt}\, dt}_{\text{Improper integral}} + i \underbrace{\sum_{n=0}^\infty y^n}_{\text{Geometric series}}$

Ans.: $D_f = x \in (0, \infty)$ and $y \in (-1,1)$

(What are the conditions that must be satisfied for x so the integration will be converging?)

Definition: A complex function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

n is a positive integer and $a_0, a_1 \dots a_n \in \mathbb{C}$, is a polynomial of degree n $(a_n \neq 0)$.

Definition: A function $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are two polynomials, is called a rational function.

Note: $D_f = \mathbb{C} \setminus \{z : Q(z) \neq 0\}$

♦ Suppose that:

$w = u + iv$ is the value of a function f at $z = x + iy$

$$\text{i.e.}: f(z) = f(x + iy) = u + i\,v$$

each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real−valued functions of real variables x and y.

$$f(z) = u(x, y) + i\,v(x, y)$$

If the polar coordinates r and θ are used instead of x and y, then

$$u + i\,v = f(re^{i\theta})$$

Where $w = u + iv$ and $z = re^{i\theta}$, in that case, we may write

$$f(z) = u(r, \theta) + i\,v(r, \theta)$$

Example: If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i\,2xy$$

Hence: $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$, when polar coordinates are used

$$f(re^{i\theta}) = (re^{i\theta})^2$$

$$= r^2 e^{i2\theta}$$

$$= r^2 \cos 2\theta + i\,r^2 \sin 2\theta$$

Therefore: $u(r, \theta) = r^2 \cos 2\theta$

$$v(r, \theta) = r^2 \sin 2\theta$$

Note: If $v(x,y)=0$ then f is real, i.e. f is real−valued function.

[1] Limits

Let f be a function at all points z in some deleted neighborhood of z_0, the statement that the limit of $f(z)$ as z approaches z_0 is a number w_0, or that

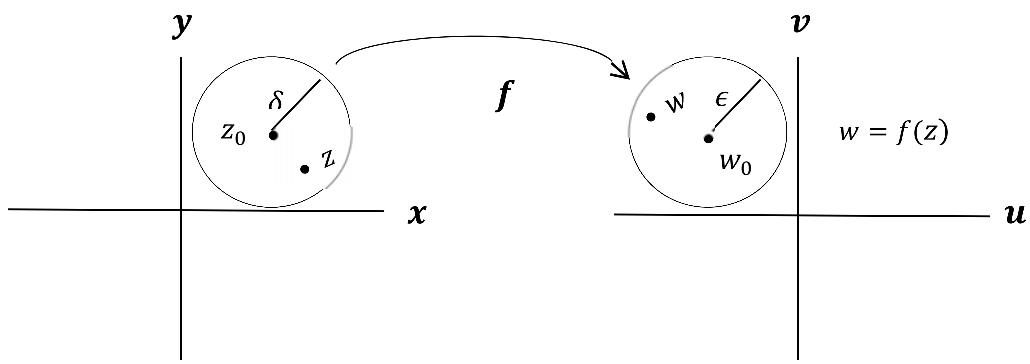
$$\lim_{z\to z_0} f(z) = w_0$$

Means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| < \delta$$

And this means: $z \to z_0$ in $z -$ plane

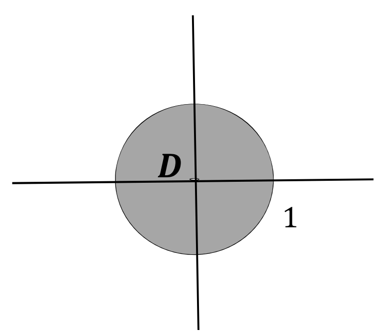
$w \to w_0$ in $w -$ plane



Example: Prove that

$$\lim_{z\to 1}\frac{iz}{2} = \frac{i}{2}$$

Such that f is defined on $|z| < 1$.



Proof: $f(z) = \dfrac{iz}{2}$

Let $\epsilon > 0$, T.p. $\exists\, \delta > 0$ such that

$$|z-1| < \delta \to \left|f(z) - \frac{i}{2}\right| < \epsilon$$

To find δ

$$\left|f(z)-\frac{i}{2}\right|=\left|\frac{iz}{2}-\frac{i}{2}\right|=\left|\frac{1}{2}i(z-1)\right|$$

Let $\delta=2\epsilon$ then:

$$\left|f(z)-\frac{i}{2}\right|=|i|\left|\frac{z-1}{2}\right|<\frac{\delta}{2}<\epsilon$$

Note: $|i|=1$

Example: If $f(z)=z^2,\ |z|<1,$ prove that

$$\lim_{z\to 1} z^2=1$$

Proof: Let $\epsilon>0$, T.p. $\exists\ \delta>0$ s.t

$$|z^2-1|<\epsilon \text{ whenever } 0<|z-1|<\delta$$

$$|z^2-1|=|z+1||z-1|\le(|z|+1)|z-1|$$

$$<2|z-1|<\epsilon$$

$$=|z-1|<\frac{\epsilon}{2}$$

$\therefore$ chose $\delta=\frac{\epsilon}{2}$

$\therefore \lim_{z\to 1} z^2=1$

Example: Prove that

$$\lim_{z\to 1+2i}[(2x+y)+i(y-x)]=4+i$$

Proof: $f(z)=(2x+y)+i(y-x)$

$z_0=1+2i\,,\qquad z=x+iy$

$L=4+i$

Let $\epsilon>0$, T.p. $\exists\ \delta>0$ s.t $0<|z-z_0|<\delta\ \to|f(z)-L|<\epsilon$

$$|z-z_0|=|x+iy-1-2i|$$

$$=|(x-1)+i(y-2)|<\delta$$

$\rightarrow |x-1| \leq |(x-1)+i(y-2)|$

$$\begin{aligned}|f(z)-L| &= |2x+y+i(y-x)-4-i| \\ &\leq |2x+y-4+i(y-x-1)| \\ &\leq |2x-2+y-2|+|i(y-x-1)| \\ &= |2x-2+y-2|+|y-2-x+1| \\ &\leq 2|x-1|+|y-2|+|y-2|+|x-1| \\ &= 3|x-1|+2|y-2|\end{aligned}$$

Let $\delta = \min\left(\frac{\epsilon}{6}, \frac{\epsilon}{4}\right) = \frac{\epsilon}{6}$

Such that $|x-1| < \delta < \frac{\epsilon}{6}$

$|y-2| < \delta < \frac{\epsilon}{4}$

$\rightarrow |f(z)-L| \leq \frac{3\epsilon}{6}+\frac{2\epsilon}{4} < \epsilon$

Exercise: Prove that

$$\lim_{z\to z_0} z^2 = z_0^2$$

Properties of Limit:

1. If $f(z)=c$ then $\lim_{z\to z_0} f(z) = c$.

2. If $f(z)=z$ then $\lim_{z\to z_0} f(z) = z_0$.

3. $\lim_{z\to z_0}\big(f(z) \mp g(z)\big) = \lim_{z\to z_0} f(z) \mp \lim_{z\to z_0} g(z)$.

4. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$

5. $\lim_{z\to z_0} f(z).g(z) = \lim_{z\to z_0} f(z).\lim_{z\to z_0} g(z)$

Proof:

1- Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$ s.t $|f(z) - c| < \epsilon$ whenever $|z - z_0| < \delta$

$\rightarrow |f(z) - c| = |c - c| = 0$

Let δ be any real number

$\therefore \lim\limits_{z \to z_0} f(z) = c$

2- Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$, $|f(z) - z_0| < \epsilon$ if $|z - z_0| < \delta$

$\rightarrow |f(z) - z_0| = |z - z_0| < \epsilon$

Chose $\epsilon = \delta$

$\therefore \lim\limits_{z \to z_0} f(z) = z_0$

Example: Find limit $f(z)$ if its exist, such that

$$f(z) = \frac{2xy}{x^2+y^2} + \frac{x^2}{1+y}\, i$$

Proof: Assume that limit $f(z)$ exists.

Let $y = 0$, we get

$$\lim_{z \to z_0 = 0} f(z) = \lim_{(x,y) \to (0,0)} f(z) = \lim_{x \to 0} x^2 i = 0$$

Let $x = 0$, we get $\lim f(z) = 0$

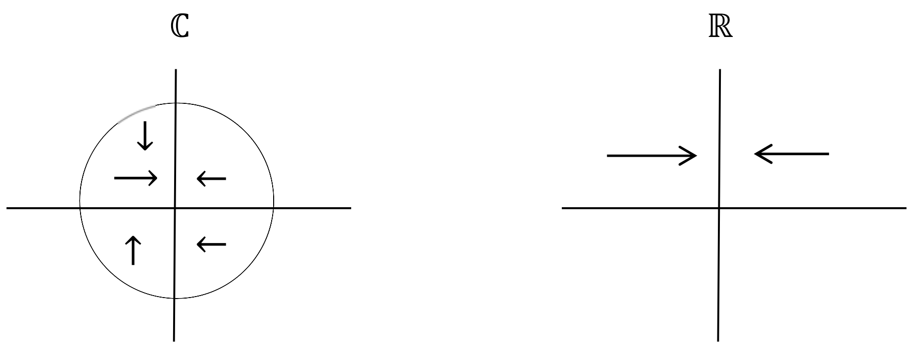
Let $y = x$, then

$$\lim_{z \to 0} f(z) = \lim_{(x,x) \to (0,0)} f(z) = \lim_{(x,x) \to (0,0)} \left(\frac{2x^2}{2x^2} + \frac{x^2}{1+x}\, i \right)$$

$$\lim_{(x,x) \to (0,0)} \left(1 + \frac{x^2}{1+x}\, i \right) = 1 + \lim_{(x,x) \to (0,0)} \frac{x^2}{1+x}\, i = 1 + 0 = 1$$

This is impossible; therefor this limit is not exist.

Note: The limit in the real numbers is studying the approaches from the right and left, but in the complex numbers is studying from every side of the circle.



Theorem: If $\lim_{z \to z_0} f(z) = w_1$, then $\lim_{z \to z_0} f(z) = w_2$

Then $w_1 = w_2$. (The limit is unique)

Proof: Let $\epsilon > 0$

Since

$\lim_{z \to z_0} f(z) = w_1 \to \exists\, \delta_1 > 0, \text{if } |z - z_0| < \delta_1$

$\to |f(z) - w_1| < \frac{\epsilon}{2}$

Since

$\lim_{z \to z_0} f(z) = w_2 \to \exists\, \delta_2 > 0, \text{if } |z - z_0| < \delta_2$

$\to |f(z) - w_2| < \frac{\epsilon}{2}$

$$
\begin{aligned}
|w_1 - w_2| &= |w_1 - f(z) + f(z) - w_2| \\
&\leq |w_1 - f(z)| + |f(z) - w_2| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}
$$

Chose $\delta = \min(\delta_1 , \delta_2)$

$\therefore |w_1 - w_2| < \epsilon$

$\rightarrow w_1 = w_2$

Theorem: Let $f(z) = u(x, y) + iv(x, y)$ such that $z = x + iy$,

$z_0 = x_0 + y_0$, $w_0 = u_0 + iv_0$, Then

$$\lim_{z \to z_0} f(z) = w_0 \text{ iff } \lim_{z \to z_0} u(x, y) = u_0, \lim_{z \to z_0} v(x, y) = v_0$$

Note: $\mathbb{C}$ is a complete space, since f is converge iff u, v are converge, but u, v are converge and u, v are real functions. Therefore it is Cauchy

$$\therefore f \text{ is converge} \rightarrow f \text{ is Cauchy}$$

$$\therefore \mathbb{C} \text{ is complete}$$

Note: $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ s.t $a_i \in \mathbb{C}, i = 0,1, \ldots, n$

Then

$$\lim_{z \to z_0} p(z) = p(z_0)$$

Example: Find limit of $f(z)$ if it's exist

1. $\lim_{z \to 3-4i} \frac{4x^2y^2 - 1 + i(x^2 - y^2) - ix}{\sqrt{x^2 + y^2}}$

Solution:

$$\lim_{z \to 3-4i} \frac{(4x^2y^2 - 1) + i(x^2 - y^2 - x)}{\sqrt{x^2 + y^2}} =$$

$$= \lim_{z \to 3-4i} \frac{4x^2y^2 - 1}{\sqrt{x^2 + y^2}} + i \lim_{z \to 3-4i} \frac{x^2 - y^2 - x}{\sqrt{x^2 + y^2}}$$

$$= 115 - 2i$$

2. $\lim_{z \to i} \frac{z - i}{z^2 + 1}$

Solution:

$$\lim_{z \to i} \frac{z - i}{z^2 + 1} = \lim_{z \to i} \frac{z - i}{z^2 - (-1)} = \lim_{z \to i} \frac{z - i}{z^2 - i^2} = \lim_{z \to i} \frac{z - i}{(z - i)(z + i)}$$

$$= \lim_{z \to i} \frac{1}{(z+i)} = \frac{1}{2i}$$

3. $\lim_{z \to (-1,i)} \frac{z^2+(3-i)\,z+2-2i}{z+1-i}$

<u>*Solution:*</u>

Note: $z^2 + (3-i)\,z + 2 - 2i = (z+1-i)(z+2)$

$$\therefore \lim_{z \to (-1,i)} \frac{z^2+(3-i)\,z+2-2i}{z+1-i} = \lim_{z \to (-1,i)} \frac{(z+1-i)(z+2)}{(z+1-i)}$$

$$= \lim_{z \to (-1,i)} (z+2)$$

$$= -1 + i + 2$$

$$= 1 + i$$

[3] Continuity

<u>**Definition:**</u>

A function f is continuous at a point z_0 if all of the three following conditions are satisfied:

1. $\lim_{z \to z_0} f(z)$ exists,

2. $f(z_0)$ exists,

3. $\lim_{z \to z_0} f(z) = f(z_0)$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point R.

<u>**Theorem:**</u> If f, g are continuous functions at z_0 then

1. $f + g$ is continuous.

2. $f.g$ is continuous.

3. $\frac{f}{g}$, $g(z_0) \neq 0$ is continuous.

4. fog is continuous at z_0 if f is continuous at $g(z_0)$.

Example: $f(z) = z^2$ is continuous in complex plane since $\forall\, z_0 \in \mathbb{C}$

1. $f(z_0) = z_0^2$

2. $\lim\limits_{z\to z_0} f(z) = z_0^2$

3. $\lim\limits_{z\to z_0} f(z) = f(z_0)$

Example: Is $f(z) = \frac{z^2-1}{z-1}$ continuous at $z = 1$

Solution: f is not continuous since $f(1)$ not exist

$f(z_0) = \frac{z_0^2-1}{z_0-1} = \frac{(z_0-1)(z_0+1)}{z_0-1} = z_0 + 1$

$\therefore \lim\limits_{z\to 1} f(z) = 2$

But $f(1) = \frac{0}{0}$

$\therefore \lim\limits_{z\to 1} f(z) \neq f(1)$

Theorem: $f(z) = u(x,y) + iv(x,y)$ is continuous at z_0 iff $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Proof: Let f be continuous at z_0 , then

$$\lim_{z\to z_0} f(z) = f(z_0)$$

That means:

$\lim\limits_{z\to z_0} (u(x,y) + iv(x,y)) = u(x_0,y_0) + i\, v(x_0,y_0)$

$\rightarrow \lim\limits_{z\to z_0} u(x,y) + i\lim\limits_{z\to z_0} v(x,y) = u(x_0,y_0) + i\, v(x_0,y_0)$

$\therefore \lim\limits_{z\to z_0} u(x,y) = u(x_0,y_0)$

$\lim\limits_{z\to z_0} v(x,y) = v(x_0,y_0)$

$\therefore u, v$ are continuous at z_0.

Example: Is $f(x+iy) = x^2 + y^2 + ixy$ continuous at $(1,1)$

Solution: $u(x,y) = x^2 + y^2,\; v(x,y) = xy$

By the above theorem

$$u(1,1) = 2\,, \qquad \lim_{\substack{x\to 1\\ y\to 1}} u(x,y) = 2 = u(1,1)$$

$$v(1,1) = 1\,, \qquad \lim_{\substack{x\to 1\\ y\to 1}} v(x,y) = 1 = v(1,1)$$

$\therefore u, v$ are continuous at $(1,1)$

$\therefore f(z)$ is continuous at $(1,1)$.

Example: Find the limit if it's exists

$$\lim_{z\to 0}\frac{\bar{z}}{z}$$

Solution:

$$\lim_{z\to 0}\frac{\bar{z}}{z} = \lim_{z\to 0}\frac{x-iy}{x+iy}$$

1. If $y = 0 \rightarrow \lim\limits_{x\to 0}\frac{x}{x} = 1$

2. If $x = 0 \rightarrow \lim\limits_{y\to 0}\frac{-iy}{iy} = -1$

$\therefore$ The limit is not exist.

Example: Discuss the continuity of

$$f(z) = \begin{cases} \dfrac{z-i}{z^2-1} & if \;\; z \neq i, -i \\ 2i & if \;\; z = \mp i \end{cases}$$

<u>*Solution*</u>: Note f is not continuous at $z = \mp i$.

(Since $f(\mp i)$ is undefined)

$$f(z) = 2i \text{ and } \lim_{z \to -i} f(z) = \lim_{z \to -i} \frac{z-i}{(z-i)(z+i)} = \lim_{z \to -i} \frac{1}{(z+i)} = \frac{1}{2i}$$

But f is not defined at $z = -i$, therefore f is not continuous at $z = i$, that is f is continuous at $\{z \in \mathbb{C} \backslash \{-i, i\}\}$

<u>**Example:**</u> Discuss the continuity of

$$f(z) = \begin{cases} \dfrac{z^2+4}{z+2i} & if \;\; z \neq -2i \\ -4i & if \;\; z = -2i \end{cases}$$

<u>*Solution:*</u> f is continuous at $\forall\, z \neq -2i$.

When $z = -2i$

$$\lim_{z \to -2i} f(z) = f(-2i) = -4i$$

$$\lim_{z \to -2i} f(z) = \lim_{z \to -2i} \frac{(z-2i)(z+2i)}{(z+2i)} = -4i$$

But f is not defined at $z = -2i$

$\therefore$ f is not continuous at $z = -2i$.

Then is f is continuous at $\{z \in \mathbb{C} : z \neq -2i\}$

<u>**Exercise:**</u> Discuss the continuity of

$$f(z) = \begin{cases} \dfrac{z+2i}{z^2+4} & if \;\; z \neq \mp 2i \\ \dfrac{1}{4} i & if \;\; z = -2i \end{cases}$$

[4] Derivative

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists. If $\Delta z = z - z_0$, then $\Delta z \to 0$ when $z \to z_0$. Thus

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Theorem: If f is differentiable at z_0, then f is continuous at z_0.

Proof: To prove f is continuous, we must prove that

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} (z - z_0) \right]$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \, . \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0) \, . \; 0$$

$$= 0$$

$$\therefore \lim_{z \to z_0} f(z) = f(z_0)$$

Differentiation Formulas:

In the following formulas, the derivative of a function f at a point z_0 is denoted by either $\frac{d}{dz}f(z)$ or $f'(z_0)$.

1. $\frac{d}{dz}\ c = 0$, c is constant

2. $\frac{d}{dz}\ z = 1$

3. $\frac{d}{dz}\ \big(c\, f(z)\big) = c\, f'(z)$

4. $\frac{d}{dz}\ [f + g] = \frac{d}{dz}\ f + \frac{d}{dz}\ g = f' + g'$

5. $\frac{d}{dz}\ [f.g] = f.g' + g.f'$

6. $\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g.f' - f.g'}{g^2}$, $g \neq 0$

7. $\frac{d}{dz}\ (z^n) = n\, z^{n-1}$

8. $(gof)'(z_0) = g'\big(f(z_0)\big)\,.f'(z_0)$

Note: If $w = f(z)$ and $W = g(w)$, then

$$\frac{dW}{dz} = \frac{dW}{dw}\cdot\frac{dw}{dz} \qquad \text{(The Chain rule)}$$

Example: Find the derivative of $f(z) = (2z^2 + i)^5$

Solution: write $w = 2z^2 + i$ and $W = w^5$

Then:

$$\frac{d}{dz}\ (2z^2 + i)^5 = 5w^4.4z = 20\, z(2z^2 + i)^4$$

Examples: Find $f'(z)$ by using the definition of derivative:

1. $f(z) = z^2$

Solution:

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z)$$

$$= 2z$$

1. $f(z) = \bar{z}$

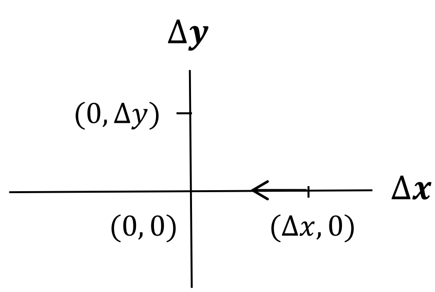
Solution:

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Let $\Delta z = (\Delta x, \Delta y)$ approach the origin $(0,0)$ in the Δz −plane. In particular, as $\Delta z \to 0$ horizontally through the point $(\Delta x, 0)$ on the real axis, then



$$\overline{\Delta z} = \overline{\Delta x + \imath\, 0} = \Delta x - i\, 0$$

$$= \Delta x + i\, 0$$

$$= \Delta z$$

$$\therefore \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

When Δz approaches $(0,0)$ vertically through the point $(0,\Delta y)$ on the imaginary axis, then

$$\overline{\Delta z} = \overline{0 + \imath\, \Delta y} = 0 - i\, \Delta y$$

$$= -(0 + i\, \Delta y)$$

$$= -\Delta z$$

$$\therefore \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta\, z} = \lim_{\Delta z \to 0} \frac{-\Delta z}{\Delta\, z} = -1$$

But the limit is unique, and then $\frac{dw}{dz}$ is not exist.

[5] Cauchy – Riemann Equations (C-R-E)

<u>Theorem:</u> Suppose that $f(z) = u(x,y) + iv(x,y)$ and $f\,'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y\,, \qquad u_y = -v_x$$

There is also

$$f'(z_0) = u_x + iv_x$$

Where these partial derivatives are to be evaluated at (x_0, y_0).

<u>Proof:</u>

Let f be differentiable at z_0 then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta\, z}\,, \qquad \Delta z = \Delta x + i\Delta y$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x,\, y_0+\Delta y)+iv(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)-iv(x_0,y_0)}{\Delta x+i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x,\, y_0+\Delta y)-u(x_0,y_0)}{\Delta x+i\Delta y} + i \lim_{\Delta z \to 0} \frac{v(x_0+\Delta x,\, y_0+\Delta y)-v(x_0,y_0)}{\Delta x+i\Delta y}$$

Let $y = 0 \Longrightarrow \Delta y = 0 \Longrightarrow \Delta z = \Delta x \to 0$

$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x}$

$= u_x(x_0, y_0) + iv_x(x_0, y_0) \quad \dots (1)$

Let $x = 0 \Longrightarrow \Delta x = 0 \Longrightarrow \Delta z = i\Delta y \to 0$

$= \lim_{i\Delta y \to 0} \frac{u(x_0, y_0+\Delta y)-u(x_0,y_0)}{i\Delta y} + i \lim_{i\Delta y \to 0} \frac{v(x_0, y_0+\Delta y)-v(x_0,y_0)}{i\Delta y}$

$= \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0)$

$= v_y(x_0, y_0) - iu_y(x_0, y_0) \quad \dots (2)$

From (1) and (2) we get

$$u_x = v_y \,, \qquad u_y = -v_x$$

Note:

1. $f'(z) = u_x + iv_x$ or $f'(z) = u_y - iv_y$.
2. If $f'(z)$ exists then C-R-Eq. are satisfied, but the converse is not true.

The converse of the above theorem is not necessary true:

Example: Let

$$f(z) = \begin{cases} 0 & if \;\; z = 0 \\ \frac{(\bar{z})^2}{z} & if \;\; z \neq 0 \end{cases}$$

Solution: The C-R-Eq. are satisfied

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z - 0}$$

$$= \lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^2$$

$$= \lim_{z \to 0} \frac{(x-iy)^2}{(x+iy)^2}$$

Let $y = 0 \to f'(0) = 1$

Let $x = 0 \rightarrow f'(0) = 1$

Let $y = x \rightarrow f'(0) = \frac{y^2(1-i)^2}{y^2(1+i)^2} = \frac{1-2i-1}{1+2i-1}$

$$= \frac{-2i}{2i}$$

$$= -1$$

$\therefore f'(z)$ is not exist at $z = 0$.

Example: $f(z) = z^2 = x^2 - y^2 + 2\,ixy$

Solution:

$u(x, y) = x^2 - y^2 \rightarrow u_x = 2\text{x}$

$v(x, y) = 2xy \quad \rightarrow v_y = 2x$

$\rightarrow u_x = v_y$

$u_y = -2\text{y}\,, \; v_x = 2y$

$\rightarrow u_y = -v_x$

$\therefore f'(\text{z}) = u_x + iv_x = 2x + i2y = 2(x + iy) = 2z$

Example: $f(z) = \bar{z} = x - iy$

Solution: $u(x, y) = x \rightarrow u_x = 1$

$v(x, y) = -y \rightarrow v_y = -1$

$\therefore u_x \neq v_y \rightarrow f$ is not differentiable at z.

Note: The following theorem gives a necessary and sufficient condition to satisfy the converse of the previous theorem.

Theorem: Let $f(z) = u(x, y) + iv(x, y)$, and

1. $u,\; v,\; u_x,\; v_x,\; u_y,\; v_y$ are continuous at $N_\epsilon(z_0)$

2. $u_x = v_y$, $u_y = -v_x$

Then f is differentiable at z_0 and

$$f'(z_0) = u_x + iv_x$$

$$f'(z_0) = v_y - iu_y$$

Example: Show that the function

$$f(z) = e^{-y}\cos x + i\, e^{-y}\sin x$$

Is differentiable z for all and find its derivative.

Solution:

Let $u(x,y) = e^{-y}\cos x$

$\rightarrow u_x = -e^{-y}\sin x$

$u_y = -e^{-y}\cos x$

$v(x,y) = e^{-y}\sin x$

$\rightarrow v_x = e^{-y}\cos x$

$v_y = -e^{-y}\sin x$

1. $u_x = v_y$ and $u_y = -v_x$

2. $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous

Then $f'(z)$ exist. To find $f'(z) = u_x + iv_x$

$$\begin{aligned} f'(z) = u_x + iv_x &= -e^{-y}\sin x + ie^{-y}\cos x \\ &= e^{-y}(i\cos x - \sin x) \\ &= ie^{-y}(\cos x + i\sin x) \\ &= ie^{-y}e^{ix} \\ &= ie^{i\,x-y} \\ &= ie^{i(x+iy)} \\ &= ie^{iz} \end{aligned}$$

[6] Polar Coordinates of Cauchy – Riemann Equations

Let $f(z) = u(r,\theta) + iv(r,\theta)$, then Cauchy-Riemann equations are:

$$u_r = \frac{1}{r} v_\theta \quad , \qquad u_\theta = -r\, v_r$$

And $f'(z_0) = e^{-i\theta}(u_r + i\, v_r)$.

Example: Use C-R equations to show that the functions

1. $f(z) = |z|^2$

2. $f(z) = z - \bar{z}$

are not differentiable at any nonzero point.

Solution:

1. $|z|^2 = x^2 + y^2$

$u(x,y) = x^2 + y^2 \ , \quad v(x,y) = 0$

$u_x = 2\text{x} \qquad , \qquad v_x = 0$

$u_y = 2y \qquad , \qquad v_y = 2x$

C-R equations are not satisfied, therefore f' is not exist.

2. $z - \bar{z} = (x + iy) - (x - iy)$

$= x + iy - x + iy$

$= 2y\, i$

$u(x,y) = 0 \qquad , \quad v(x,y) = 2y$

$u_x = 0 \qquad , \qquad v_x = 0$

$u_y = 0 \qquad , \qquad v_y = 2$

C-R equations are not satisfied, hence f' is not exist.

Example: Use C-R equations to show that $f'(z)$ and $f''(z)$ are exist everywhere

1. $f(z) = z^3$

Solution:

$$f(z) = z^3 = (x + iy)^3$$

$$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$$

$$= x^3 + 3i\, x^2 y - 3xy^2 - iy^3$$

$$= x^3 - 3xy^2 + i\,(3x^2 y - y^3)$$

$$u(x, y) = x^3 - 3xy^2 \rightarrow u_x = 3x^2 - 3y^2$$

$$u_y = -6xy$$

$$v(x, y) = 3x^2 y - y^3 \rightarrow v_x = 6xy$$

$$v_y = 3x^2 - 3y^2$$

$$\therefore u_x = v_y \ , \qquad u_y = -v_x$$

$\therefore$ C-R equations are satisfied

$$f'(z) = u_x + iv_x$$

$$= 3x^2 - 3y^2 + i\,6xy$$

$$= 3(x^2 + i^2 y^2 + 2i\, xy) = 3(x + iy)^2 = 3z^2$$

$$f''(z) = u'_x + iv'_x$$

$$= 6x + i\,6y$$

$$= 6(x + iy)$$

$$= 6z$$

2. $f(z) = \cos x \cosh y - i \sin x \sinh y$

Solution:

$u(x,y) = \cos x \cosh y \quad \rightarrow u_x = -\sin x \cosh y$

$u_y = \cos x \sinh y$

$v(x,y) = -\sin x \sinh y \rightarrow v_x = -\cos x \sinh y$

$v_y = -\sin x \cosh y$

$\therefore u_x = v_y \ , \qquad u_y = -v_x$

$\therefore$ C-R equations are satisfied

$f'(z) = u_x + iv_x$

$= -\sin x \cosh y - i \cos x \sinh y$

$f''(z) = u'_x + iv'_x$

$= -\cos x \cosh y + i \sin x \sinh y$

<u>**Example:**</u> Let $f(z) = z^3$, write f in polar form and then find f'(z)

<u>*Solution*</u>: $f(z) = z^3 = \left(re^{i\theta}\right)^3 = r^3 e^{3i\theta}$

$= r^3 \cos 3\theta + i\, r^3 \sin 3\theta$

$u(r,\theta) = r^3 \cos 3\theta \rightarrow u_r = 3r^2 \cos 3\theta$

$u_\theta = -3r^3 \sin 3\theta$

$v(r,\theta) = r^3 \sin 3\theta \ \rightarrow v_r = 3r^2 \sin 3\theta$

$v_\theta = 3r^3 \cos 3\theta$

Now, $u_r = \frac{1}{r}\, v_\theta \ , \ \ u_\theta = -rv_r$

$f'(z) = e^{-i\theta}[u_r + i\, v_r]$

$= e^{-i\theta}[3r^2 \cos 3\theta + i3r^2 \sin 3\theta]$

$= 3r^2 e^{-i\theta}[\cos 3\theta + i \sin 3\theta]$

$= 3r^2 e^{-i\theta} e^{3\theta i}$

Example: Let $f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$, $z \neq 0$, $f'(\text{z})$.

Solution:

$u(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta$

$v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta$

$\rightarrow u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta$, $u_\theta = -\left(r + \frac{1}{r}\right)\sin\theta$

$\rightarrow v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta$, $v_\theta = \left(r - \frac{1}{r}\right)\cos\theta$

Since $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous and C-R equations holds then

$$f'(z) = e^{-i\theta}[u_r + i\, v_r]$$

$$= e^{-i\theta}\left[\left(1 - \frac{1}{r^2}\right)\cos\theta + i\ \left(1 + \frac{1}{r^2}\right)\sin\theta\right]$$