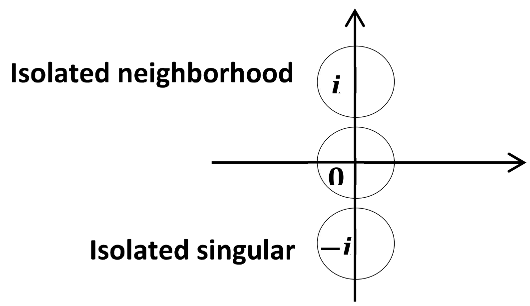
<u>Note</u>: not every singular point is isolated, as in example 3, 5.

6. $f(z) = \frac{1}{z(z^2+1)}$, has singular and isolated points at $z = 0, i, -i$.



Let z_0 be any isolated singular point of f, then f is analytic at each point z, when $0 < |z - z_0| < R$, so $f(z)$ can be represented by a Laurent series

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \qquad \ldots (1)$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$

and $\quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}} dz$

Hence:

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz \qquad \ldots (2)$$

Or

$$\int_C f(z)\, dz = 2\pi i b_1$$

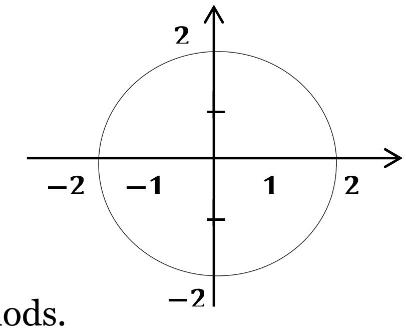
where C is any simple closed contour around z_0 described in positive sense. The coefficient $b_1$of $\frac{1}{z-z_0}$ in expansion (1) is called the residue of f at the isolated singular point z_0. Formula (2) gives us a powerful method for evaluating certain integrals around simple closed contours and it is denoted by

$$b_1 = Res[f, z_0]$$

<u>Example</u>: Evaluate

$$\oint_C \frac{e^{-z}}{(z-1)^2}\,dz$$

such that $C : |z| = 2$.



<u>Solution</u>:

<u>Note</u>: we can solve this integral by two methods.

i. By Cauchy integral formula

$$\oint_C \frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i f'(z_0)$$

$$f(z) = e^{-z} \rightarrow f'(z) = -e^{-z} \rightarrow f'(1) = -e^{-1}$$

$$\oint_C \frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i f'(1)$$

$$= 2\pi i(-e^{-1})$$

$$= -\frac{2\pi i}{e}$$

ii. Note that $f(z) = \frac{e^{-z}}{(z-1)^2}$ is analytic over C except $z_0 = 1$, so by Laurent theorem

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}e^{-(z-1)}}{(z-1)^2}$$

$$= \frac{e^{-1}}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{n!}\ ,|z-1| < \infty$$

$$= \frac{1}{e}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^{n-2}}{n!}$$

$$= \frac{1}{e}\left[\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{2!} - \frac{z-1}{3!} + \cdots\right]$$

where the coefficient of $(z-z_0)^{-1} = (z-1)^{-1}$ is $\frac{-1}{e} = b_1$, so:

$$\oint_C \frac{e^{-z}}{(z-1)^2}\,dz = 2\pi i(b_1)$$

$$= \frac{-2\pi i}{e}$$

Note: if z_0 is an isolated point, the we can find the integral by Laurent and then we find the residue of the function at $-z$.

Example: Evaluate

$$\oint_C \frac{e^z}{z} dz$$

such that $C : |z| = 1$.

Solution: Note $z_0 = 0$ is a singular point of f.

$$\frac{1}{z} e^z = \frac{1}{z}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

Note that $b_1$is the coefficient of $\frac{1}{z}$, then $b_1 = 1$ and

$$\oint_C \frac{e^z}{z} dz = 2\pi i b_1$$

$$= 2\pi i$$

Example: Evaluate

$$\oint_C e^{1/z^2} dz$$

such that $C : |z| = 2$.

Solution: Note that there is no fraction so we cannot solve by the two previous methods that is Cauchy integral formula cannot be applied here, so we will solve by residue.

$z_0 = 0$ is a singular and isolated point of f.

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $|z| < \infty$, so

$$e^{1/z^2} = \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!} , \left|\frac{1}{z^2}\right| < \infty$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!\, z^{2n}} \rightarrow \left|\frac{1}{z}\right| < \infty$$

$$= 1 + \frac{1}{z^2} + \frac{1}{2!\, z^4} + \cdots , 0 < |z| < \infty$$

The coefficient of $(z-z_0)^{-1}=(z-0)^{-1}$ is 0, then $b_1=0$ so that:

$\oint_C e^{1/z^2}\,dz = 2\pi i b_1 = 0$

And this is clear, since f is analytic on C and so by Cauchy $\oint_C f(z)\,dz = 0$.

Example: Evaluate the following integral by using residues:

$$\oint_C z^3\cos\left(\frac{1}{z}\right)dz\ ;\ C : |z+1+i\,| = 4$$

Solution:

The point $z_0=0$ is an isolated singularity of $\cos\left(\frac{1}{z}\right)$ and lies in the given contour of integration; we want a Laurent series expansion of $z^3\cos\left(\frac{1}{z}\right)$ about this point (i.e. $z_0=0$), since:

$\cos z = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$, we have

$$\cos\left(\frac{1}{z}\right) = 1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \frac{\left(\frac{1}{z}\right)^6}{6!} + \cdots$$

$$\rightarrow z^3\cos\left(\frac{1}{z}\right) = z^3 - \frac{1}{2!}z + \frac{1}{4!}\frac{1}{z} - \frac{1}{6!}\frac{1}{z^3} + \cdots$$

$$\rightarrow b_1 = \frac{1}{4!}$$

$$\oint_C z^3\cos\left(\frac{1}{z}\right)dz = 2\pi i b_1 = \frac{2\pi i}{4!}$$

$$= \frac{\pi i}{12}$$

Example: Let C be a positively oriented unit circle $z_0=0$. Evaluate

$$\oint_C \frac{dz}{z^3+z^2}$$

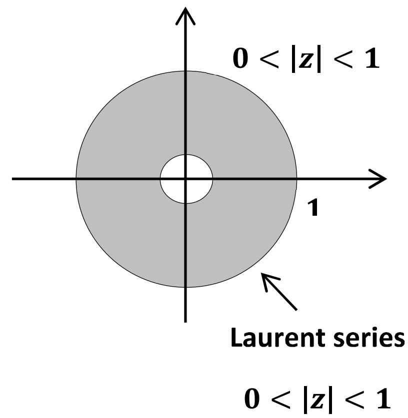
Solution: The isolated singular points are $z=0$ and $z=-1, -1 \notin 0<|z|<1$

$f(z) = \frac{1}{z^3+z^2} = \frac{1}{z^2(z+1)}$

$= \frac{1}{z^2}\left(\frac{1}{1+z}\right)$

$= \frac{1}{z^2}\ (1 - z + z^2 - z^3 + \cdots)$

$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \cdots$



$\rightarrow b_1 = -1$, so

$\oint_C f(z)\, dz = \oint_C \frac{1}{z^3+z^2}\, dz$

$= 2\pi i b_1$

$= -2\pi i$

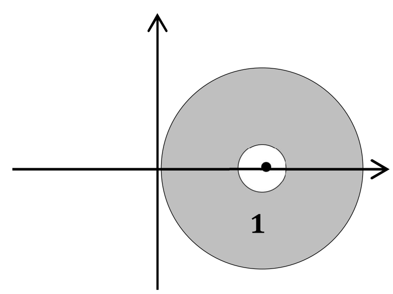
<u>Example</u>: Evaluate

$$\oint_C \frac{e^{-z}}{(z-1)^2}\, dz$$

where C is the circle $|z| = 2$,described in the positive sense.

<u>Solution</u>:

$f(z) = \frac{e^{-z}}{(z-1)^2}$ is analytic on C and its interior except at the isolated singular point $z = 1$, now



$e^{-z} = e^{-z+1-1}$

$= e^{-1}e^{1-z}$

$= e^{-1}\sum_{n=0}^{\infty}\frac{(1-z)^n}{n!}$

$= e^{-1}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{n!}$

$= e^{-1}\left[1 - (z-1) + \sum_{n=2}^{\infty}(-1)^n\frac{(z-1)^n}{n!}\right]$

$\therefore\ e^{-z} = e^{-1} - e^{-1}(z-1) + e^{-1}\sum_{n=2}^{\infty}(-1)^n\frac{(z-1)^n}{n!}$

Since $|z-1| > 0$, we can divide both sides by $(z-1)^2$

$$\therefore \frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{(z-1)} + e^{-1}\sum_{n=2}^{\infty}(-1)^n \frac{(z-1)^{n-2}}{n!}$$

$\therefore b_1$ at $z = 1$ is equal to $-e^{-1}$, so

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i b_1$$

$$= -\frac{2\pi i}{e}$$

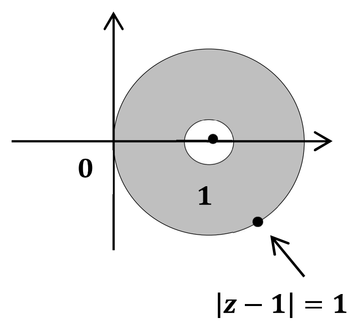
<u>Example</u>: Evaluate

$$\oint_C \frac{dz}{z(z-1)}$$

where C is the circle $|z-1| = 1$ (i.e.: or described in the positive sense as shown in the following figure).

<u>Solution</u>:

$f(z) = \frac{1}{z(z-1)}$, which is analytic on C and at all points inside C except at $z = 1$, which is an isolated singular point. The Laurent series expansion of $f(z)$ that converges in the annular region centered at $z = 1$, is



$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$= \frac{1}{z-1} - \frac{1}{(z-1)+1}$$

$$= (z-1)^{-1} - \sum_{n=0}^{\infty}(-1)^n (z-1)^n$$

$$= (z-1)^{-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

$\therefore b_1 = 1$, so

$$\oint_C \frac{dz}{z(z-1)} = 2\pi i b_1 = 2\pi i$$

<u>Example</u>: Evaluate $\oint_C \frac{\sin z}{z \sinh z} dz$, around $|z| = 1$.

<u>Solution</u>:

$z = 0$ is the only isolated singular point inside $|z| = 1$, recall that: