التحليل الدالي

المحاضرة الخامسة

قسم الرياضيات

الصف الرابع

Chapter Two: Banach spaces

Definition 2.1. *A normed linear space X is said to be* ***complete*** *if all Cauchy convergent sequences in X are convergent in X. The complete normed space is called* ***Banach space****.*

Examples 2.2.

[1] *The space* F^n *with the norm* $\| x \| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$, $\forall x=(x_1,x_2,\ldots,x_n) \in F^n$ *is a Banach space.*

Proof : F^n *is a normed space ,*

let $\{x_n\}$ *is Cauchy sequence in* $F^n \Rightarrow x_m \in F^n \Rightarrow x_m=(x_{1m},x_{2m},\ldots,x_{nm})$

let $\varepsilon>0 \Rightarrow \exists k \in Z^+$ s.t. $\| x_m-x_l \| < \varepsilon \quad \forall m.l > k$

$\Rightarrow \| x_m-x_l \|^2 < \varepsilon^2 \quad \forall m.l > k \quad \ldots\ldots (1)$

$x_m - x_l = (x_{1m}-x_{1l}, x_{2m}-x_{2l}, \ldots, x_{nm}-x_{nl})$

$\| x_m- x_l \|^2 = \sum_{i=1}^{n} |x_{im} - x_{il}|^2 \quad \ldots\ldots (2)$

from (1) & (2) , we get:

$\sum_{i=1}^{n} |x_{im} - x_{il}|^2 < \varepsilon^2 \quad \forall m.l \geq k$

then $|x_{im}- x_{il}|^2 < \varepsilon^2 \quad \forall m.l \geq k \Rightarrow |x_{im}- x_{il}| < \varepsilon \quad \forall m.l \geq k$

$\Rightarrow \forall i, \{x_{im}\}$ *is a Cauchy sequence in F*

Since F is complete (because F is IR or C)

$\Rightarrow \forall i, \exists x_i \in F$ *s.t.* $x_{im} \to x_i$

Put $x=(x_1,x_2, \ldots ,x_n) \Rightarrow x \in F$, *T.P.* $x_m \to x$.

Let $\varepsilon > 0$, $\forall m>k$ *, we get:*

$\| x_m-x \|^2 = \sum_{i=1}^{n} |x_{im} - x_i|^2 < \varepsilon^2 \Rightarrow \| x_m- x \| < \varepsilon \quad \forall m > k \Rightarrow \{x_m\}$ *convergent* $\Rightarrow F^n$ *is complete*

Since F^n *is normed space* $\Rightarrow F^n$ *is a Banach space*

[2] *The space* l^p $(1 \leq p < \infty)$ *with the norm* $\| x \| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}, x=(x_1,x_2,\ldots) \in l^p$ *, is a Banach space.* ***H.W.***

[3] *The space l^∞ with the norm $\| x \| = \sup_i | x_i |$ is a Banach space.*

Proof: Since l^∞ *is a normed space*

Let $\{ x_m \}$ *is a Cauchy sequence in* $l^\infty \Rightarrow x_m \in l^\infty \Rightarrow x_m = (x_{1m}, x_{2m}, \ldots, x_{nm}, \ldots)$

Let $\varepsilon > 0$, $\exists\, k \in Z^+$ *s.t.*

$\| x_m - x_l \| < \varepsilon, \quad \forall m, l > k \qquad \ldots\ldots(1)$

$x_m - x_l = (x_{1m} - x_{1l}, \ldots, x_{nm} - x_{nl}, \ldots)$

$\| x_m - x_l \| = \sup_i | x_{im} - x_{il} | \qquad \ldots\ldots..(2)$

From (1) *and* (2), *we have:*

$\sup_i | x_{im} - x_{il} | < \varepsilon \quad , \forall m , l > k$

then for all i , $| x_{im} - x_{il} | < \varepsilon \quad , \forall m , l > k \qquad \ldots\ldots (3)$

$\Rightarrow \forall i$, *then* $\{ x_{im} \}$ *is Cauchy sequence in F*

Since F is complete $\Rightarrow \{ x_{im} \}$ *is convergent* $\Rightarrow \exists\, x_i \in F$ *s.t.* $x_{im} \to x_i$

Put $x = (x_1, x_2, \ldots)$, *we must prove that* $x \in l^\infty$, $x_m \to x$

From (3) , *we get:*

$| x_{im} - x_i | < \varepsilon, \quad \forall m > k \quad \ldots.. (4)$

Since $x_m \in l^\infty \Rightarrow \exists\, k_m \in IR$ *s.t.:* $| x_{im} | \le k_m , \quad \forall i$

$x_i = (x_i - x_{im}) + x_{im}$

$| x_i | \le | x_i - x_{im} | + | x_{im}$

[4] *Let* $X = C[a, b]$, $\|x\|_1 = \sup\{| f(x)| : a \le x \le b\}$, $\forall x \in [a, b]$ *is a Banach space.*

Proof:

T.P. $(C[a, b], \|.\|_1)$ *is Banach space*

1. $C[a, b]$ *is v.s. over IR*
2. $(C[a, b], \|.\|_1)$ *is normed space*
3. *T.P.* $(C[a, b], \|.\|_1)$ *is complete*

Let $\{ f_m\}$ *be a Cauchy seq. in* $C[a, b]$

Given $\varepsilon > 0$, $\exists\, k \in Z^+$ *s.t.* $\| f_m - f_n \|_1 < \varepsilon, \ \forall m, n > k$

$\| f_m - f_n \|_1 = \sup\{| (f_m - f_n)(x)| : a \le x \le b\} = \sup\{| f_m(x) - f_n(x)| : a \le x \le b\} < \varepsilon, \ \forall m, n > k$

$\Rightarrow | f_m(x) - f_n(x) | < \varepsilon, \ \forall x \in [a, b], \ \forall m, n > k$

Since $\{f_m\}$ *is Cauchy seq. in IR,*

Since IR is complete Then $\{f_m\}$ *is convergent*

i.e. $\exists f \in IR$ *(f cont's & bounded) s.t.* $f_m \to f$

$\Rightarrow$ *(C[a, b],* $||.||_1$*) is complete normed space*

$\Rightarrow$ *(C[a, b],* $||.||_1$*) is Banach space*

[5] *Let X=C[0, 1],* $||\ .\ ||_2$*: C[0, 1]* $\to$ *IR defined by*

$$||f||_2 = \int_0^1 |f(x)|\,dx, \forall f \in C[0,1]$$

Then (C[0, 1], $||f||_2$*) is not Banach space because it is normed space but not complete*

Proof:

Let $\{f_n\}$ *is Cauchy seq. in C[0, 1], where:*

$$f_n = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -nx + \frac{1}{2}n + 1 & \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

let m, n >3, then:

$$||f_m - f_n|| = \int_0^1 |(f_m - f_n)(x)|\,dx = \int_0^1 |f_m(x) - f_n(x)|\,dx$$

$$= \int_0^{1/2} |(f_m(x) - f_n(x)|\,dx + \int_{1/2}^1 |f_m(x) - f_n(x)|\,dx$$

$$\le \int_{1/2}^1 |(f_m(x)|\,dx + \int_{1/2}^1 |f_n(x)|\,dx$$

$$\le \int_{1/2}^1 |(f_m(x)|\,dx + \int_{1/2}^1 |f_n(x)|\,dx$$

$$\le \frac{1}{2m} + \frac{1}{2n} \to 0 \quad \text{as} \quad m, n \to \infty$$

$\Rightarrow ||f_m - f_n|| \to 0$ *as* $m,n \to \infty$

$\Rightarrow$ $\{f_m\}$ *is Cauchy convergent seq. but not convergent*

Suppose $\exists f \in$ *C[0, 1] s.t.* $f_m \to f$

i.e. $\lim_{m\to\infty} f_m(x) = f(x), \forall x \in [0,1]$

$$\Rightarrow f(x)=\begin{cases} 0 & ,0\le x\le \frac{1}{2} \\ 1 & ,\frac{1}{2}<x\le 1 \end{cases} \quad C!$$

Since f is not continuous at x= 1/2

$\Rightarrow$ ($C[0, 1]$, $||f||_2$) *is not complete* $\Rightarrow$ *not Banach space·*

Lemma (linear combination) 2.3.: ***Let X be a normed space, {*** $x_1, x_2, \ldots, x_n$***}*** *linearly independent set in X, then* $\exists c>0$ *s.t.:*

$$\| \sum_{i=1}^{n} \lambda_i x_i \| \ge c \sum_{i=1}^{n} |\lambda_i|, \quad \forall \lambda_i \in F, \quad 1\le i\le n$$

Theorem 2.4.: *If X is finite dimension normed space then X is complete.*

Proof: Let dim X=n > 0 and $\{x_1,x_2,\ldots,x_n\}$ *is a base to X.*

T.P. X is complete space we must prove every Cauchy sequence in X is convergent.
Suppose that $\{y_n\}$ *is Cauchy sequence,*
$||y_m - y_l|| \to 0$ *when* $m,l\to\infty$ ….. (1)
since $y_m, y_l \in X$ *then:*

$$y_m = \sum_{i=1}^{n} \lambda_{im} x_i \quad , \quad \lambda_{im}\in F$$

$$y_l = \sum_{i=1}^{n} \lambda_{il} x_i \quad , \quad \lambda_{il}\in F$$

$$\Rightarrow \quad y_m - y_l = \sum_{i=1}^{n} (\lambda_{im}-\lambda_{il}) x_i \quad , \quad \lambda_{im}\in F$$

Since the set $\{x_1,x_2,\ldots,x_n\}$ *is linear independent , then* $\exists\, c>0$ *such that*

$$|| y_m - y_l || = ||\sum_{i=1}^{n} (\lambda_{im}-\lambda_{il}) x_i || \ge c \sum_{i=1}^{n} (\lambda_{im}-\lambda_{il}) x_i \quad \ldots\ldots (2)$$

From (1) & (2) *, we get* $\sum_{i=1}^{n} |\lambda_{im}-\lambda_{il}| \to 0$ *when* $m,l\to\infty$*, then:*

$|\lambda_{im}-\lambda_{il}| \to 0$ *when* $m,l\to\infty$, $\forall i$.

$\therefore$ $\forall i= 1, \ldots, n$, $\{\lambda_{im}\}$ *is Cauchy sequence in F.*

Since F is IR or C and both of them is complete

Then $\forall i$, $\exists\, \lambda_i \in F$ *s.t.* $\lambda_{im} \to \lambda_i$

Put $y = \sum_{i=1}^{n} \lambda_i x_i \Rightarrow y\in X$, $y_m \to y \Rightarrow$ *X is complete.*

Corollary (2.5.)

Every finite dimensional subspace M of a normed space X is closed

Proof:

Since M is a finite dimensional subspace of a normed space $X \Rightarrow M$ is a complete space $\Rightarrow$ M is closed

Note that, infinite dimensional subspace of Banach space need not be closed.

Example (5.8)

Let $X = C[0,1]$ and let $M = [\{f_0, f_1, \cdots\}]$ where $f_i(x) = x^i$ so that M is the set of all polynomials. M is an infinite dimensional subspace of X but not closed in X.