$$|R_N(z)| \le \frac{r^N}{2\pi} \frac{2\mu r_1 \pi}{(r_1 - r)\, r_1^N}$$

$$= \frac{\mu r_1}{r_1 - r} \left(\frac{r}{r_1}\right)^N \ , \ \frac{r}{r_1} < 1$$

So, when $N \to \infty$, we have $R_N(z) \to 0$. Therefore, for each point z inside C_0, the limit of the sum for the first N terms on the right in Eq.(2) as $N \to \infty$, is $f(z)$. That is, if f is analytic inside a circle centered at z_0 with radius r_0, then $f(z)$ is represented by a Taylor series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \text{where } |z - z_0| < r_0. \blacksquare$$

<u>Important Note</u>:

The special case in which $z_0 = 0$; i.e.:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= f(0) + f'(0)z + \frac{f''(0)z^2}{2!} + \cdots + \frac{f^{(n)}(0)z^n}{n!} + \cdots$$

is called a Maclaurin series.

<u>Example</u>: Find the Maclaurin series expansion for the following:

$$\sin z \, , \cos z \, , \sinh z \, , \cosh z \text{ and } e^z$$

<u>Solution</u>:

❖ Let $f(z) = \sin z$, then

$f(0) = \sin 0 = 0$

$f'(z) = \cos z \to f'(0) = 1$

$f''(z) = -\sin z \to f''(0) = 0$

$f^{(3)}(z) = -\cos z \to f^{(3)}(0) = -1$

$f^{(4)}(z) = \sin z \to f^{(4)}(0) = 0$

$\vdots$

$$f(z) = \sin z = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots$$

$$= 0 + z + 0 - \frac{z^3}{3!} + 0 + \frac{z^5}{5!} + \cdots$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

i.e.:

$$\sin z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}\ ,\ |z| < \infty \qquad \dots (1)$$

❖ To find the series of $\cos z$:

Differentiating both sides of (1) with respect to z, we get:

$$\cos z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!}\ ,\ |z| < \infty \qquad \dots (2)$$

❖ To find the series of $\sinh z$:

Since $\sinh z = -i \sin iz$, it follows from (1), that

$$\sinh z = -i \sum_{k=0}^{\infty}(-1)^k \frac{(iz)^{2k+1}}{(2k+1)!}$$

$$= -i \sum_{k=0}^{\infty}(-1)^k (i)^{2k+1} \frac{z^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k (-i)(i)(i^2)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad [((-1)^2)^k = 1]$$

$$\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}\ ,\ |z| < \infty \qquad \dots (3)$$

❖ To find the series of $\cosh z$:

Differentiating both sides of (3) with respect to z, we get:

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}\ ,\ |z| < \infty \qquad \dots (4)$$

❖ To find the series of e^z:

When $f(z) = e^z$, then $f^{(n)}(z) = e^z$

$f^{(n)}(0) = 1$, since e^z is analytic for all z, so:

$$e^z = e^0 + e^0 z + \frac{e^0}{2!}z^2 + \cdots + \frac{e^0}{n!}z^n + \cdots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad \ldots (5)$$

Example: Expand $\cos z$ into a Taylor series about the point $z = \frac{\pi}{2}$.

Solution: let $f(z) = \cos z$, then

$$f(z) = \cos z = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)}{1!} + \frac{f''\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)^2}{2!} + \cdots + \frac{f^{(n)}\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)^n}{n!} + \cdots$$

Now,

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(z) = -\sin z \rightarrow f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(z) = -\cos z \rightarrow f''\left(\frac{\pi}{2}\right) = 0$$

$$f^{(3)}(z) = \sin z \rightarrow f^{(3)}\left(\frac{\pi}{2}\right) = 1$$

$$f^{(4)}(z) = \cos z \rightarrow f^{(4)}\left(\frac{\pi}{2}\right) = 0$$

$$\vdots$$

$$\rightarrow \cos z = 0 - \frac{\left(z-\frac{\pi}{2}\right)}{1!} + 0 + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} + 0 - \frac{\left(z-\frac{\pi}{2}\right)^5}{5!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!}$$

Example: Show that

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (n+1)(z+1)^n$$

where $|z+1| < 1$.

Solution:

Since $|z+1| < 1 \to z_0 = -1$ and,

$f(z) = \frac{1}{z^2} \to f(-1) = 1$

$f'(z) = \frac{-2}{z^3} \to f'(-1) = 2$

$f''(z) = \frac{2.3}{z^4} \to f''(-1) = 3!$

$f^{(3)}(z) = \frac{-2.3.4}{z^5} \to f^{(3)}(-1) = 4!$

$\vdots$

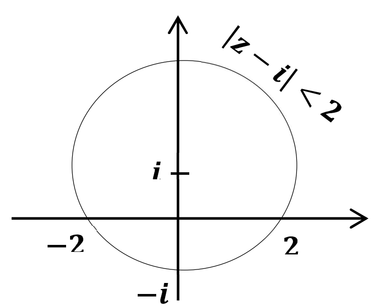
$f^{(n)}(z) = \frac{(-1)^n.2.3...(n+1)}{z^{n+2}} \to f^{(n)}(-1) = (n+1)!$

$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(z+1)^n$

$= \sum_{n=0}^{\infty}(n+1)(z+1)^n$

Example: Expand $f(z) = \frac{3}{z+i}$ into a Taylor series about $|z-i| < 2$.

Solution:



Note that $-i$ is a singular point located on the perimeter. The largest size circle that can be found is the one that the function is not analytic at it, which is $-i$. The distance between i and $-i$ represents the radius of convergence which is 2, and that's why we have the circle $|z-i| < 2$. And if we have $|z-i| < 3$ then the Taylor series cannot be applied, since the function will not be analytic and one of its conitions is that the function must be analytic inside C.

$\frac{3}{z+i} = \frac{3}{z+2i-i}$

$= \frac{3}{2i+(z-i)}$

$= \frac{3}{2i\left(1+\frac{z-i}{2i}\right)}$

$= \frac{3}{2i}\left[\frac{1}{1+\frac{z-i}{2i}}\right]$ (Geometric series $a = 1, r = \frac{z-i}{2i}$)

$= \frac{3}{2i}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n\right]$

Note: $\left|\frac{z-i}{2i}\right| < 1 \to |z-i| < 2.$

Example:

1. Expand $f(z) = \frac{1}{1+z}$ about $z = 0$.

Solution:

$f(z) = \frac{1}{1-(-z)}$

$\quad = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots \ , |z| < 1$

$\quad = \sum_{n=0}^{\infty}(-1)^n z^n$

2. Expand $f(z) = \frac{1}{1-z^2}$ about $z = 0$.

Solution:

$f(z) = \sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} z^{2n} \ , \ |z| < 1$

Note: to find the radius of convergence $= \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$, such as in the previous example,

$$\left.\begin{array}{l} a_{n+1} = -1 \to |a_{n+1}| = 1 \\ a_n = 1 \quad \to |a_n| = 1 \end{array}\right\} \to \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{1}{1} = 1$$

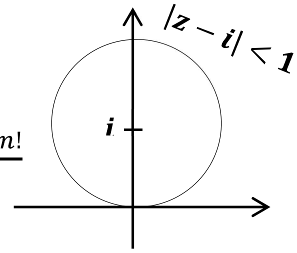
$\therefore r_0 = 1$

$|z - 0| < r_0 \to |z| < 1.$

Example: Write $f(z) = \frac{1}{z}$ into a Taylor series about $z = i, r_0 = 1$.

Solution: from Taylor's theorem $|z - z_0| < r_0 \to |z - i| < 1$

$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-i)^n \ , \ |z-i| < 1$



$f(i) = \frac{0!}{i}, f'(i) = \frac{-1!}{i^2}, f''(i) = \frac{2!}{i^3}, \dots, f^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$

$\therefore \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}}$

Or:

$\frac{1}{z} = \frac{1}{z-i+i} = \frac{1}{i+(z-i)}$

$= \frac{1}{i\left(1+\frac{z-i}{i}\right)}$

$= \frac{1}{i}\left[\frac{1}{1+\frac{z-i}{i}}\right]$, since $|z-i| < 1$

$= \frac{1}{i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^n}$

$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}}$

<u>Example</u>: Write $f(z) = \frac{1}{z}$ into a power series for $(z-1)$.

<u>Solution</u>:

$\frac{1}{z} = \frac{1}{z-1+1}$

$= \frac{1}{1+(z-1)}$ (Geometric series $a = 1, r = (z-1)$)

$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n \ , \ |z-1| < 1$

<u>Example</u>: Represent the function

$$f(z) = \frac{z}{(z-3)(z-1)}$$

into a series of negative power of $(z-1)$, which converges to $f(z)$ where $0 < |z-1| < 2$

<u>Solution</u>:

$f(z) = \frac{z}{(z-3)(z-1)} = \frac{A}{z-1} + \frac{B}{z-3}$