التحليل الدالي

المحاضرة الاولى

قسم الرياضيات

الصف الرابع

What is Functional Analysis?

Functional analysis was born in the works of Italian mathematician Vito Volterra (Volterra 1913, Volterra and Peres 1935). He was the first who considered functions as the points of some space. The spaces whose points are functions are called function spaces.

Functional Analysis is a branch of mathematics that studies vector spaces endowed with a topology, particularly spaces of functions, and the continuous linear operators acting upon these spaces. It lies at the intersection of analysis, topology, and linear algebra, playing a critical role in modern mathematical analysis.

Functional Analysis is a powerful and versatile field that bridges various areas of mathematics and science. Its abstract framework and robust theorems provide essential tools for understanding complex systems, solving equations, and modeling real-world phenomena across disciplines. The interplay between theoretical insights and practical applications makes Functional Analysis a cornerstone of modern mathematical analysis.

In this course we studied the following subjects:

Chapter One: *Vector Spaces: Finite and Infinite Dimentional, Metric Spaces, Norms & Normed Spaces.*

Chapter Two: *Banach Spaces: Some Important Inequalities (Cauchy, Holder and Minkowski's inequalities), Examples of Banach Spaces, Quotient Space of a Normed Linear Space, Continuous and Bounded Linear Transformations, Norm of Bounded Linear Transformations, Linear Operator on a Normed Space. Equivalent Norms, Continuous Linear Functional, Dual Spaces, The Hahan-Banach Theorem.*

Chapter Three: *Hilbert Spaces: Definitions, Pre-Hilbert Spaces, Chauchy- Schwarz Inequality, orthogonal, Gram- Schmidt Theorem.*

References:

1- *Introductionary Functional Analysis and Application, By E. Kreyzig, 1978.*

2- *Introduction to Hilbert Space, by S. K. Berberian, 1976.*

3- *Introduction to Functional Analysis, by Daniel Daners*
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Chapter One: Vector Space

Definition 1.1.

A vector space over F is a non-empty set V together with two functions, one from V x V to V, and the other from F x V to V, denoted by x + y and αx respectively, for all x, y $\in$ V and α $\in$F, such that, for any $\alpha,\beta \in F$ and any x, y, z $\in$ V,

(a) x+y=y+x, x+(y+z)=(x+y)+z;

(b) there exists a unique 0 $\in$ V (independent of x) such that x + 0 = x;

(c) there exists a unique -x $\in$ V such that x + (-x) = 0;

(d) 1x = x, $\alpha(\beta x) = (\alpha\beta)x$;

(e) $\alpha(x + y) = \alpha x + \alpha y$, $(\alpha + \beta)x = \alpha x + \beta x$.

If F = R (respectively, F = C) then V is a real (respectively, complex) vector space. Elements of F are called scalars, while elements of V are called vectors. The operation x + y is called vector addition, while the operation ax is called scalar multiplication.

Some important inequalities

1- *Holder's inequality : if p, q $\in$ IR such that $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\sum_{i=1} |x_i y_i| \le (\sum_{i=1} |x_i|^p)^{1/p} (\sum_{i=1} |y_i|^q)^{1/q}$$

2- *If p=2 then q=2 and:*

$$\sum_{i=1} |x_i y_i| \le (\sum_{i=1} |x_i|^2)^{1/2} (\sum_{i=1} |y_i|^2)^{1/2}$$

and is called Cauchy - Schwar's inquality.

3- *MinKowsk's inquality: if $p \ge 1$, then:*

$$(\sum_{i=1} |x_i + y_i|^p)^{1/p} \le (\sum_{i=1} |x_i|^p)^{1/p} + (\sum_{i=1} |y_i|^p)^{1/p}$$

Example 1.2. *[H.W.2-6]*

[1] S={x= $\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n\}$ is a vector space over R or C (sequence space).

[2] $l_p = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n \text{ s.t. } \sum_{n=1}^{\infty} |\alpha_n|^p < \infty\}$, l_p is a vector space over R or C ($1 \le p \le \infty$)

[3] $l_\infty = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n \text{ s.t. } \sum_{n=1}^{\infty} |\alpha_n|^p \leq m\}$ *is a vector space over R or C.*

[4] C[a, b]={f : [a, b]→R : f is continuous and C[a, b]} is a vector space over R or C.

[5] $L^p[a, b]$= {f :[a, b] →R , f is Lebesgue integrable on [a, b] s.t. $\int_a^b |f(x)|\,dx < \infty$ *} is a vector space over R or C.*

[6] Let V be the set M(m, n)(C) of complex{valued m x n matrices, with usual addition of matrices and scalar multiplication.

Sol.

[1] Let $x = (\alpha_n)_{n=1}^{\infty}$, $y = (\beta_n)_{n=1}^{\infty}$ *∈S, λ is a scalar, then*

1. $x + y = (\alpha_n)_{n=1}^{\infty} + (\beta_n)_{n=1}^{\infty} = (\alpha_n + \beta_n)_{n=1}^{\infty} \in S$

2. $\lambda x = \lambda(\alpha_n)_{n=1}^{\infty} = (\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n, ...) = (\lambda\alpha_n)_{n=1}^{\infty} \in S$

Definition 1.3

Let V be a vector space. A non-empty set U ⊂ V is a linear subspace of V if U is itself a vector space (with the same vector addition and scalar multiplication as in V). This is equivalent to the condition that:

$\alpha x + \beta y \in U$*, for all α, β ∈F and x, y ∈U*

(which is called the subspace test).

Example 1.4.

[1] The set of vectors in R^n of the form (x_1, x_2, x_3, 0, .. , 0) forms a three-dimensional linear subspace.

[2] The set of polynomials of degree ≤ r forms a linear subspace of the set of polynomials of degree ≤ n for any r ≤ n.

Definition 1.5. *Linear independence and dependence of a given set M of vectors $x_1, \ldots, x_r$ (r ≥1) in a vector space V are defined by means of the equation*

$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r = 0$ .... (*)

where α_1, α_2, ..., α_r are scalars. Clearly, equation (*) holds for $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$. If this is the only r-tuple of scalars for which (*) holds, the set M is said to be linearly independent. M *is said to be linearly dependent if M is not linearly independent, that is , if (*) also holds for some r-tuple of scalars, not all zero.*

Definition 1.6.: *Let V be a vector space over a field F, $x \in V$ is called linear combination of $x_1, x_2, \ldots, x_n \in V$ if $x = \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n = \sum_{i=1}^{n} \lambda_i \alpha_i,\ \lambda_i \in F,\ \ 1 \le i \le m$.*

Definition 1.7.: *Let V be a vector space over a field F, and let $S = \{x_1, x_2, \ldots, x_n\} \subseteq V$, S is said to be generated V if $x = \sum_{i=1}^{n} \lambda_i \alpha_i,\ \forall x_i \in S, \lambda_i \in F,\ \ 1 \le i \le m$.*

Definition 1.8.: *Let V be a vector space over a field F, and A be a non-empty subset of V ($\phi \neq A \subseteq V$), A is said to be basis of V if :*

1- *A linearly independent set.*
2- *A generated V.*

Definition 1.9. *A vector space V is said to be finite dimensional if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of n+1 or more vectors of X is linearly dependent. n is called the dimension of X, written n=dim X. By definition, X={0} is finite dimensional and dim X=0. If X is not finite dimensional, it is said to be infinite dimensional.*

Examples 1.10.: *dim R=1, dim R^2=2, dim R^n=n .*

Remarks

1- *Let V(F) be a finite dimensional V.S. over a field F, and let W subspace of V(F), then dim W $\le$ dim V, If dim W=dim V then W=V.*
2- *Let ($\phi \neq S \subseteq V$) then if $0 \in S$ then S is linear dependent subspace.*
3- *The singleton $\{x\}$ is linear dependent iff $x \neq 0$.*
4- *Any subset of linear dependent set is linear dependent.*
5- *Any set containing a linearly dependent subset is linearly dependent too.*