Chapter Five

Sequences and Series

This chapter is devoted mainly to series representations of analytic functions. To begin with, we shall give some definitions and results concerning the sequences of complex numbers.

Definition:

The function $f(n)$ defined for every positive integer $n = 1,2,3, \ldots$, is a sequence writing $z_n = f(n)$, the sequence z_0 is denoted by $\{z_n\} = \{z_1, z_2, \ldots, z_n\}$. For example, if $f(n) = n$, then the sequence is denoted by $\{n\} = \{1,2,3, \ldots, n, \ldots\}$.

If g is a function defined by $g(n) = i^n$, then the sequence is denoted by

$$\{i^n\} = \{i, -1, -i, 1, i, \ldots\}$$

The range R of a sequence $\{z_n\}$ is the set of distinct values of $\{z_n\}$. A sequence $\{z_n\}$ has a limit z and written as

$$\lim_{n\to\infty} z_n = z$$

If for every $\epsilon > 0$, there exists a positive integer N, such that $|z_n - z| < \epsilon$ whenever $n > N$. When the limit z exists, $\{z_n\}$ is called convergent, otherwise it is called divergent.

Theorem 1:

If $\{z_n\}$ is convergent to z, then z is unique.

Proof: let $z_n \to z$ and $z_n \to z^*$, to prove that $z = z^*$. Now,

$$\begin{aligned} |z - z^*| &= |z - z_n + z_n - z^*| \\ &\le |z - z_n| + |z_n - z^*| \\ &< \epsilon_1 + \epsilon_2 = \epsilon \end{aligned}$$

Hence, the limit is unique. ■

Theorem 2:

Suppose that $z_n = x_n + iy_n$, $z = x + iy$, then $\lim_{n\to\infty} z_n = z$ if and only if $x_n \to x$ and $y_n \to y$.

Proof: let $z_n \to z$, so $|z_n - z| < \epsilon$, whenever $n > N$, i.e.:

$$|x_n + iy_n, -x - iy| = |x_n - x + i(y_n - y)| < \epsilon, \text{ whenever } n > N$$

Now,

$$|x_n - x| = \sqrt{(x_n - x)^2}$$
$$\leq \sqrt{(x_n - x)^2 + (y_n - y)^2}$$
$$= |x_n - x + i(y_n - y)| < \epsilon, \text{ whenever } n > N$$

Similarly,

$|y_n - y| < \epsilon$, whenever $n > N$

Hence, $x_n \to x$ and $y_n \to y$. Conversely, if $x_n \to x$, i.e.:

$$|x_n - x| < \frac{\epsilon}{2}, \text{ whenever } n > N_1$$

And $y_n \to y$, i.e.:

$$|y_n - y| < \frac{\epsilon}{2}, \text{ whenever } n > N_2$$

Now,

$$|z_n - z| = |x_n - x + i(y_n - y)|$$
$$\leq |x_n - x| + |y_n - y|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever $n > N$, where $N = \max\{N_1, N_2\}$, thus $z_n \to z$. ■

Definition:

The sequence $\{z_n\}$ diverges to infinity if there exists a positive integer μ, such that $|z_n| > \mu$, $\forall n > N$.

Example:

1. $\{1+ni\} = \{1+i, 1+2i, \dots, 1+ni, \dots\}$, is divergent to ∞.
2. $\{ni\} = \{i, 2i, 3i, \dots\}$, is divergent.
3. $\left\{\frac{i}{n}\right\} = \left\{i, \frac{i}{2}, \frac{i}{3}, \frac{i}{4}, \dots\right\}$, converges to 0.
4. $\sum_{n=1}^{\infty} \frac{3i}{2^n}$,

Note: $S_1 = \frac{3i}{2}, S_2 = \frac{3i}{2^2}, \dots, S_n = 3i\left(\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^n}\right)$ is geometric series, the first term is $\frac{1}{2}$, i.e.:

$$S = 3i\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) = 3i, \left(\text{Geometric series and its sum } S = \frac{a}{1-r}\right)$$

This series is convergent, since $\left|\frac{1}{2}\right| < 1$.

Note:

1. If $\sum z_n$ is convergent then $\lim_{n\to\infty} z_n = 0$, but the converse is not true. For example, $\sum \frac{1}{n}$ is divergent, but $\lim_{n\to\infty} \frac{1}{n} = 0$.
2. We say that $\sum z_n$ is convergent (absolute convergent) if $\sum |z_n|$ is convergent.

Note:

Every absolute convergent series is convergent, but the converse is not true.

Definition:

The series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$

is called a power series, where z_0 and a_n are complex constants and z may be any point in a stated region containing z_0. If $z_0 = 0$, then the series is called a Maclaurin series.

Note:

If the function is analytic somewhere then it can be represented by a power series and vice versa.

Example: Let $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

1. $z = i$
2. $z = 2$

Is convergent or divergent series?

Solution:

1. When $z = i$

$\therefore \sum_{n=1}^{\infty} \left|\frac{i^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}\ ,\left(1, \frac{1}{4}, \frac{1}{9}, \ldots \to 0\right)$

Then the series is convergent.

2. When $z = 2$

$\therefore \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$, then by ratio test we get:

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^2}{2^n/n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2+2n+1}$$

$$= 2 > 1$$

The series is divergent.

[1] Taylor Series

Taylor's Theorem:

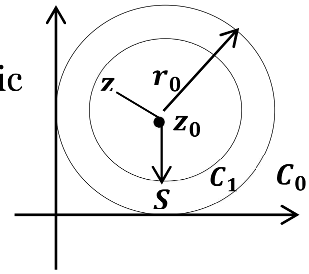
Let f be analytic everywhere inside a circle C_0 with center z_0 and radius r_0. Then at each point z inside C_0

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

that is, the power series converges to $f(z)$ when $|z - z_0| < r_0$.

<u>*Proof*</u>:

Let z be any point inside C_0 and $|z - z_0| = r$, where $r < r_0$. Let S be any point lying on a circle C_1 centered at z_0 and with radius r_1 where $r < r_1 < r_0$.



Thus $|S - z_0| = r_1$, since z inside C_1 and f is analytic within and on that circle, it follows that by (C.I.F):

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{S-z} \quad \dots (1)$$

Now,

$$\frac{1}{S-z} = \frac{1}{S-z_0+z_0-z}$$

$$= \frac{1}{(S-z_0)-(z-z_0)}$$

$$= \frac{1}{(S-z_0)\left[1-\frac{z-z_0}{S-z_0}\right]}$$

$$= \frac{1}{S-z_0}\left[1 + \frac{z-z_0}{S-z_0} + \left(\frac{z-z_0}{S-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{S-z_0}\right)^{N-1} + \frac{\left(\frac{z-z_0}{S-z_0}\right)^N}{1-\left(\frac{z-z_0}{S-z_0}\right)}\right]$$

Since $\left[\frac{1}{1-c} = 1 + c + c^2 + \cdots + c^{N-1} + \frac{c^N}{1-c}\right]$

$$\to \frac{f(S)}{S-z} = \frac{f(S)(z-z_0)}{(S-z_0)^2} + \cdots + \frac{f(S)(z-z_0)^{N-1}}{(S-z_0)^N} + \frac{f(S)(z-z_0)^N}{(S-z)(S-z_0)^N}$$

Integrating around C_1 and dividing by $2\pi i$, we get:

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{S-z} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{(S-z_0)^2}(z-z_0) + \cdots + \frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{(S-z_0)^N}(z-z_0)^{N-1}$$

$$+ \frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{(S-z)(S-z_0)^N}(z-z_0)^N$$

$$\to f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(N-1)}(z_0)}{(N-1)!}(z-z_0)^{N-1} + R_N(z)$$

Where $R_N(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{(S-z)(S-z_0)^N}(z-z_0)^N \quad \dots (2)$

Note that: