The set of all positive rational number is
$$\{1, \tfrac{1}{2}, 2, \tfrac{1}{3}, \tfrac{3}{2}, 3, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{5}{2}, 4, \tfrac{1}{3}, \dots\}$$

$\therefore$ each rationl numbe in $\{0, 1, -1, \tfrac{1}{2}, \tfrac{-1}{2}, 2, -2, \tfrac{1}{3}, \tfrac{-1}{3}, \dots\}$

$\therefore \exists$ bijective $f: Q \longrightarrow N$ such that:

$$0 \longleftrightarrow 0$$
$$1 \longleftrightarrow 1$$
$$-1 \longrightarrow 2$$
$$\tfrac{1}{2} \longrightarrow 3$$
$$-\tfrac{1}{2} \longrightarrow 4$$
$$\vdots$$

$\therefore Q$ is Countable set

Dense ordered الترتيب الكثيف

Definition Let "$\leq$" is partially ordered relation on the set A. Then "$\leq$" is said to be dense iff

$$\boxed{(a, b \in A) \wedge (a < b) \longrightarrow (\exists\, c \in A \text{ such that } a < c < b)}$$

Remark If $n \in Z \Rightarrow \nexists\ t \in Z$ Such that $n < t < n+1$.

Proof. If $\exists\, t \in Z$ such that $n < t < n+1$

$\therefore \exists\ p \in Z^+$ such that $t = n + p$

$\therefore$

$\exists\ m \in Z^+$ such that $n+1 = m + t$

$\therefore \cancel{n}+1 = \cancel{n} + p + m$. But $p, m \in Z^+$ and

$\therefore p + m = 1 \Rightarrow$ either $p = 0 \Rightarrow t = n$
or $m = 0 \Rightarrow n+1 = t$

$\therefore \nexists\ t \in Z$ such that $n < t < n+1$

Binary operations: العمليات الثنائية

Definition Let A be a nonempty set. The function $* : A \times A \longrightarrow A$ is said to be $\boxed{\text{binary operation}}$ on A defined by $*((a,b)) = a * b$.

Definition

Let A be a nonempty set and $B \subseteq A$ and $*$ binary operation on A. Then B is said to be closed subset with the respect to "$*$" iff

$\forall\, a,b \in B \quad , \quad a * b \in B.$

Remark

if "$*$" is binary operation on A, then A is closed set w.r.t. "$*$"

Example

1- Let $A = N$ and "$*$" = "$+$" : That is

$\forall \quad (a,b) \in N \times N \quad , \quad a * b = a + b \in N$

$\therefore \; + : N \times N \longrightarrow N$ is binary operation on N and N is closed set w.r.t "$+$"

2. The function "$+$" is binary operation on $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$

3- Let $A = \{-2, -1, 0, 1, 2\}$ and $* = +$.

Then the operation "+" is not binary operation on A. where

if $a = 2$, $b = 1$, then $a + b = 2 + 1 = 3 \notin A$

$\Rightarrow 2, 1 \in A$ but $2 + 1 \notin A$.

H.W.

① Is the operation " − " is binary operation on N?

② Let $*$ defined by the following table on the set $A = \{1, 2, 3\}$

*	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

Is "$*$" binary operation on A?

Mathematical System

Definition: A nonempty set A with one binary or more than is said to be Mathematical System

Example (IR, +), (N, +), (Q, +), (Z, +) are mathematical systems.

H.W.

Are (N, −) and (N, ÷) mathematical systems?

Field الحقل

Definition Let A be the nonempty set and "$+$", "$\cdot$" be two binary operation on A. Then the mathematical system $(A, +, \cdot)$ is said to be <u>Field</u> if: $\forall\, a, b, c \in A$;

1. $a + b = b + a$
2. $\exists$ "0" the identity element such that $\forall\, a \in A$, $a + 0 = 0 + a = a$
3. $\forall\, a \in A$, $\exists$ "$-a$" $\in A$ such that $a + (-a) = (-a) + a = 0$
4. $(a + b) + c = a + (b + c)$
5. $a \cdot b = b \cdot a$
6. $\forall\, a (\neq 0) \in A$, $\exists\, a^{-1} \in A$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$
7. $\exists\, 1$ the identity element s.t. $\forall\, a \in A$ $a \cdot 1 = 1 \cdot a = a$
8. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
9. $a \cdot (b + c) = a \cdot b + a \cdot c$
   $(b + c) \cdot a = b \cdot a + c \cdot a$
10. $1 \neq 0$

Theorem The mathematical system $(Q, +, \cdot)$ field (and it's said to be the field of rational numbers).

Notation Let $x \in Q$, $y \in Q - \{0\}$, then

$$x \cdot y^{-1} = \frac{x}{y}$$

Theorem

If the algabric system $(A, +, \cdot, \leq)$ ordered field then the relation " $\leq$ " is dense

Proof. $\because a < b \Rightarrow a + b < b + b$

and $a + a < b + a = a + b$

$\therefore \Rightarrow a + a < a + b < b + b$

$\longrightarrow 2a < a + b < 2b$

$\because (2 = 1 + 1 > 0) \Rightarrow \frac{1}{2} > 0$

$\Rightarrow a < \frac{a+b}{2} < b$

$\therefore \forall a, b \in A$ s.t $a < b$ $\exists \frac{a+b}{2} \in A$

s.t $a < \frac{a+b}{2} < b$.

$\therefore \leq$ dense.

Remark

the relation "$\leq$" on $\mathbb{Q}$ is dense. that is

$$\forall\ p, q \in \mathbb{Q}, \exists\ r \in \mathbb{Q} \text{ s.t } p < r < q$$

such that $p < q$

Theorem.

$$\nexists\ x \in \mathbb{Q} \text{ such that } x^2 = 2$$

proof.

Suppose that $x \in \mathbb{Q}$ such that $x^2 = 2$.

$\Rightarrow x = \frac{a}{b}$; $a, b \in \mathbb{Z}^+$ and $b \neq 0$ with $\gcd(a,b) = 1$

$\Rightarrow a^2 = 2b^2$ $\left(\text{where } x^2 = \frac{a^2}{b^2} = 2\right)$

$\Rightarrow a^2$ is even $\Rightarrow a$ is even $\Rightarrow a = 2k$; $k \in \mathbb{Z}^+$

$\Rightarrow k < a$

$\Rightarrow a = 2k \Rightarrow a^2 = 4k^2 = 2b^2$

$\Rightarrow b^2 = 2k^2 \Rightarrow b^2$ even $\Rightarrow b$ is an even $c!$

لانه لا يمكن ان تكون العددان زوجيان في نفس الوقت

$\Rightarrow b = 2t$; $t \in \mathbb{Z}$

$\Rightarrow x = \frac{a}{b} = \frac{2k}{2t} = \frac{k}{t}$

$\Rightarrow \exists\ k < a$ such that $x = \frac{k}{t}$ $c!$

$\Rightarrow \nexists\ x \in \mathbb{Q}$ s.t $\boxed{x^2 = 2}$

Archimedean Order الترتيب الارخميدي

للحقل المرتب Q خاصية مهمة والبيان : كل عنصر موجب
مضاعفات صحيحة كبيرة وعالي التحيث .

Def The ordered field $(A, +, \cdot, \leq)$ is Archimed iff $\forall\ a, b \in A$ such tha $0 < a < b$, then $\exists$ $n \in N - \{0\}$ such that $na \geq b$

Theorem

The ordered field $(Q, +, \cdot, \leq)$ is Archimed field.

Proof let $0 < x < y \Rightarrow \exists\ p, q, r \in Z$ s.t

$y = \frac{q}{r}$, $x = \frac{p}{r} \Rightarrow 0 < \frac{p}{r} < \frac{q}{r} \Rightarrow$

Let $n = rq \Rightarrow$ $nx = rq \cdot \frac{p}{r} = pq \geq q \geq \frac{q}{r} = y$

$\therefore$ $nx \geq y$