## [7] Analytic Functions

**Definition:**

A function $f$ is said to be analytic at $z_0$ if $f'(z_0)$ exists and $f'(z)$ exists at each point $z$ in the same neighborhood of $z_0$.

**Note:** $f$ is analytic in a region $R$ if it is analytic at every point in $R$.

**Definition:**

If $f$ is analytic at each point in the entire plane, then we say that $f$ is an entire function.

**Example:** $f(z) = z^2$ , is an entire function since it is a polynomial.

**Definition:**

If $f$ is analytic at every point in the same neighborhood of $z_0$ but $f$ is not analytic at $z_0$, then $z_0$ is called singular point.

**Example:** Let $f(z) = \frac{1}{z}$, then $f'(z) = \frac{-1}{z^2}$ $(z \neq 0)$

Then $f$ is not analytic at $z_0 = 0$, which is a singular point.

**Note:** If $f$ is analytic in $D$, then $f$ is continuous through $D$ and C-R equations are satisfied.

**Note:** A sufficient conditions that $f$ be analytic in $\mathbb{R}$ are that C-R equations are satisfied and $u_x,\ v_x,\ u_y,\ v_y$ are continuous in $\mathbb{R}$.

## [8] Harmonic Functions

**Definition:**

A function $h$ of two variables x and y is said to be harmonic in $D$ if the first partial derivatives are continuous in $D$ and

$$h_{xx} + h_{yy} = 0 \quad \text{(Laplace equation)}$$

**Example:** Show that $u(x,y) = 2x(1-y)$ is harmonic in some domain $D$.

*Solution*:

$u_x = 2(1-y) \to u_{xx} = 0$

$u_y = -2x \quad \to u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$

Since $u, u_x, u_y$ are continuous and satisfied Laplace equation then the function is harmonic.

**Definition:**

Let $w = u + iv$, we say that $w$ is harmonic function if $u, v$ are also harmonic functions and we say $v$ is a harmonic conjugate of $u$ and $u$ is a harmonic conjugate of $v$.

**Theorem:** If a function $f(z) = u(x,y) + i\,v(x,y)$ is analytic in a domain $D$ then its component functions $u$ and $v$ are harmonic in $D$.

*Proof:*

Since $f$ is analytic then it satisfies C-R equations

i.e.: $u_x = v_y$ , $\quad u_y = -v_x$

$\to u_{xx} = v_{yx}$ , $\quad u_{yy} = -v_{xy}$

$\therefore u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

$\to u$ is harmonic function. By the same way we prove that $v$ is harmonic function.

**Note:** The converse of the above theorem is not true, which means that if $u$ and $v$ are harmonic functions then $f$ is not necessary analytic function.

**Example:** $u(x,y) = 2xy$ , $v(x,y) = x^2 - y^2$

*Solution:* $u$ , $v$ are harmonic functions, but

$$f(z) = u + iv = 2xy + i(x^2 - y^2)$$

is not analytic function since it doesn't satisfy C-R equations

$u_x = 2y$ , $v_x = 2x$

$u_y = 2x$ , $v_y = -2y$

$\rightarrow u_x \neq v_y$

$\therefore f$ is not analytic function.

**Definition:**

Let $u, v$ be two harmonic functions and $u_x = v_y$ , $u_y = -v_x$, then we say that $v$ is a harmonic conjugate of $u$ .

**Note:**

1. If $v$ is a harmonic conjugate of $u$ and $u$ is a harmonic conjugate of $v$ then $u$ , $v$ are constant functions.
2. If $v$ is a harmonic conjugate of $u$ then $u$ is a harmonic conjugate of $-v$.
3. $f = u + iv$ is analytic iff $v$ a harmonic conjugate of $u$.

**Example:** Show that $u(x, y) = \sin x \cosh y$ is harmonic and find the harmonic conjugate.

*Solution*:

$u_x = \cos x \cosh y \rightarrow u_{xx} = -\sin x \cosh y$

$u_y = \sin x \sinh y \rightarrow v_{yy} = \sin x \cosh y$

$\rightarrow u_{xx} + v_{yy} = 0 \rightarrow u$ is harmonic

To find the harmonic conjugate $v$ we must satisfy

$$u_x = v_y , \ u_y = -v_x$$

1. $u_x = \cos x \cosh y = v_y$

2. $v = \cos x \sinh y + \emptyset_x$

We obtain $\emptyset_x$ by integration and using the second equation of C-R:

$$v_x = -\sin x \sinh y + \emptyset'_x$$

But $-v_x = u_y$, then

$$-\sin x \sinh y + \emptyset'_x = -\sin x \sinh y \;\rightarrow\; \emptyset'_x = 0 \overset{\int}{\rightarrow} \emptyset_x = c$$

$\therefore\ v = \cos x \sinh y + c$

**Example:** Let $u(x, y) = xy$, find $v$ such that $f(z) = u + iv$ is analytic.

*Solution*: Since $f$ is an analytic, then C-R equation are satisfied

$u_x = v_y \rightarrow y = v_y \rightarrow v = \frac{y^2}{2} + \emptyset(x)$

But $u_y = -v_x \rightarrow x = -\emptyset'(x)$

$$\rightarrow \emptyset'(x) = -x$$

$$\overset{\int}{\rightarrow}\ \emptyset(x) = \frac{-x^2}{2} + c$$

$$\therefore\ v = \frac{y^2}{2} - \frac{x^2}{2} + c$$

If $c = 0$, then $f(z) = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right)$