The Natural Numbers الأعداد الطبيعية

Definition:
Let $0=\varphi$, where φ is empty set.
$1=\{0\}$
Likewise define,
$2=\{0,1\}$, $3=\{0,1,2\}, 4=\{0,1,2,3\}$,
and hence
$0=\varphi,\quad 1=\{\varphi\},\ 2=\{\varphi,\{\varphi\}\},\ 3=\{\varphi,\{\varphi\},\{\varphi,\{\varphi\}\}\},\ 4=\{\varphi,\{\varphi\},\{\varphi,\{\varphi\}\},\{\varphi,\{\varphi\},\{\varphi,\{\varphi\}\}\}\}$.

Definition :
Let S be any set. The successor of S is denoted by S^+ and defined by $S^+=S\cup\{S\}$.

Remarks.
1. $S\subset S^+$
2. $S\in S^+$

Example.
1. If $S=\{a,b\}$, then $S^+=S\cup\{S\}=\{a,b,\{a,b\}\}$
2. $0=\varphi$
3. $1=\{0\}=\{\varphi\}=\varphi\cup\{\varphi\}=\varphi^+=0^+$
3. $2=\{0,1\}=\{0\}\cup\{1\}=1\cup\{1\}=1^+$
4. $3=\{0,1,2\}=\{0,1\}\cup\{2\}=2\cup\{2\}=2^+$
5. $4=\{0,1,2,3\}=\{0,1,2\}\cup\{3\}=3\cup\{3\}=3^+$
6. $5=\{0,1,2,3,4\}=\{0,1,2,3\}\cup\{4\}=4\cup\{4\}=4^+$

السؤال الذي يطرح هنا: هل يوجد شي ممكن تسميته مجموعة الأعداد الطبيعية؟ ان الطريقة التي ذكرناها سابقا لا نستطيع بواسطتها تكوين مجموعة كل الأعداد الطبيعية حيث يمكن فقط تكوين الأعداد 0,1,2, ... , n . لذا لا نستطيع أن نتكلم عن مجموعة كل الأعداد الطبيعية.

Definition.
Let S be any set. We say that S is successor set if
1. $\varphi\in S$ 2. If $x\in S$, then $x^+\in S$

Remark.
Any successor set contain the numbers $0,1,2,\dots,n$
Proof. Since if S is successor set, $\varphi\in S \Rightarrow 0\in S \Rightarrow 0^+\in S \Rightarrow 1^+\in S \Rightarrow 2\in S$
then $\Rightarrow 0,1,2,\dots,n\in S$

Axiom of Infinity (بديهية المالانهاية)
There exists successor set (توجد مجموعة تابعية)

Theorem.
1. The family of successor sets is nonempty

2. The intersection of any nonempty family of successor sets is also successor set.

Proof:

1. Direct from Axiom of Infinity.
2. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be nonempty family of successor sets

$\Rightarrow \varphi \in A_\lambda$ for all $\lambda \in \Lambda \Rightarrow \varphi \in \bigcap_{\lambda\in\Lambda} A_\lambda$

Let $x \in \bigcap_{\lambda\in\Lambda} A_\lambda \Rightarrow x \in A_\lambda$ for all $\lambda \in \Lambda$

$\Rightarrow x^+ \in A_\lambda$ for all $\lambda \in \Lambda \Rightarrow x^+ \in \bigcap_{\lambda\in\Lambda} A_\lambda$ so that $\bigcap_{\lambda\in\Lambda} A_\lambda$ is successor set.

Definition.

The intersection of all successor sets is called the set of natural numbers and denoted by $\mathbb{N}$. Each element of $\mathbb{N}$ is called the natural number. The set of natural numbers is smallest successor set.

Definition.

The set S is said to be **transitive set** if the following condition hold

If $x \in S$, then $x \subseteq S$

Examples.

1. The set $A = \{a,b\}$ is not transitive , since $a \in A$, but $a \not\subset A$
2. The natural number 3 is transitive set , since $3 = \{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}\}$, and

$\varphi \in 3 \Rightarrow \varphi \subseteq 3$

$\{\varphi\} \in 3 \Rightarrow \{\varphi\} \subseteq 3$, since $\varphi \in 3$

$\{\varphi, \{\varphi\}\} \in 3 \Rightarrow \{\varphi, \{\varphi\}\} \subseteq 3$, since $\varphi \in 3$ and $\{\varphi\} \in 3$

3. Zero is transitive set , since

If 0 is not transitive set, then there exists $x \in 0$ such that $x \not\subset 0$.

This contradiction because $0 = \varphi$, i.e. there is no $x \in 0$.

Lemma (*): The set S is said to be transitive set is it satisfy $x \in n \to x \subseteq n$

Remark. Every natural number satisfy the property $x \in n \to x \subseteq n$**.**

Peano's Axioms

The Peano's axioms (postulates) for natural numbers are:

1. $0 \in \mathbb{N}$
2. If $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$
3. If $n \in \mathbb{N}$, then $n^+ \neq 0$
4. If X is a successor subset of $\mathbb{N}$, then $X = \mathbb{N}$
5. If $n, m \in \mathbb{N}$ such that $n^+ = m^+$, then $n = m$.

Proof 1. Since $\mathbb{N}$ is successor set , then $0 \in \mathbb{N}$

Proof 2. Let $n \in \mathbb{N}$, since $\mathbb{N}$ is successor set , then $n^+ \in \mathbb{N}$

Proof 3. Let $n \in \mathbb{N}$, $n^{+} = n \cup \{n\} \Rightarrow n \in n^{+} \Rightarrow n^{+} \neq \varphi \Rightarrow n^{+} \neq 0$

Proof 4. $\because \mathbb{N}$ is intersection of successor set and X is successor set,
$\therefore \mathbb{N} \subseteq X$, but $X \subseteq \mathbb{N}$, then $X = \mathbb{N}$.

Proof 5. Let $n, m \in \mathbb{N}$ such that $n^{+} = m^{+}$
Since $n \in n^{+}$ and $n^{+} = m^{+}$, then $n \in m^{+}$, but $m^{+} = m \cup \{m\}$, then either $n \in m$ or $n = m$
If n=m, we are done.
Or $n \in m$, by lemma (*), we have $n \subseteq m$
by the same argument we have $m \subseteq n$, $\rightarrow$ n=m.

Remark.
Axiom (4) is called The Principle of Mathematical Induction.

Example:
1. The set A={a, b} is not transitive since $a \in A$ but $a \not\subset A$.
2. The natural number 4 is transitive set,
Since $4 = \{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}, \{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}\}\}$,
$\varphi \in 4 \rightarrow \varphi \subseteq 4$
$\{\varphi\} \in 4 \rightarrow \{\varphi\} \subseteq 4$
$\{\varphi, \{\varphi\}\} \in 4 \rightarrow \{\varphi, \{\varphi\}\} \subseteq 4$ (since $\in 4, \{\varphi\} \in 4$)
$\{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}, \{\varphi, \{\varphi\}\}\}$ $4 \rightarrow \{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}, \{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}\}\}$
(since $\in 4, \{\varphi\} \in 4, \{\varphi, \{\varphi\}\} \in 4$ & $\{\varphi, \{\varphi\}, \{\varphi, \{\varphi\}\}\} \in 4$)
3. Zero is transitive set. If not:
φ is not transitive set $\rightarrow \exists x \in 0$ but $x \not\subseteq 0$ C !, because $0 = \varphi$ (That means $\nexists x \in 0$).

Arithmetic of the Natural Numbers حساب الأعداد الطبيعية

Recursion Theorem مبرهنة التكرار

Let $a \in X$ (X is nonempty) and f: X→X be a function, then $\exists!$ function $\alpha: \mathbb{N} \rightarrow X$ such that $\forall n \in \mathbb{N}$,
1. $\alpha(0) = a$
2. $\alpha(n^{+}) = f(\alpha(n))$.

Example.
Let $c \neq 1$ and $\mathbf{f} : \mathbb{R} \rightarrow \mathbb{R}$ function defined by $\mathbf{f}(\mathbf{x}) = \mathbf{cx}, \forall \mathbf{x} \in \mathbb{R}$. Define $\boldsymbol{\alpha}: \mathbb{N} \rightarrow \mathbf{X}$ by $\boldsymbol{\alpha}(\mathbf{n}) = \mathbf{c}^{\mathbf{n}}$, then
1. $\alpha(0) = c^{0} = 1$

2. $\alpha(n^+) = f(\alpha(n)) = f(c^n) = c^n c = c^{n+1}\ \forall n \in \mathbb{N}$

$= c^{n^+}$

Generalization Recursion Theorem تعميم مبرهنة التكرار

Let $a \in X$. For each $m \in \mathbb{N}$, $f_m: X \to X$, $\exists!\ \alpha: \mathbb{N} \to X$ such that $\forall n \in \mathbb{N}$,

1. $\alpha(0) = a$
2. $\alpha(n^+) = f_n(\alpha(n))$.

Addition on $\mathbb{N}$ الجمع على الأعداد الطبيعية

Theorem.

Let $m \in \mathbb{N}$, , $\exists!\ \alpha: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

1. $\alpha(m, 0) = m\ , \forall n \in \mathbb{N}$
2. $\alpha(m, n^+) = (\alpha(m, n^+))^+, \forall n \in \mathbb{N}$.

$$\boldsymbol{\alpha(m, n) = m + n}, \qquad \boldsymbol{\forall n, m \in \mathbb{N}}$$

Definition.

We write $\alpha(m, n) = m+n$ name "addition" for the binary operation "+".

Theorem.

1. $m + 0 = m\ , \forall m \in \mathbb{N}$
2. $m + n^+ = ((m + n^+))^+, \forall n, m \in \mathbb{N}$.

Example:

$2 + 1 = 2 + 0^+ = (2 + 0)^+ = 2^+ = 3$

$1 + 2 = 1 + 1^+ = (1 + 1)^+ = (1 + 0^+)^+ = ((1 + 0)^+)^+ = (1^+)^+ = 2^+ = 3$

Cancellation law for addition

Let $n, m, k \in \mathbb{N}$ and $m + k = n + k$, then $n = m$.

Properties of addition on $\mathbb{N}$

Theorem: for all $n, m, k \in \mathbb{N}$:

1. $n^+ = 1 + n$
2. $(m + n) + k = m + (n + k)$ (Associative property)
3. $0 + n = n$
4. $m + n = n + m$. (Commutative property)

Proof.

1. Let $X = \{n \in \mathbb{N} : n^+ = 1 + n\} \Rightarrow X \subseteq \mathbb{N}$

Since $0^+ = 1 = 1 + 0$, then $0 \in X$

Let $n \in X$. To prove $n^+ \in X$

Since $n \in X \Rightarrow n^+ = 1 + n$

$\Rightarrow (n^+)^+ = (1 + n)^+ = 1 + n^+ \Rightarrow n^+ \in X$. By the axiom of induction $X = \mathbb{N}$.

2. Let $X_{mn} = \{k \in \mathbb{N} : (m + n) k = m + (n + k)\} \Rightarrow X_{mn} \subseteq \mathbb{N}$

Since $(m+n)+0 = m+n$, $m+(n+0) = m+n$, then $0 \in X_{mn}$

Let $k \in X_{mm}$. To prove $k^+ \in X_{mm}$

Since $k \in X_{mn} \Rightarrow (m+n)+k = m+(n+k)$

$$(m+n)+k^+ = ((m+n)+k)^+ = (m+(n+k))^+ = m+(n+k)^+ = m+(n+k)$$

$\Rightarrow k^+ \in X_{mn}$. By the axiom of induction $X_{mn} = \mathbb{N}$.

3. Let $X = \{n \in \mathbb{N} : 0+n = n\} \Rightarrow X \subseteq \mathbb{N}$

Since $0+0=0$, then $0 \in X$

Let $n \in X$. To prove $n^+ \in X$

Since $n \in X \Rightarrow 0+n = n \Rightarrow 0+n^+ = (0+n)^+ = n^+ \Rightarrow n^+ \in X$

By the axiom of induction $X = \mathbb{N}$.

4. Let $X_m = \{n \in \mathbb{N} : m+n = n+m\} \Rightarrow X_m \subseteq \mathbb{N}$

Since $n+0 = n$, $0+n = n$, then $0 \in X_m$

Let $n \in X_m$. To prove $n^+ \in X_m$

Since $n \in X_m \Rightarrow m+n = n+m$

$m+n^+ = (m+n)^+ = 1+(n+m) = (1+n)+m = n^+ + m \Rightarrow n^+ \in X_m$. By the axiom of induction $X_m = \mathbb{N}$

Theorem.
There is a unique binary operation on $\mathbb{N}$ (called addition) such that
1. $m+0=m$ for all $m \in \mathbb{N}$
2. $m+n^+=(m+n)^+$ for all $m, n \in \mathbb{N}$.

Multiply Natural Numbers.

Theorem. Let $m \in \mathbb{N}$, by recursion theorem $\exists!\ \gamma_m: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that
1. $\gamma_m = 0$ for all $m \in \mathbb{N}$
2. $\gamma_m(n^+) = \gamma_m(n) + m$ for all $m, n \in \mathbb{N}$.

Definition. Let $m \in \mathbb{N}$, by recursion theorem $\exists!\ \gamma_m: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\gamma_m(n) = m.n \quad n \in \mathbb{N}$$

Then m.n is said to be multiply natural numbers.

Remark. let $n, m \in \mathbb{N}$
1. $m.0 = 0$
2. $m.n^+ = m.n + m$

Properties of multiplication on $\mathbb{N}$

Theorem.
1. $0 \cdot n = 0$, for all $n \in \mathbb{N}$
2. $1 \cdot n = n$, for all $n \in \mathbb{N}$
3. $m \cdot (n+k) = m \cdot n + m \cdot k$ for all $n, m, k \in \mathbb{N}$ (Left distributive over addition)
4. $(n+k) \cdot m = n \cdot m + k \cdot m$ for all $n, m, k \in \mathbb{N}$ (right distributive over addition)

5. $(m \cdot n) \cdot k = m \cdot (n \cdot k)$ for all $n, m, k \in \mathbb{N}$ (Associative Properties)
6. $m \cdot n = n \cdot m$ for all $n, m \in \mathbb{N}$ (Commutative Properties)

Proof:

1. Let $X = \{n \in \mathbb{N} : 0 \cdot n = 0\} \Rightarrow X \subseteq \mathbb{N}$
   Since $0 \cdot 0 = 0$, then $0 \in X$
   Let $n \in X$. To prove $n^+ \in X$
   Since $n \in X \Rightarrow 0 \cdot n = 0$
   $0 \cdot n^+ = 0 \cdot n + 0 = 0 + 0 = 0 \Rightarrow n^+ \in X$. By the axiom of induction $X = \mathbb{N}$.

2. Let $X = \{n \in \mathbb{N} : 1 \cdot n = n\} \Rightarrow X \subseteq \mathbb{N}$
   Since $1 \cdot 0 = 0$, then $0 \in X$
   Let $n \in X$. To prove $n^+ \in X$
   Since $n \in X \Rightarrow 1 \cdot n = n$
   $1 \cdot n^+ = 1 \cdot n + 1 = n + 1 = 1 + n = n^+ \Rightarrow n^+ \in X$. By the axiom of induction $X = \mathbb{N}$.

3. Let $X_{mn} = \{k \in \mathbb{N} : (m \cdot n) \cdot k = m \cdot (n \cdot k)\} \Rightarrow X_{mn} \subseteq \mathbb{N}$
   Since $(m \cdot n) \cdot 0 = 0$, $m \cdot (n \cdot 0) = m \cdot 0 = 0$, then $0 \in X_{mn}$
   Let $k \in X_{mm}$. To prove $k^+ \in X_{mn}$
   Since $k \in X_{mn} \Rightarrow (m \cdot n) \cdot k = m \cdot (n \cdot k)$
   $(m \cdot n) \cdot k^+ = (m \cdot n) \cdot k + m \cdot n = m \cdot (n \cdot k) + m \cdot n = m \cdot (n \cdot k + n) = m \cdot (n \cdot k^+)$
   $\Rightarrow k^+ \in X_{mn}$. By the axiom of induction $X_{mn} = \mathbb{N}$.

4. Let $X_m = \{n \in \mathbb{N} : m \cdot n = n \cdot m\} \Rightarrow X_m \subseteq \mathbb{N}$
   Since $n \cdot 0 = 0$, $0 \cdot n = 0$, then $0 \in X_m$
   Let $n \in X_m$. To prove $n^+ \in X_m$
   Since $n \in X_m \Rightarrow m \cdot n = n \cdot m$
   $m \cdot n^+ = m \cdot n + m = n \cdot m + m \cdot 1 = (n + 1) \cdot m = (1 + n) \cdot m = n^+ \cdot m$
   $\Rightarrow n^+ \in X_m$. By the axiom of induction $X_m = \mathbb{N}$.

Definition (3.23):

Let $n, m \in \mathbb{N}$. Define m^n as follows

1. $m^0 = 1$ for all $m \in \mathbb{N}$
2. $m^{n^+} = m^n \times n$ for all $n, m \in \mathbb{N}$

Theorem (3.24):

1. $m^{n+k} = m^n \times m^k$ for all $n, m, k \in \mathbb{N}$
2. $(m \times n)^k = m^n \times n^k$ for all $n, m, k \in \mathbb{N}$
3. $(m^n)^k = m^{n \times k}$ for all $n, m, k \in \mathbb{N}$