Chapter One

Complex Numbers

[1] Definition:

A *complex number* ***z*** is an ordered pair $(a,\ b)$ of real numbers such that

$$\mathbb{C} = \{\ \mathbb{R} \times \mathbb{R}\ \} = \{(a, b): a, b \in \mathbb{R}\ \}$$

where $\mathbb{R}$ denotes the Real Numbers set. The real numbers a, b are called the real and imaginary parts of the complex number $z = (a, b)$, that is $a = Re(z)$ and $b = Im(z)$. If $b = Im(z) = 0$ then $z = (a, 0) = a$ so that the set of complex numbers is a natural extension of real numbers, then we have:

$a = (a, 0)$ for any real number a. Thus

$$0 = (0,0), \qquad 1 = (1,0), \qquad 2 = (2,0), \dots$$

A pair $(0,\ b)$ is called a pure imaginary number and the pair $(0,\ 1)$ is called the imaginary $\boldsymbol{i}$, that is

$$(0,1) = i$$

Now any complex number z can be written as:

$$(a, 0) + (0, b) = (a, b) = z$$

The operation of addition $(z_1 + z_2)$ and multiplication $(z_1 . z_2)$ are defined as follows

$z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

$z_1 . z_2 = (a_1, b_1) . (\ a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2)$

Such that $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$

Now,

$z = (a, 0) + (0, b) = (a, 0) + (0,1)(b, 0)$

Hence $(a, 0) + (0,1)(b, 0) = (a, b) = z$ where $(0,1) = i$

Then $z = a + ib$

Now, $z^2 = z.z,\ z^3 = z.z.z,\ \ z^n = \underbrace{z.z \ldots..z}_{n-\text{times}}$

$i^2 = i.i = (0,1).(0,1) = -1$ or $i = \sqrt{-1}$

Then $i^2 = -1,\ i = \sqrt{-1}$

[2] Basic Algebraic Properties:

The following algebraic properties hold for all $z_1, z_2, z_3 \in \mathbb{C}$

1. $z_1 + z_2 = z_2 + z_1$
2. $z_1.z_2 = z_2.z_1$

(Commutative laws under addition and multiplication)

3. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associative under addition)
4. $(z_1.z_2).z_3 = z_1.(z_2.z_3)$ (Associative under multiplication)
5. $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$ (Distribution laws)
6. $z_1 + z_3 = z_3 + z_2$ iff $z_1 = z_3$
7. $z_1.z_2 = z_3.z_2$ iff $z_1 = z_3$

(Cancelation law)

Note: the additive identity $0 = (0,0)$ and the multiplication identity $1 = (1,0)$, for any complex number. That is

$$z + 0 = 0 + z = z$$

$$1.z = z.1 = z$$

for any complex number.

Definition:

The additive inverse z^* of z is a complex number with the property that

$z + z^* = 0$ (1)

It is clear that (1) is satisfied if $z^* = (-x, -y)$, has an additive inverse.

Definition:

The multiplication inverse $z^{-1}(z \neq 0)$ of z is a complex number with the property that

$z.z^{-1} = z^{-1}.z = 1$ (2)

Such that:

$$z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \quad \text{(H.w)}$$

Note: the additive and multiplication identity are unique.

Note: if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}, \frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$$

Exercise: show that $z = 0$ iff $Re(z) = 0$ and $Im(z) = 0$.

Example: verify that

1. $\left(\sqrt{2} - i\right) - i(1 - \sqrt{2}\, i)$

Solution:

$\sqrt{2} - i - i - \sqrt{2} = -2i$

2. $(2, -3)(-2,1)$

Solution:

$(2, -3)(-2, 1) = (-4 + 3, 2 + 6) = (-1, 8)$

3. $(3,1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right)$

Solution:

$(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right) = (9+1,-3+3)\left(\frac{1}{5},\frac{1}{10}\right)$

$= (10,0)\left(\frac{1}{5},\frac{1}{10}\right)$

$= \left(\frac{10}{5}-0,\ \frac{10}{10}+0\right)$

$= (2,1)$

Example: show that each of the two numbers $z = 1 \mp i$ satisfies the equation

$$z^2 - 2z + 2 = 0$$

Proof: for $z = 1+i$

$(1+i)^2 - 2(1+i) + 2 = 1 + 2i - 1 - 2 - 2i + 2 = 0$

for $z = 1 - i$ (H.w)

Example: show that $(1-i)^4 = -4$

Proof: $((1-i)^2)^2 = (1-2i-1)^2$

$= 4i^2 = -4$

Example: prove that $(1+z)^2 = 1 + 2z + z^2$

Proof: L.H.S → $(1+z)^2 = (1+z)(1+z)$

$= \big((1,0)+(x,y)\big).((1,0)+(x,y))$

$= (1+x,y)(1+x,y)$

$= (1+2x+x^2-y^2, 2y+2xy)$

R.H.S → $1 + 2z + z^2 = (1,0) + 2(x,y) + (x,y).(x,y)$

$= (1,0) + (2x,2y) + (x,y).(x,y)$

$= (1+2x+x^2-y^2, 2y+2xy)$

$= (1+z)^2$

= L.H.S

Note: $(-z)$ is the only additive inverse of a given complex number.

[3] Properties of Complex Numbers:

1. $Im(iz) = Re(z)$

2. $Re(iz) = Im(z)$

3. $\frac{1}{1/z} = z, \ z \neq 0$

4. $(-1)z = -z$

5. $(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4)$

6. $\frac{z_1+z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}, z_3 \neq 0$

Note:

$$(1+z)^n = 1 + nz + \frac{n(n+1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \cdots + z^n$$

[4] Vectors and Moduli

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment or vector from the origin to the point $(x,\ y)$ that represents z in the complex plane. In fact, we can often refer to z as the point z or the vector z, in Fig. 1 the number $z = x + iy$ and $-2 + i$ are displayed graphically as both two points and radius vector.

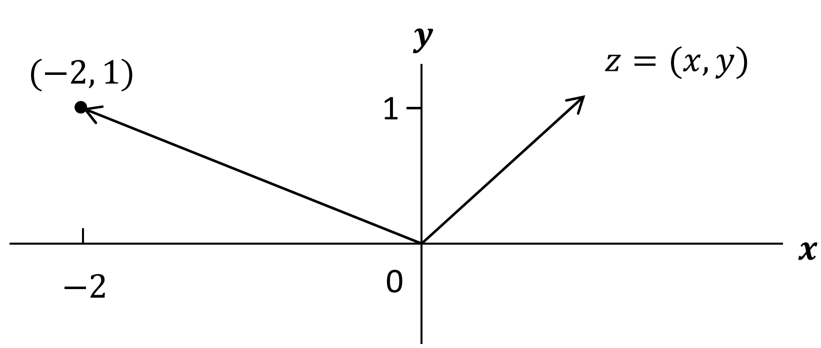


Figure 1

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Corresponds to the point $(x_1 + x_2, y_1 + y_2)$, it is also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 2.

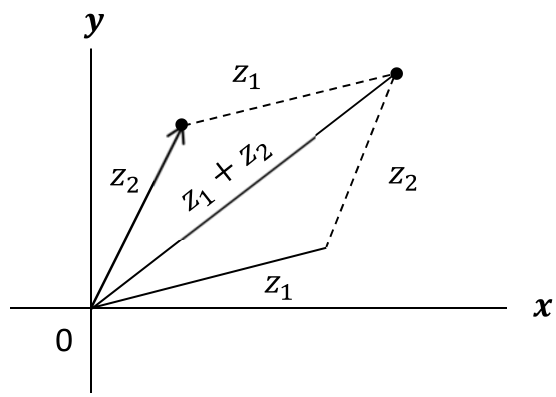


Figure 2

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$, this is clear from Fig. 3, since $|z_1 - z_2|$ is the length of the vector representing the number $z_1 - z_2 = z_1 + (-z_2)$,

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

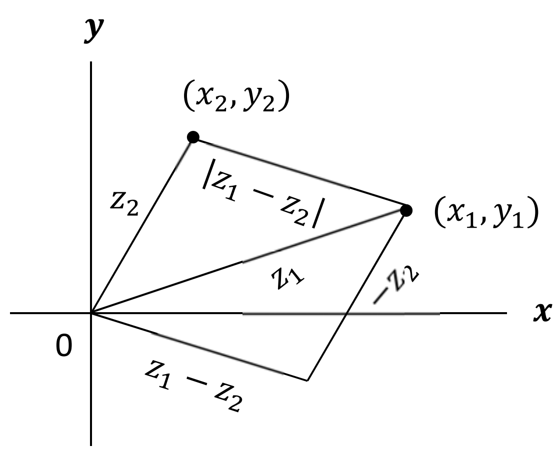


Figure 3

Example: the equation $|z - 1 + 3i| = 2$ represents the circle whose center is $z_0 = (1, -3)$ and whose radius is $R = 2$.

$|z - z_0| = R$, where z_0 represents the center of circle with radius R.

Definition: (The Absolute Value)

The modulus or absolute value of a complex number $z = x + iy$ is defined by $\sqrt{x^2 + y^2}$ and also by $|z|$, such that

$$|z| = \sqrt{x^2 + y^2}$$

we notice that the modulus $|z|$ is a distance from (0,0) to (x, y), the statement $|z_1| < |z_2|$ means that z_1 is closer to (0,0) than z_2. The distance between z_1 and z_2 is given by

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

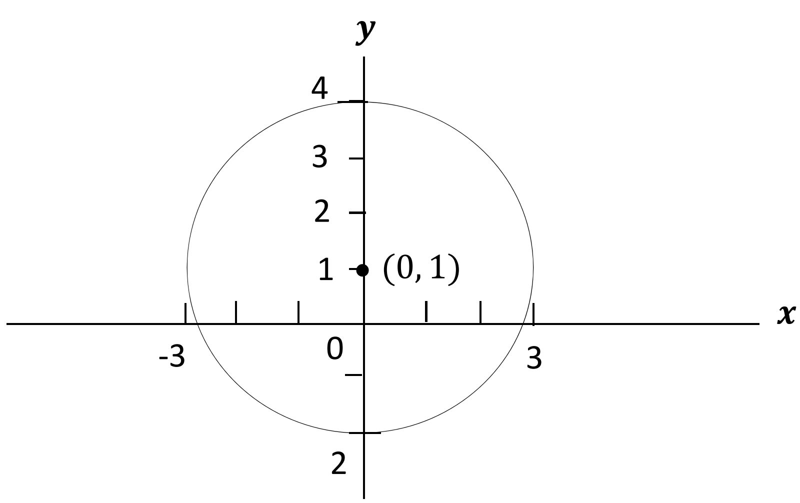
Example: $|z - i| = 3$

Solution: we refer to $|z - i| = 3$ as $|x + iy - i| = 3$

$|x + i(y - 1)| = 3 \rightarrow \sqrt{x^2 + (y - 1)^2} = 3$

$x^2 + (y - 1)^2 = 9 \Leftrightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$

The complex number corresponding to the points lying on the circle with center (0,1) and radius 3



Note: the real numbers $|z|$, $Re(z)$ and $Im(z)$ are related by the equation:

$$|z|^2 = (Re(z))^2 + (Im(z))^2$$

As follows

$$|z| = \sqrt{x^2 + y^2} \rightarrow |z|^2 = x^2 + y^2 = (Re(z))^2 + (Im(z))^2$$

Since $y^2 \geq 0$, we have

$$|z|^2 \geq x^2 = \left(Re(z)\right)^2 = |Re(z)|^2$$

And since $|z| \geq 0$, we get

$$|z| \geq |Re(z)| \geq Re(z)$$

Similarly $|z| \geq |Im(z)| \geq Im(z)$.

[5] Complex Conjugates

The complex conjugate of z is defined by

$$\bar{z} = x - iy$$

The number is $\bar{z}$ represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 4), note that

$$\bar{\bar{z}} = z \text{ and } |\bar{z}| = |z|, \qquad \text{for all } z$$

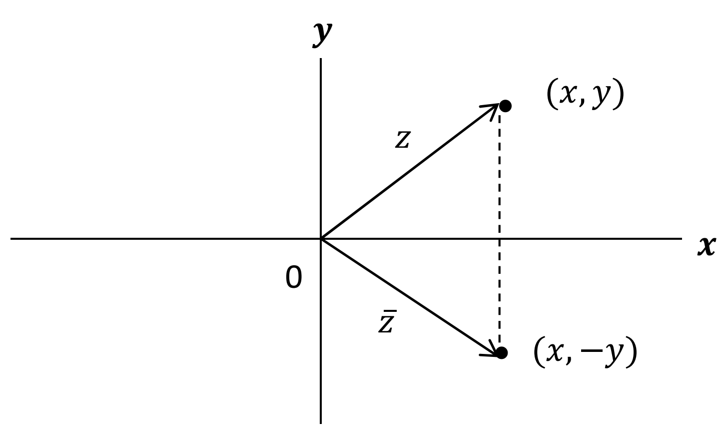


Figure 4

Some Properties of Complex Conjugates:

1. $\bar{\bar{z}} = z$

2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

3. $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad , z_2 \neq 0$

Note:

1. $z + \bar{z} = x + iy + x - iy = 2x = 2Re(z)$

$$Re(z) = \frac{z+\bar{z}}{2}$$

2. $z - \bar{z} = x + iy - x + iy = 2iy = 2Im(z)$

$$Im(z) = \frac{z-\bar{z}}{2}$$

Some Properties of Moduli

1. $|z_1 z_2| = |z_1||z_2|$

2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \; , z_2 \neq 0$

3. $|z_1 + z_2| \leq |z_1| + |z_2|$

4. $|z_1 + z_2 + \cdots z_n| \leq |z_1| + |z_2| \cdots |z_n|$

5. $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

6. $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

Example: If a point z lies on the unite circle $|z| = 1$ about the origin, show that $|z^2 - z + 1| \leq 3$ and $|z^3 - 2| \geq ||z|^3 - 2|$

Proof: $|z^2 - z + 1| = |(z^2 + 1) - z| \leq |z^2 + 1| + |z|$

$\leq |z^2| + 1 + |z|$

$= |z|^2 + 1 + |z|$

$= 1^2 + 1 + 1$

$= 3$

$\rightarrow |z^2 - z + 1| \leq 3$

Prove that $\sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

Solution:

$\left(\sqrt{2}\,|z|\right)^2 = 2|z|^2 = 2(x^2 + y^2)$

$= (x^2 + y^2) + (x^2 + y^2)$

$\geq (x^2 + y^2) + 2|x||y| \cdots$ (by $*$)

$= (|x| + |y|)^2$

$\therefore \left(\sqrt{2}\,|z|\right)^2 \geq (|x| + |y|)^2$

$\rightarrow \sqrt{2}\,|z| \geq |x| + |y| = |Re(z)| + |Im(z)|$

$\therefore \sqrt{2}\,|z| \geq |Re(z)| + |Im(z)|$

Note: $(|x| - |y|)^2 \geq 0$

$\rightarrow |x|^2 + |y|^2 - 2|x||y| \geq 0$

$\rightarrow x^2 + y^2 \geq 2|x||y| \quad \ldots (*)$

Prove that:

1. z is real iff $\bar{z} = z$ (H.w)

2. z is either real or pure imaginary iff $(\bar{z})^2 = z^2$

Prove that: if $|z_2| \neq |z_3|$ then

$$\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|}$$

Proof:

$$\left|\frac{z_1}{z_2+z_3}\right| = \frac{|z_1|}{|z_2+z_3|} \qquad \dots (1)$$

Since $|z_2 + z_3| \geq \big||z_2| - |z_3|\big|$

$$\rightarrow \frac{1}{|z_2+z_3|} \leq \frac{1}{\big||z_2|-|z_3|\big|}$$

$$\rightarrow \frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|} \quad \dots (2)$$

From (1) and (2) we have

$$\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{\big||z_2|-|z_3|\big|}$$

Example: If a point z lies on the unite circle $|z| = 2$ then show that

$$\frac{1}{|z^4-4z^3+3|} \leq \frac{1}{3}$$

Proof: $|z^4 - 4z^3 + 3| = |(z^2 - 1)(z^2 - 3)|$

$$= |z^2 - 1||z^2 - 3|$$

$$\geq \big||z|^2 - 1\big| \, \big||z|^2 - 3\big|$$

$$= |4 - 1| \, |4 - 3|$$

$$= 3$$

$\therefore \ |z^4 - 4z^3 + 3| \geq 3$

$$\rightarrow \frac{1}{|z^4-4z^3+3|} \leq \frac{1}{3}$$

Exercises:

1. Show that the hyperbola $x^2 - y^2 = 1$, can be written as

$$z^2 + \bar{z}^2 = 2$$

2. Show that $|z - 4i| + |z + 4i| = 10$ is an ellipse whose foci are $(0, \mp 4)$.

Proof: 1. $x^2 - y^2 = 1,\ x = \frac{z+\bar{z}}{2},\ y = \frac{z-\bar{z}}{2i}$

$$\left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 = 1$$

$$\frac{z^2+2z\bar{z}+\bar{z}^2}{4} - \frac{z^2-2z\bar{z}+\bar{z}^2}{4i^2} = 1$$

$$\frac{z^2+2z\bar{z}+\bar{z}^2}{4} + \frac{z^2-2z\bar{z}+\bar{z}^2}{4} = 1$$

$$\to 2z^2 + 2\bar{z}^2 = 4$$

$$\to 2(z^2 + \bar{z}^2) = 4$$

$$\to z^2 + \bar{z}^2 = 2$$

[6] Polar Form of Complex Numbers: (Exponential Form)

Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero complex number $z = x + iy$,

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

The number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$tan\theta = \frac{y}{x}\ ,\ x \neq 0,\ r^2 = x^2 + y^2,\ i\theta = \cos\theta + i\sin\theta$$

This implies that for any complex number $z = x + iy$, we have

$$|z| = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

In fact r is the length of the vector represent z. In particular, since $z = x + iy$ we may express z in polar form by

$$z = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

The real number θ represents the angle, measured in radians, that z makes with the positive real axis (Fig. 5).

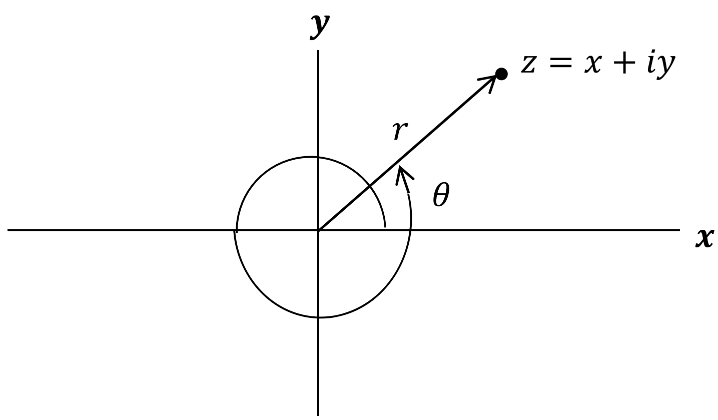


Figure 5

Each value of θ is called an argument of z and the set of all such values is denoted by $\arg z = \theta$.

<u>Note:</u> $arg\, z$ is not unique.

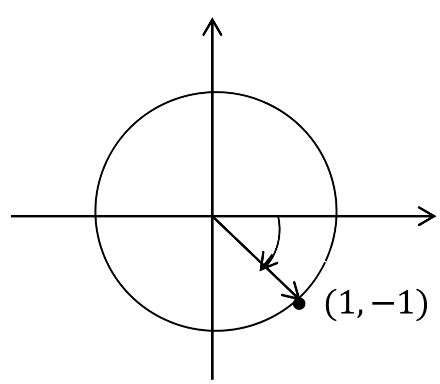
<u>Definition:</u> The principal value of $\arg z$ (Arg z)

If $-\pi < \theta < \pi$ and satisfy

$$\arg z = \text{Arg}\, z + 2n\pi,\ n = 0, \mp 1, \mp 2, \ldots$$

Then this value of θ (which is unique) is called the principal value of $\arg z$ and denoted by Arg z.

<u>Example:</u> Write $z = 1 - i$ in polar form



<u>Solution:</u> $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$x = r\cos\theta \to 1 = \sqrt{2}\cos\theta \to \cos\theta = \frac{1}{\sqrt{2}}$

$y = r\sin\theta \to -1 = \sqrt{2}\sin\theta \to \sin\theta = \frac{-1}{\sqrt{2}}$

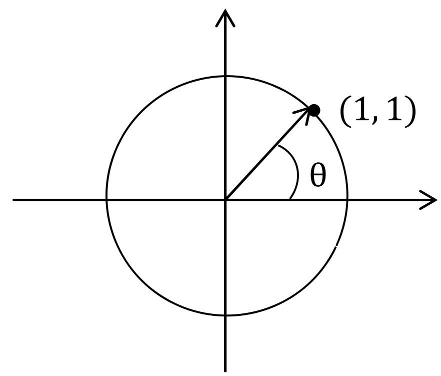
$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$

$\theta = \tan^{-1}(-1) = \frac{-\pi}{4}$

$z = 1 - i = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

$= \sqrt{2}\left(\cos\left(\frac{-\pi}{4} + 2n\pi\right) + i\sin\left(\frac{-\pi}{4} + 2n\pi\right)\right)$

Example: Write $z = 1 + i$ in polar form



Solution: $r = \sqrt{2}$, $\tan\theta = \frac{y}{x} = 1$

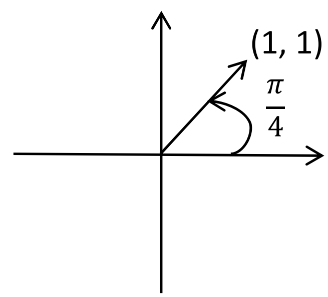
$\rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \theta = arg\, z = \frac{\pi}{4} + 2n\pi$

$\therefore 1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2n\pi\right) + i\sin\left(\frac{\pi}{4} + 2n\pi\right)\right)$

Example: Find the principal argument Arg z when

1. $z = 1 + i$



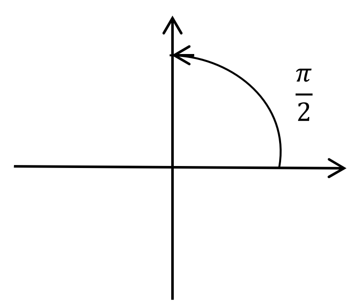
Solution: $\arg z = \text{Arg}\, z + 2n\pi$

$= \frac{\pi}{4} + 2n\pi$

$\therefore \text{Arg}\, z = \frac{\pi}{4}$

2. $z = i$

Solution: $r = 1$, $\theta = \frac{\pi}{2} + 2n\pi = arg\, i$



$\arg z = \text{Arg}\, z + 2n\pi$

$= \frac{\pi}{2} + 2n\pi$

$\therefore \text{Arg}\, z = \frac{\pi}{2}$

$\therefore i = z = 1.\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{2}\right)$

Exercises: Find the principal argument Arg z when $z = -i, 1, -1$.

Example: Let $z=-1-i$, write z in polar form and find $\operatorname{Arg} z$.

Solution: $r=\sqrt{1+1}=\sqrt{2}$

$x=r\cos\theta \rightarrow -1=\sqrt{2}\cos\theta \rightarrow \cos\theta=\frac{-1}{\sqrt{2}}$

$y=r\sin\theta \rightarrow -1=\sqrt{2}\sin\theta \rightarrow \sin\theta=\frac{-1}{\sqrt{2}}$

$\theta=\tan^{-1}(1)=\frac{\pi}{4}$

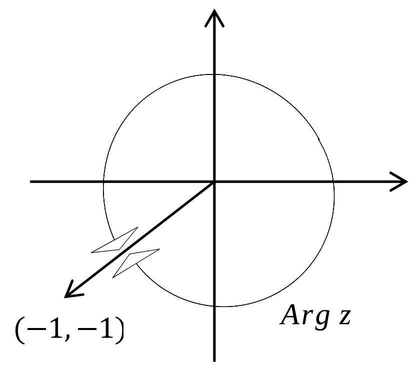
$\theta=\frac{\pi}{4}+\pi=\frac{5\pi}{4}+2n\pi$ (Since θ is located in the third quarter)

$=argz$

$\therefore \operatorname{Arg} z=\arg z-2\pi$

$=\frac{5\pi}{4}-2\pi=\frac{-3\pi}{2}\in[-\pi,\pi]$

$z=-1-i=\sqrt{2}\left(\cos\frac{-3\pi}{2}+i\sin\frac{-3\pi}{2}\right)$



Example: Let $z_1=1+\sqrt{3}\,i$, $z_2=-1-\sqrt{3}\,i$, write z_1, z_2 in polar form and find $\operatorname{Arg} z_1$, $\operatorname{Arg} z_2$.

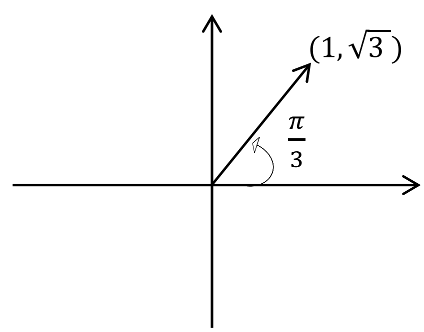
Solution: $z_1=r_1=\sqrt{x^2+y^2}=\sqrt{1^2+\left(\sqrt{3}\right)^2}=\sqrt{1+3}=2$

$x=r\cos\theta \rightarrow 1=2\cos\theta \rightarrow \cos\theta=\frac{1}{2}$

$y=r\sin\theta \rightarrow \sqrt{3}=2\sin\theta \rightarrow \sin\theta=\frac{\sqrt{3}}{2}$

$\therefore \theta=\tan^{-1}\frac{y}{x}=\frac{\pi}{3}+2n\pi$

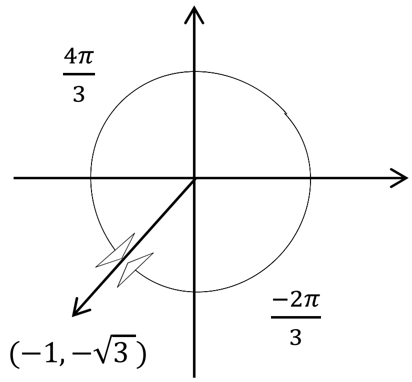
$z_1=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$



$\rightarrow z_2=r_2=\sqrt{(-1)^2+\left(-\sqrt{3}\right)^2}=2$

$x=r\cos\theta \rightarrow -1=2\cos\theta \rightarrow \cos\theta=\frac{-1}{2}$

$y = r\sin\theta \to -\sqrt{3} = 2\sin\theta \to \sin\theta = \frac{-\sqrt{3}}{2}$



$\therefore \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{-1} = \tan^{-1}\sqrt{3}$

$= \left(\pi + \frac{\pi}{3}\right) + 2n\pi$

$= \frac{4\pi}{3} + 2n\pi$

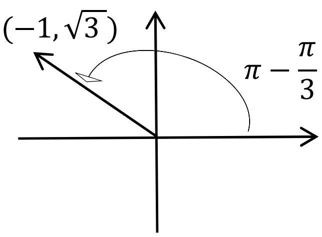
$\text{Arg } z_2 = \frac{4\pi}{3} - 2\pi$

$= \frac{-3\pi}{3}$

$z_2 = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

Example: $z_3 = -1 + \sqrt{3}\, i,\ z_4 = 1 - \sqrt{3}\, i$

Solution:

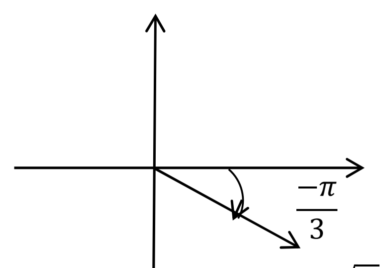


$\text{Arg } z_3 = \frac{2\pi}{3}$

$z_3 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\to z_4 = 1 - \sqrt{3}\, i$

$= 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$



Note:

$\left.\begin{matrix} 1 \mp i \\ -1 \mp i \end{matrix}\right\}$ Angle 45°

$\left.\begin{matrix} 1 \mp \sqrt{3}\, i \\ -1 \mp \sqrt{3}\, i \end{matrix}\right\}$ Angle 60°

$\left.\begin{matrix} \sqrt{3} \mp i \\ -\sqrt{3} \mp i \end{matrix}\right\}$ Angle 30°

- **Properties of $\arg z$:**

1. $\arg(z_1.z_2) = \arg z_1 + \arg z_2$

2. $\arg\left(\frac{1}{z}\right) = -\arg z$

3. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

4. $\arg \bar{z} = -\arg z$

Proof:

1. Let $z_1 = r_1(\cos\theta_1 + i\ \sin\theta_1)$

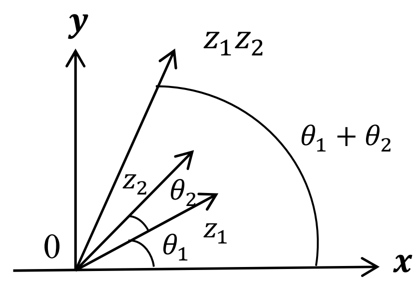
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1.z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\cos\theta_1\ \sin\theta_2 + i\ \sin\theta_1\cos\theta_2)$

$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$\therefore \arg z_1 z_2 = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$



Example: Find $\arg\left(i\left(1+\sqrt{3}\,i\right)\right)$

Solution:

$\arg\left(i\left(1+\sqrt{3}\,i\right)\right) = \arg i + \arg(1+\sqrt{3}\,i)$

$$= \left(\frac{\pi}{2} + 2n\pi\right) + \left(\frac{\pi}{3} + 2n\pi\right)$$

$$= \frac{5}{6}\pi + 2k\pi\,, \quad k = n + m$$

2. Let $z = r(\cos\theta + i\sin\theta)$

$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} \cdot \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta - i\sin\theta)}$

$= \frac{r(\cos\theta - i\sin\theta)}{r^2(\cos^2\theta + \sin^2\theta)}$

$= \frac{r(\cos\theta - i\sin\theta)}{r^2}$

$\frac{1}{z} = \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right)$

$\therefore \arg\left(\frac{1}{z}\right) = -\arg z$

<u>**Note:**</u> $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$

<u>For example:</u> Let $z_1 = i\,,\; z_2 = -1 + \sqrt{3}\,i$

$\arg z_1 = \left(\frac{\pi}{2} + 2n\pi\right),\; \arg z_2 = \left(\frac{\pi}{3} + 2n\pi\right)$

$\operatorname{Arg} z_1 = \frac{\pi}{2}$ $\qquad ,\operatorname{Arg} z_2 = \frac{\pi}{3}$

$z_1 z_2 = i\left(-1 + \sqrt{3}\,i\right) = -\sqrt{3} - i$

$\arg z_1 z_2 = \pi + \frac{\pi}{6} = \frac{7}{6}\pi + 2n\pi$

$\operatorname{Arg} z_1 z_2 = \left(\pi + \frac{\pi}{6}\right) - 2\pi = \frac{-5}{6}\pi$

$\therefore \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7}{6}\pi \;\notin [-\pi, \pi]$

[7] Powers and Roots

Let $z = re^{i\theta}$ be a nonzero complex number, let n be an integer number then

$$z^n = r^n e^{in\theta}$$

<u>**Example:**</u> Find $(1+i)^{25}$

<u>*Solution*</u>: $r = \sqrt{x^2 + y^2} = \sqrt{2}\,,\; \theta = \frac{\pi}{4}$

$z^{25} = \left(re^{i\theta}\right)^{25}$

$\quad = \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{25}$

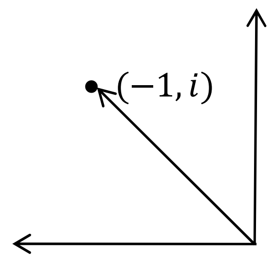
$\quad = \left(\sqrt{2}\right)^{25} e^{i\,25.\frac{\pi}{4}}$

$= 12\sqrt{2}\left(cos\frac{\pi}{4} + i\ sin\frac{\pi}{4}\right)$

$= 12\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= 12(1 + i)$

Example: Find $(-1 + i)^4$



Solution: $r = \sqrt{2}\ ,\ \theta = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$

$z^n = r^n e^{in\theta} = \left(\sqrt{2}\right)^4 e^{i\,4.\frac{3\pi}{4}}$

$= 4e^{i3\pi}$

$= 4(\cos 3\pi + i \sin 3\pi)$

$= 4(-1 + 0) = -4$

[8] De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof: by mathematical induction

1. If $n = 1 \rightarrow (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

2. Let it be true if $n = k$, we get

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \ldots (*)$

3. We must proof it is true if $n = k + 1$

Multiplying $(*)$ by $(\cos\theta + i\sin\theta)$

$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k = (\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta)$

$= (\cos\theta\cos k\theta + i\cos\theta\sin k\theta + i\sin\theta\cos k\theta - \sin\theta\sin k\theta)$

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1) + i\sin(k+1)$

$\therefore$ It is true if $n = k + 1$

Note: If $z^n = z_0$ then $z = z_0^{\frac{1}{n}}$ and $z = re^{i\theta} = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0+2k\pi}{n}\right)} = z^{1/n}$ is called nth − root of z.

Example: Calculate root of $z^3 = i$

Solution: $z^3 = i \to z = (i)^{1/3}$

$$\to re^{i\theta} = \left(1.e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right)^{1/3}$$

$$\text{s.t } \theta_0 = \frac{\pi}{2} + 2k\pi\ ,\ k = 0, \mp1, \mp2, \dots$$

$$\to re^{i\theta} = e^{i\,\frac{\pi}{6}+\frac{2}{3}k\pi}$$

$$\therefore r = 1\ ,\ \theta = \frac{\pi}{6} + 2k\pi,\ k = 0, \mp1, \mp2, \dots$$

To find the roots:

If $k = 0 \to \theta_1 = \frac{\pi}{6}$ (in the first quarter)

$$\to z_1 = 1.e^{i\,\frac{\pi}{6}}$$

If $k = 1 \to z_2 = 1.e^{i\,\frac{\pi}{6}+\frac{2\pi}{3}}$ (in the second quarter)

$$= \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi$$

$$= \frac{-\sqrt{3}}{6} + \frac{i}{2}$$

If $k = 2 \to z_3 = 1.e^{i\,\frac{\pi}{6}+\frac{4\pi}{3}}$

$$= \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$$

$$= -i$$

Note:

1. If the complex number was raised to a fraction whether it was $\frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$ then the number of roots is $3, 4, \dots, n$. In the above example the number of roots is 3.

2. $z^n = z_0$ has n different roots only and they are located on the vertices of a regular polygon centered at the origin.

Example: $z^2 = 1 + i$ has two different roots

Solution:

$z^2 = 1 + i \rightarrow z = (1+i)^{1/2}$

$r_0 = \sqrt{2}\,,\ \theta_0 = \frac{\pi}{4} + 2n\pi$

Since $z = (1+i)^{1/2}$

$$\therefore re^{i\theta} = \left(\sqrt{2}\right)^{\frac{1}{2}}\left(e^{i\,\frac{\pi}{4}+2n\pi}\right)^{\frac{1}{2}}$$

$$= \sqrt[4]{2}\, e^{i\,\frac{\pi}{8}+\, n\pi}$$

$r = \sqrt[4]{2}\,,\quad \theta = \frac{\pi}{8} + k\pi$

If $k = 0 \rightarrow\ z_1 = \sqrt[4]{2}\, e^{i\,\frac{\pi}{8}}$

$$= \sqrt[4]{2}\left(\sqrt{\frac{1+\cos\frac{\pi}{8}}{2}} + i\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}\right)$$

If $k = 1 \rightarrow\ z_2 = \sqrt[4]{2}\, e^{i\,\frac{\pi}{8}+\,\pi}$

$$= \sqrt[4]{2}\left(\cos\left(\frac{\pi}{8}+\pi\right) + i\sin\left(\frac{\pi}{8}+\pi\right)\right)$$

$$= \sqrt[4]{2}\left(-\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$$

$$= -\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Note:

$$\cos\frac{\theta}{2} = \mp\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \mp\sqrt{\frac{1-\cos\theta}{2}}$$

Note: Let $m, n \neq 0$ be any integer numbers, let z be any complex number then

$$(z)^{m/n} = \left(z^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{r_0}\, e^{\left(\frac{i\theta_0 + 2k\pi}{n}\right)}\right)^m$$

$$= \left(\sqrt[n]{r_0}\right)^m e^{i\,\frac{m(\theta_0+2k\pi)}{n}}, \ k = 0, \mp 1, \mp 2, \ldots$$

Example: Solve the following equation

$$z^{2/3} = i$$

Solution: $z^{3/2} = i \rightarrow z = (i)^{2/3} = \left(i^{1/3}\right)^2$

$$= (i)^{1/3}(i)^{1/3}$$

That is each one has three roots.

Let $w = (i)^{1/3} \rightarrow z = w^2$

Now, we find the roots of w

$r_0 = 1\,, \theta_0 = \frac{\pi}{2} + 2k\pi\,, k = 0, \mp 1, \mp 2, \ldots$

$$w = re^{i\theta} = 1.\left(e^{i\,\frac{\pi}{2} + 2k\pi}\right)^{1/3}$$

$$= e^{i\,\frac{\pi}{6} + \frac{2k\pi}{3}}$$

$\therefore w_1 = e^{i\,\frac{\pi}{6}} = \cos\left(\frac{\pi}{6} + i \sin\frac{\pi}{6}\right), k = 0$

$w_2 = e^{i\,\frac{\pi}{6} + \frac{2\pi}{3}} = e^{i\,\frac{5\pi}{6}}\,, k = 1$

$w_3 = e^{i\,\frac{\pi}{6} + \frac{4\pi}{3}} = e^{i\,\frac{3\pi}{2}}\,, k = 2$

$\therefore z = w^2$

$\therefore z_1 = (w_1)^2 = \left(e^{i\,\frac{\pi}{6}}\right)^2 = e^{i\,\frac{\pi}{3}}$

$= cos\frac{\pi}{3} + i\ sin\frac{\pi}{3}$

$= \frac{1}{2} + i\ \frac{\sqrt{3}}{2}$

$z_2 = (w_2)^2 = \left(e^{i\frac{5\pi}{6}}\right)^2 = e^{i\frac{5\pi}{3}}$

$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$z_3 = (w_3)^2 = \left(e^{i\frac{3\pi}{2}}\right)^2 = e^{i3\pi}$

$$= \cos 3\pi + i\sin 3\pi$$

<u>H.w:</u> Find the roots of $(-8i)^{1/3}$.

[9] Regions in the Complex Plane

Some definitions and concepts:

<u>Definition:</u> Let z be any point in the z-plane, let $\epsilon > 0$ then

1. $N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$

This set is called a neighborhood of z_0.

2. $S_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| = \epsilon\}$

This set is called sphere with center z_0.

3. $D_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$

This set is called the Disk with center z_0 and radius ϵ.

<u>Definition:</u> Let $U \subseteq \mathbb{C}$, we say that U is open set if

$$\forall\, w \in U, \exists\, N_\epsilon(w)\ \text{ s.t }\ N_\epsilon(w) \subseteq U.$$

<u>For example:</u> $\emptyset$, $\mathbb{C}$ are open sets.

<u>Definition:</u> Let $F \subseteq \mathbb{C}$, we say that F is closed set if $\mathbb{C} - F$ is open set.

Definition: An open set $S \subseteq \mathbb{C}$ is connected if each pair of points z_1, z_2 in it can be joined by a polygon line, consisting of a finite number of line segments joined end to end that lies entirely in S.

Definition: Let $S \subseteq \mathbb{C}$, we say that S is Region if it is open and connected.

Example:

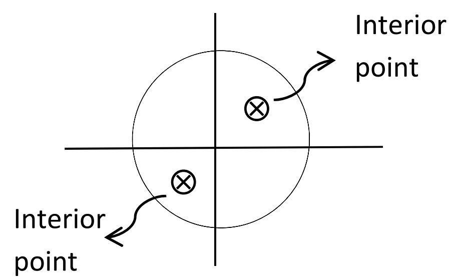
1. $|z| > 1, |z| < 1$ is Region.

2. Let $|z| = 0$ is not Region, since it is connected but not open set.

3. $\mathbb{R} \subset \mathbb{C}$ is connected but not open, since $\forall\, r \in \mathbb{R}, \exists\, N_\epsilon(r)$ contain some of complex points.

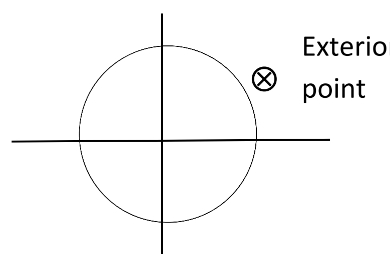
Definition: Let $z_0 \in S$, we say that z_0 is interior point if there exist a neighborhood $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \subseteq S$.

Example: $|z| < 1$

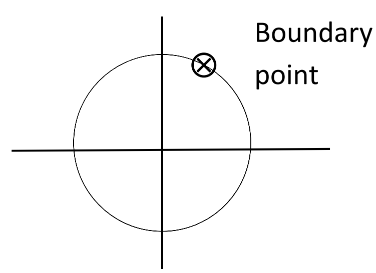


Definition: Let $z_0 \in S$, we say that z_0 is exterior point if there exist a neighborhood $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \cap S = \emptyset$.

Example: $|z| > 1$



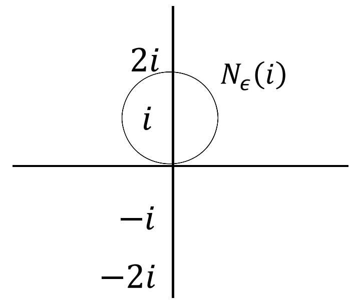
Definition: Let $z_0 \in S$, we say that z_0 is Boundary point if $\forall\, N_\epsilon(z_0)$ contain points from inside S and outside it.



Note: S is close set iff it contains all the boundary points.

Example: $S = \{\mp i, \mp 2i\}$, is S open set ?

Note $N_\epsilon(i) \nsubseteq S$, therefore S is not open.



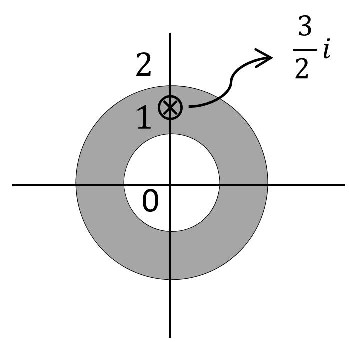
Example: $S = \{z \in \mathbb{C} : 1 < |z| < 2\}$

Note

0 is exterior point of S

1, 2 are boundary points of S

$\left(\frac{3}{2} i\right)$ is interior point of S

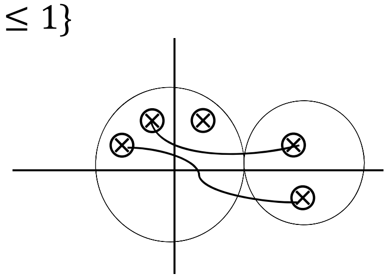


Example: $D = \{z \in \mathbb{C} : 2 < |z| \leq 3\}$

D is not open set since it contain all the boundary points.

Example: $S = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 2| \leq 1\}$

Note S is connected set.



But if

$S = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 2| < 1\}$,

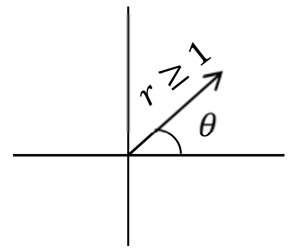
then S is not a connected set.

Definition: Let $S \subseteq \mathbb{C}$, we say that S is bounded set if $\exists$ Disk D,

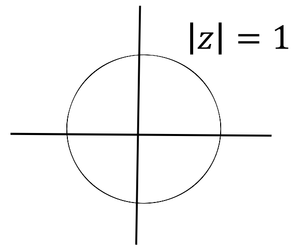
$D = \{\text{z} : |z| \leq \mathbb{R}\}$ such that $S \subseteq D$.

<u>Example</u>: $S = \left\{z \in \mathbb{C} : r \geq 1,\ 0 \leq \theta \leq \frac{\pi}{4}\right\}$

S is not bounded set since $\nexists$ Disk contain S.



<u>Example:</u> $|z| = 1$ is bounded set



<u>Example</u>: $S = \{\mp i, \mp 2i\}$

1. S is not open set since every point of S is boundary point.

2. S is close set since every point of S is boundary point.

3. S is not connected set.

4. S is not bounded set.

<u>Definition:</u> Let $z_0 \in S$, we say that z_0 is limit point if

$$N_\epsilon(z_0) \cap (S - z_0) \neq \emptyset$$

<u>Example</u>: $S = \left\{z \in \mathbb{C} : z = \frac{1}{n},\ n = 1, 2, \ldots\right\}$, 0 is the only limit point.