التحليل الدالي

المحاضرة الرابعة

قسم الرياضيات

الصف الرابع

Convergence in Normed Spaces

Let X be a non-empty set. A sequence in X is any function from N (the set of all natural numbers) into X . If f is a sequence in X , the image f (n) of $n \in N$ is usually, denoted by x_n . It is customary to denote the sequence of the classical symbol $\{x_n\}$. We say that $\{x_n\}$ the sequence of real numbers if $X = R$. Sometime, we write it as $\{x_1 , x_2 , ..., x_n ,..\}$ .The image x_n of n is called the nth term of the sequence.

Note that, there is difference between the sequence and its range. For example, the rang of the sequence $\{(-1)^n\}$ is $\{x_n : n \in N \} = \{-1,1\}$ but the sequence is $\{x_n\}=\{(-1)^n\}=\{1,-1,1,-1, \ldots\}$.

Definition(1.32)

A sequence $\{x_n\}$ in a normed space X is said to be

1. *Converge to the point $x \in X$, if $\lim_{n\to\infty} \|x_n - x\| = 0$. i.e. for each $\varepsilon > 0$,*

   there exist $k \in Z^+$ such that $\|x_n - x\| < \varepsilon$ for all $n \geq k$.

   and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

   It follows that $x_n \to x$ iff $\| x_n - x\| \to 0$

2. *Cauchy sequence in X , if for every $\varepsilon > 0$, there exists $k \in Z^+$ such that $\|x_n - x_m\| < \varepsilon$ for all $n, m \geq k$*

3. *Bounded, if there is $M > 0$ such that $\|x_n \| \leq M$ for all n.*

Theorem (1.33)

In a normed space X and $A \subseteq X$

1. limit point of sequence is unique.

2. Every convergence sequence is Cauchy sequence, but the converse not true.

3. Every Cauchy sequence is bounded, but the converse not true.

4.Every convergent sequence in the normed space X is bounded.

5. $x \in \bar{A}$ iff there exists a sequence $\{x_n\}$ in A such that $x_n \to x$.

6. If a Cauchy sequence in X has a convergent subsequence, then the sequence is convergent.

7. If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in R , then $\{\| x_n - y_n \|\}$ is convergent in R.

Proof:

1-Suppose that $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$, and put $\| x-y \| = \varepsilon \to \varepsilon > 0$

since $x_n \to x \Rightarrow \exists k_1 \in Z^+$ *s.t.* $\| x_n - x\| < \varepsilon/2$, $\forall n > k_1$

and $x_n \to y \Rightarrow \exists k_2 \in Z^+$ *s.t.* $\| x_n - y\| < \varepsilon/2$, $\forall n > k_2$

put $k=\max\{k_1, k_2\}$. *Then* $\| x_n - x\| < \varepsilon/2$, $\| x_n - y\| < \varepsilon/2$ $\forall n > k$.

$\varepsilon = \| x-y \| = \| (x - x_n) + (x_n - y) \| \le \| (x_n - x) \| + \| (x_n - y) \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$!

and this contradiction then x=y.

2- Suppose that $\{x_n\}$ *is a convergent sequence in the normed space X, then* $\exists x \in X$ *s.t.* $x_n \to x$

Let $\varepsilon > 0$, *since* $x_n \to x \Rightarrow \exists k \in Z^+$ *s.t.* $\| x_n - x \| < \varepsilon/2$ $\forall n > k$

If $n,m \ge k$,*then* $\| x_n - x_m \| = \| (x_n - x) + (x - x_m) \| \le \| x_n - x\| + \| x - x_m \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Then $\{x_n\}$ *is a Cauchy sequence.*

Remark:

The converse to above theorem may not be true. For example:

Let $X = IR - \{0\}$, $\{x_n\} = \{1/n\}$

$\{x_n\}$ *Cauchy convergent sequence in IR*

Since IR complete $\Rightarrow \{x_n\} = \{1/n\} \to 0$ *convergent in IR*

But $\{x_n\}$ *not convergent in* $IR-\{0\}$, *since* $0 \notin IR-\{0\}$.

3- Let $\{x_n\}$ *be a Cauchy sequence in X*

Given $\varepsilon=1$, $\exists k \in Z^+$ *s.t.* $\| x_n - x_m \| < 1$, $\forall n, m > k$.

Let $m = k+1 \Rightarrow \| x_n - x_{k+1} \| < 1$

Since $| \| x_n\| - \| x_{k+1} \| | \le \| x_n - x_{k+1} \| < 1$

$\Rightarrow | \| x_n\| - \| x_{k+1} \| | < 1 \Rightarrow \| x_n\| < 1 + \| x_{k+1} \|$, $\forall n > k$

Put $M = \max \{ \|x_1\|, \|x_2\|, \ldots, \|x_k\|, \|x_{k+1}\|\} \Rightarrow \| x_n \| \le M$, $\forall n \in Z^+$.

4-Let $\{x_n\}$ *be a convergent sequence in X* $\Rightarrow \{x_n\}$ *a Cauchy convergent sequence in X*

$\Rightarrow \{x_n\}$ *bounded.*

H.W. prove 5,6 and 7

Theorem 1.34.: *Let X be a normed space,* $\{ x_n \}$, $\{ y_n \}$ *be a sequence in X such that* $x_n \to x_0$, $y_n \to y_0$, *then:*

1- $x_n \pm y_n \to x_0 \pm y_0$

2- $\| x_n \| \to \| x_0 \|$

3- $\| x_n - y_n \| \to \| x_0 - y_0 \|$

4- $\alpha x_n \to \alpha x_0 \quad \forall \alpha \in F$

Proof:

1- Since $x_n \to x_0$, $y_n \to y_0$, then:

if $\varepsilon > 0$

$\exists k_1(\varepsilon) \in Z^+$ s.t. $\| x_n - x_0 \| < \varepsilon / 2$, $\forall n > k_1(\varepsilon)$

$\exists k_2(\varepsilon) \in Z^+$ s.t. $\| y_n - y_0 \| < \varepsilon / 2$, $\forall n > k_2(\varepsilon)$

Define $k_3(\varepsilon) = \max \{ k_1(\varepsilon), k_2(\varepsilon) \}$

$$\| (x_n + y_n) - (x_0 + y_0) \| = \| x_n + y_n - x_0 - y_0 \|$$
$$\leq \| x_n - x_0 \| + \| y_n - y_0 \|$$
$$< \varepsilon / 2 + \varepsilon / 2 = \varepsilon, \ \forall n > k_3(\varepsilon)$$

$\to x_n + y_n \to x_0 + y_0$

2- Since $x_n \to x_0$ T.P. $\| x_n \| \to \| x_0 \|$ i.e. T.P. $| \, \| x_n \| - \| x_0 \| \, | \to 0$

By Theorem (1.13.)-4 : $| \, \| x_n \| - \| x_0 \| \, | \leq \| x_n - x_0 \|$ ….. (1)

Since $x_n \to x_0 \to \| x_n - x_0 \| \to 0$ …..(2)

By (1) & (2) we get: $| \, \| x_n \| - \| x_0 \| \, | \to 0$

Then $\| x_n \| \to \| x_0 \|$

3- T.P. $\| x_n - y_n \| \to \| x_0 - y_0 \|$, i.e. T.P. $| \, \| x_n - y_n \| - \| x_0 - y_0 \| \, | \to 0$

Since $x_n \to x_0 \Rightarrow \| x_n - x_0 \| \to 0$

& $y_n \to y_0 \Rightarrow \| y_n - y_0 \| \to 0$

$$| \, \| x_n - y_n \| - \| x_0 - y_0 \| \, | \leq \| x_n - y_n - x_0 + y_0 \|$$
$$\leq \| x_n - x_0 \| + \| y_n - y_0 \|$$

$\Rightarrow | \, \| x_n - y_n \| - \| x_0 - y_0 \| \, | \to 0 \ \Rightarrow \| x_n - y_n \| \to \| x_0 - y_0 \|$

4- $\| \alpha x_n - \alpha x_0 \| = \| \alpha (x_n - x_0) \| = |\alpha| \, \| x_n - x_0 \|$

since $\| x_n - x_0 \| \to 0$ where $n \to \infty \Rightarrow \| \alpha x_n - \alpha x_0 \| \to 0$ where $n \to \infty \Rightarrow \alpha x_n \to \alpha x_0$

<u>Definition 1.35.:</u> *Let X be a normed space, $x_0 \in X$, a function f is said to be **continuous** at x_0 if:*

$\forall \ \varepsilon > 0, \exists \ \delta(x_0, \varepsilon) > 0$ s.t. $\| f(x) - f(x_0) \| < \varepsilon$ whenever $\| x - x_0 \| < \delta$.

<u>Theorem 1.36.</u> : *Let X, Y be two Normed space, a function $f : X \to Y$ continuous at $x_0 \in X$ iff for each sequence $\{x_n\}$ in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

Theorem 1.37.: *Let X be a normed space, A subspace in X, then the two following statements are equivalent:*

1- *A is bounded.*

2- *If* $\{x_n\}$ *seq. in X and* $\{\lambda_n\}$ *seq. in F such that* $\lambda_n \to 0$ *as* $n \to \infty$, *then* $x_n\lambda_n \to 0$ *as* $n \to \infty$.

Proof:

T.P. $1 \to 2$

$\| x_n\lambda_n - 0\| = \| x_n\lambda_n \| = | \lambda_n| \, \| x_n \| \to 0.$ as $n \to \infty$.

T.P. $2 \to 1$

Suppose A unbounded

i.e. $\exists\, x_n \in A$ s.t. $\| x_n \| > M$, $\forall n \in Z^+$

put $\lambda_n = 1/n \to 0$ as $n \to \infty$

but $\lambda_n x_n \to 0$ C!

then A is bounded.

Definition 1.38.: *The linear vector space X /Y is called quotient or factor space formed as follows:*

The elements of X / Y are cosets of Y {sets of the form x + Y for x $\in X$*}. The set of cosets is a linear v. space under the operations:*

$(x_1 + Y) \oplus (x_2 + Y) = (x_1 + x_2) + Y;$

$\lambda(x + Y) = \lambda x + Y.$

So for example $Y + Y = Y$ *and* $\lambda Y = Y$ *for* $\lambda \neq 0$. *Two cosets* $x_1 + Y$ *and* $x_2 + Y$ *are equal if assets* $x_1 + Y = x_2 + Y$, *which is true if and only if* $x_1 + x_2 \in Y$.

Definition 1.39.: *A quotient vector space X/Y is called quotient normed space if there exists norm define on X/Y.*

Theorem 1.40.: *If X is a normed space, and Y is a normed linear subspace, then X/Y is a normed space under the norm:*

$\| x + Y \| = \inf \{\| x+y \| : y \in Y\}$. **H.W.**