Construction of the rational numbers

إن المعادلات من نوع $AX = B$ حيث إن $A, B \in Z$ ، $A \neq 1$
قد لا يكون لها حل في نظام الاعداد الصحيحة ولهذا تم توسيع الاعداد
الصحيحة وانشاء الاعداد النسبية .

Notation Let

$$A = \{ (a,b) \mid a, b \in Z , b \neq 0 \}$$

$\therefore A = Z \times Z^* \quad ; Z^* = Z - \{0\}$

Theorem $\exists$ equivalence relation R on the set A

$(a,b) \; R \; (c,d)$ iff $ad = cb$ for all $(a,b),(c,d) \in A$

(i.e $a, b, c, d \in Z$ and $b, d \neq 0$)

Proof

① R is ref. (H.w.)

② R is Symm. (H.w.)

③ R is transitive

Let $(a,b), (c,d), (e,f) \in A$ such that

$(a,b) \; R \; (c,d) \wedge (c,d) \; R \; (e,f)$

$\Rightarrow ad = cb \wedge cf = ed$

$\Rightarrow adf = cbf = bcf = bed$

$\therefore afd = adf = bed$

$\because d \neq 0 \Longrightarrow af = be$

$\Rightarrow (a,b) \; R \; (e,f)$

$\Rightarrow R$ is transitive on A

$\Rightarrow R$ is equivalence relation on A.

Remark: we will write $(a,b) \sim (c,d)$ instead of $(a,b) R (c,d)$ and Read (a,b) equivalence to (c,d)

Example

Let $(5,6), (20,9), (3,4), (15,20) \in A$

Then

$(3,4) \sim (15,20)$ because $(3)\cdot(20) = (4)\cdot(15)$

$\Rightarrow (3,4) = (15,20)$

but $(5,6) \nsim (20,9)$ because $(5)\cdot(9) = 45$
and $(6)(20) = 120$

$\Rightarrow 5 \cdot 9 \neq 6 \cdot 20$

$\Rightarrow (5,6) \neq (20,9)$

Definition

The equevalence class Contains on (a,b) is said to be rational number (define by $[a,b]$)

i.e

$$[a,b] = \{ (c,d) \mid (c,d) \sim (a,b) \}$$

~~The set of all rational numbers~~

The set of all equivalence class is said to be rational numbers. That is

$$\boxed{Q = A/R} = \frac{Z \times Z^*}{R} = \{ [(a,b)] \mid (a,b) \in A \}$$

Example

$[0,1] = \{ (c,d) \mid (c,d) \sim (0,1) \}$

$= \{ (c,d) \mid c \cdot 1 = d \cdot 0 \}$

$= \{ (c,d) \mid c = 0 \}$

$= \{ (0,1), (0,2), (0,3), \dots \}$

$= (0,-1), (0,-2), (0,-3), \dots \}$

Definitions 1. Let $x \in Q$, x is said to be negative rational number if $-x \in Q^+$ (i.e $x > 0$)

i.e if $x = (a, b) \Rightarrow -x = (-a\ b)$

2. if $x \neq 0 \Rightarrow x^{-1} = (b, a)$

Example

$(-3,5)$ is negative rational number because

$-(-3,5) = (3,5) \in Q^+$

Ordered on Q

$Q^+ = \{ x \in Q \mid ab > 0 \ ; (a,b) \in X \}$

Theorem Let $x \in Q$ and $(a,b) \sim (c,d)$. Then if $ab > 0$, then $cd > 0$.

proof. $\because (a,b) \sim (c,d) \Longrightarrow ad = cb$

ضرب الطرفين ب (cb) $\Rightarrow (cb)(ad) = (cb)(cb)$

$\Rightarrow (a\ b)(c\ d) = (cb)(cb) > 0$

$\therefore (ab)(cd) > 0$. But $ab > 0 \Rightarrow cd > 0$

Definition

Let $x, y \in \mathbb{Q}$, then x less than y $(x<y)$ iff $y-x \in \mathbb{Q}^+$

Theorem

The system $(\mathbb{Q}, \leq)$ is totally ordered set.

Proof

① reflaxive

$x \leq x \quad \forall x \in \mathbb{Q} \Rightarrow$ "$\leq$" is reflexive

② Unti-symmtric

if $x \leq y \wedge y \leq x$, then

$x \leq y \Rightarrow x < y$ or $x = y$

$y \leq x \Rightarrow y < x$ or $x = y$.

Now, ① if $x < y \wedge y < x \Rightarrow y-x \in \mathbb{Q}^+ \wedge x-y \in \mathbb{Q}^+$

$\Longrightarrow y-x \in \mathbb{Q}^+ \wedge \underbrace{-(y-x)}_{x-y} \in \mathbb{Q}^+ \quad C!$

$\rightarrow x = y$

② if $x < y \wedge y = x \Rightarrow$ ~~$x<y$~~ $x = y$

③ if $x = y \wedge y < x \Rightarrow x = y$

④ if $x = y \wedge y = x \Rightarrow x = y$

$\therefore$ "$\leq$" is unti-symmtric

③ <u>transitive</u>

Let $x \leq y \wedge y \leq k$, then

① if $x < y \wedge y < k \Rightarrow y - x \in Q^{+} \wedge k - y \in Q^{+}$

$\Rightarrow (y-x) + (k-y) \in Q^{+}$

$\Rightarrow k - x \in Q^{+} \Rightarrow x < k \Rightarrow x \leq k$

② if $x < y \wedge y = k \Rightarrow x < k$

③ if $x = y \wedge y < k \Rightarrow x = y \wedge k - y \in Q^{+}$

$\Rightarrow k - x \in Q^{+}$

$\Rightarrow x < k \Rightarrow x \leq k$

④ if $x = y \wedge y = k \Rightarrow x = y \Rightarrow x \leq y$

$\Rightarrow$ transitive

$\therefore$ "$\leq$" is partially operation on Q

<u>Now</u>

let $x, y \in Q$, then $(y - x \in Q^{+}) \vee (y - x = 0) \vee (-(y-x) \in Q^{+})$

$\Rightarrow (x \leq y) \vee (y = x) \vee (y < x)$

$\Rightarrow x \leq y \quad \vee \quad y \leq x$

$\Rightarrow$ for each $x, y \in Q$, is Camparable.

$\therefore$ "$\leq$" is totally ordered on Q.

Arithmetic of the Rational Numbers

لأجل تعريف عمليتي الجمع والضرب في الأعداد النسبية نحتاج إلى

Lemma Let $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then

① $(ad+cb,\ bd) \sim (a'd'+c'b',\ b'd')$

② $(ac,\ bd) \sim (a'c',\ b'd')$

Proof ①

$\because (a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$

then ① $ab' = a'b$ and ② $cd' = dc'$

$\therefore$

$$\begin{aligned}(ad+cb)(b'd') &= adb'd' + cbb'd' \\ &= ab'dd' + cd'bb' \\ &= a'bdd' + dc'bb' \\ &= a'd'bd + c'b'bd \\ &= (a'd' + c'b')bd\end{aligned}$$

$\therefore (ad+cb,\ bd) \sim (a'd'+c'b',\ b'd')$

Proof ②

$$\begin{aligned}(ac)(b'd') &= (ab')(cd') \\ &= (a'b)(c'd) \\ &= (a'c')(bd)\end{aligned}$$

$\because b'd' \neq 0$ and $bd \neq 0$

$\therefore (ac,\ bd) \sim (a'c',\ b'd')$

Theorem

There exist f, g functions defined on Q such that: for all $(a,b) \in X$, $(c,d) \in Y$, then

① $f(x,y) = (ad + cb, bd)$

② $g(x,y) = (ac, db)$

Definition

Let $x, y \in Q$; $x = (a,b)$, $y = (c,d)$, then

① the function f which defined on Q by

$$f(x,y) = [ad + cb, bd]$$

is said to be $\boxed{\text{addition in } Q}$ and write تسمى عملية الجمع على Q وتكتب بالشكل

$$f(x,y) = x \underset{Q}{+} y \quad \forall\, x, y \in Q$$

② The function g is said to be Multiplication on Q such that

$$g(x,y) = (ac, bd)$$ عملية الضرب على Q

and write وتكتب

$$g(x,y) = x \underset{Q}{\cdot} y \quad \forall\, x, y \in Q$$

Example

1. Let $x = [1, 2]$, $y = [3, 5] \in Q$

then $x + y = [1 \cdot 5 + 3 \cdot 2, 2 \cdot 5]$
$= [11, 10]$

$xy = [1 \cdot 3, 2 \cdot 5]$
$= [3, 10] \in Q$

Definition Let $x, y \in Q$ where $(a,b) \in x$ and $(c,d) \in y$, then the product (حاصل طرح)
subtraction

$$x - y = x + (-y) = [a,b] + [-c,d]$$
$$= [ad - cb, bd]$$

Remarks 1. $0 = [0,1]$
2. $1 = [1,1]$

Example
Let $x = [4,9]$, $y = [6,13]$

$$x - y = [4,9] - [6,13]$$
$$= [4,9] + [-6,13]$$
$$= [4 \cdot 13 - 9 \cdot (6), 9 \cdot 13]$$
$$= [52 - 54, 117]$$
$$= [-2, 117]$$

Definition Let $x \in Q$, $y \in Q - \{0\}$, then

① $\frac{x}{y} = x \cdot y^{-1}$

② $x^{-1} = \frac{1}{x}$

Example 1. Let $x = [3,4]$, $y = [1,1]$, then

① $\frac{1}{x} = [1,1] \cdot x^{-1} = [1,1] \cdot [\ ,\ ] = [4,3]$

② $x \cdot x^{-1} = [3,4] \cdot [4,3] = [12,12] = [1,1]$

③ $X = [2, 5]$, $Y = [-3, 11]$, then

$$\frac{Y}{X} = Y \cdot X^{-1} = [-3, 11] , [5, 2]$$
$$= [-15, 22]$$

<u>Remark</u> The set $V = \{x \in Q \mid x \leq 1\}$ has no minimal elements.

<u>Theorem</u> $(Q, \leq)$ is not well-ordered.

<u>Proof</u>. $(Q, \leq)$ is poset but $\exists$ the set
$V = \{x \in Q \mid x \leq 1\}$ has no minimal element.
$\therefore (Q, \leq)$ is not well-ordered.

« Properties of Q »

<u>Theorem</u>. Q is Countable

<u>proof</u>. سندرج الاعداد النسبية الموجبة في عدد منته من المتتاليات (دون تكرار)

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \cdots \\
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\
\frac{1}{4} & \frac{3}{4} & \frac{5}{4} & \frac{7}{4} & \cdots \\
\vdots & & & &
\end{array}$$

The set of all positive rational number is

$$\{1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, 3, \frac{1}{4}, \frac{2}{3}, \frac{5}{2}, 4, \frac{1}{3}, \dots\}$$

$\therefore$ each rationl numbe in $\{0, 1, -1, \frac{1}{2}, \frac{-1}{2}, 2, -2, \frac{1}{3}, \frac{-1}{3}, \dots\}$

$\therefore \exists$ bijective $f: \mathbb{Q} \to N$ such that:

$$0 \longleftrightarrow 0$$
$$1 \longleftrightarrow 1$$
$$-1 \longrightarrow 2$$
$$\frac{1}{2} \longrightarrow 3$$
$$-\frac{1}{2} \longrightarrow 4$$
$$\vdots$$

$\therefore \mathbb{Q}$ is Countable set

Dense ordered الترتيب الكثيف

Definition Let "$\leq$" is partially ordered relation on the set A. Then "$\leq$" is said to be dense iff

$$\boxed{(a, b \in A) \wedge (a < b) \longrightarrow (\exists\, c \in A \text{ such that } a < c < b)}$$

Remark

the relation "$\leq$" on $\mathbb{Q}$ is dense. that is

$\forall\ p, q \in \mathbb{Q}, \exists\ r \in \mathbb{Q}$ s.t $p < r < q$

such that $p < q$

Theorem. $\nexists\ x \in \mathbb{Q}$ such that $x^2 = 2$

Proof.

Suppose that $x \in \mathbb{Q}$ such that $x^2 = 2$.

$\Rightarrow x = \frac{a}{b}$; $a, b \in \mathbb{Z}$ and $b \neq 0$.

$\Rightarrow a^2 = 2b^2$ (where $x^2 = \frac{a^2}{b^2} = 2$)

$\Rightarrow a^2$ is even $\Rightarrow a$ is even $\Rightarrow a = 2k$; $k \in \mathbb{Z}^+$

$\Rightarrow k < a$

$\Rightarrow a = 2k \Rightarrow a^2 = 4k^2 = 2b^2$

$\Rightarrow b^2 = 2k^2 \Rightarrow b^2$ even $\Rightarrow b$ is an even

$\Rightarrow b = 2t$; $t \in \mathbb{Z}$

$\Rightarrow \underset{x}{\|} \frac{a}{b} = \frac{2k}{2t} = \frac{k}{t}$

$\Rightarrow \exists\ k < a$ such that $x = \frac{k}{t}$ C!

$\Rightarrow \nexists\ x \in \mathbb{Q}$ s.t $\boxed{x^2 = 2}$