Linear Algebra

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Chapter One
Matrices

Definition:

An array of **mn** numbers arranged in **m** rows and **n** columns is said to be **m×n** matrix.

$$\begin{array}{cc}
 & \text{j-th column} \\
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix} & \text{i-th row}
\end{array}$$

- **m×n** is the order (or size or dimension or degree) of the matrix
- The number which appears at the intersection of the i-th row and j-th column is usually referred as the (i,j)-th entry of the matrix A and denoted by a_{ij} and the matrix A denoted by $[a_{ij}]_{m\times n}$ or $A_{m\times n}$.
- The entries a_{ij} of a matrix A may be real or complex (or any field).
- If all entries of the matrix are real, then the matrix is called real matrix.
- If all entries of the matrix are complex, then the matrix is called complex matrix.
- If $A=[a_{ij}]_{m\times n}$ then $-A=[-a_{ij}]_{m\times n}$.

Examples:

(1) $A=\begin{bmatrix} 2 & -3 & 4 \\ 7 & 0 & -5 \end{bmatrix}$ is a real matrix, A has 2-rows and 3-columns.

$a_{11}=2$, $a_{12}=-3$, $a_{13}=4$, $a_{21}=7$, $a_{22}=0$, $a_{23}=-5$

(2) $B=\begin{bmatrix} 0 & i & \frac{1}{2} \\ 2+i & 4 & -3 \\ -1 & -i & 2+3i \end{bmatrix}$ is a complex matrix, B has 3-rows and 3-columns.

Types of Matrices

(1) Equal Matrices: The matrix A = $[a_{ij}]_{m\times n}$ equal to the matrix B = $[b_{ij}]_{m\times n}$ if they have the same size and the corresponding elements of A and B are equal $a_{ij} = b_{ij}$, $1 \le i \le m$ and $1 \le j \le n$.

Examples: (1) The matrices $\begin{bmatrix} 1 & -2 & 3 \\ 0 & \frac{3}{2} & 0.25 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1.5 & \frac{1}{4} \end{bmatrix}$ are equal matrices.

(2) The matrices $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 5 \\ 5 & 4 & 2 \end{bmatrix}$ are not equal matrices.

(2) Square matrix: A matrix having n rows and n columns and we say that it is of order n.

Examples: (1) $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}_{2\times 2}$ square matrix of order 2.

(2) $\begin{bmatrix} 5 & -6 & 9 \\ -3 & 2 & 1 \\ 0 & 8 & 0 \end{bmatrix}_{3\times 3}$ square matrix of order 3.

Note: The main diagonal in the square matrix contains the elements a_{ii} where $i = 1,2,\ldots,n$ (beginning from top left upper to bottom right).

(3) Zero matrix: the matrix all elements are zero denoted by $O_{m\times n}$.

Examples: (1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$ square zero matrix of order 2.

(2) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{2\times 4}$ zero matrix of order 2×4.

(4) Identity Matrix: A square matrix all elements of the main diagonal are equal 1 and other elements are equal to 0 denoted by I_n.

Other Definition: A matrix A=$[a_{ij}]_{n\times n}$ is said to be identity matrix if

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}.$$

Examples: **(1)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ square identity matrix of order 2×2 (I_2).

(2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ square identity matrix of order 4×4 (I_4).

(5) Transpose of Matrix: The matrix resulting from replacing the rows of the matrix by the column of it and denoted by A' or A^t or A^T.

Examples: **(1)** If $A = \begin{bmatrix} 2 & 0 & 3 & -1 \\ 4 & 1 & 5 & 7 \end{bmatrix}_{2\times4}$ then $A^t = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}_{4\times2}$.

(2) If $B = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 3 & 0 & -2 \\ 1 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}_{5\times3}$ then $B^t = \begin{bmatrix} 2 & -1 & 3 & 1 & 3 \\ 0 & 1 & 0 & 1 & 2 \\ -1 & 0 & -2 & -1 & 0 \end{bmatrix}_{3\times5}$.

(6) Symmetric Matrix: A square matrix A is called symmetric matrix if it is equal to its transpose (i.e. $A = A^t$).

Examples: **(1)** $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $A^t = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ so A is symmetric matrix since $A = A^t$.

(2) $B = \begin{bmatrix} 6 & 3 & -2 \\ 3 & 0 & 5 \\ 2 & 5 & -4 \end{bmatrix}$ and $B^t = \begin{bmatrix} 6 & 3 & 2 \\ 3 & 0 & 5 \\ -2 & 5 & -4 \end{bmatrix}$ so B is not symmetric matrix since $B \neq B^t$.

(7) Skew Symmetric Matrix: A square matrix A is called skew symmetric matrix if it is equal to the negative of its transpose (i.e. $A = -A^t$).

Examples: (1) $A=\begin{bmatrix}0 & 2 & -1\\ -2 & 0 & 3\\ 1 & -3 & 0\end{bmatrix}$, $A^t=\begin{bmatrix}0 & -2 & 1\\ 2 & 0 & -3\\ -1 & 3 & 0\end{bmatrix}$ and $-A^t=\begin{bmatrix}0 & 2 & -1\\ -2 & 0 & 3\\ 1 & -3 & 0\end{bmatrix}$ so A is skew symmetric matrix since $A = -A^t$.

(2) $B=\begin{bmatrix}0 & -8\\ 8 & 1\end{bmatrix}$, $B^t=\begin{bmatrix}0 & 8\\ -8 & 1\end{bmatrix}$ and $-B^t=\begin{bmatrix}0 & -8\\ 8 & -1\end{bmatrix}$ so B is not skew symmetric matrix since $B \neq -B^t$.

Note: All the elements of the main diagonal in the skew symmetric matrix are zero.

(8) Conjugate Matrix: A matrix A is called conjugate matrix to the matrix B if the elements of the matrix A are the complex conjugated numbers with the elements of the matrix B and denoted by $\overline{A}$.

Examples: (1) The matrix $C=\begin{bmatrix}2i & 4\\ 1 & 3\end{bmatrix}$ is conjugate matrix to the matrix $D=\begin{bmatrix}-2i & 4\\ 1 & 3\end{bmatrix}$.

(2) The conjugate matrix to the matrix $A=\begin{bmatrix}2-i & 1+i & i\\ 2-3i & 4+i & 2i\\ 1-2i & -i & -2i\end{bmatrix}$ is the matrix $\overline{A}=\begin{bmatrix}2+i & 1-i & -i\\ 2+3i & 4-i & -2i\\ 1+2i & i & 2i\end{bmatrix}$

Note: The conjugate transpose of the matrix is equal to the transpose conjugate matrix.

i.e. $(\overline{A})^t = (\overline{A^t}) = A^*$.

Example: $A=\begin{bmatrix}1+i & 3\\ -3 & -i\end{bmatrix} \Rightarrow A^t=\begin{bmatrix}1+i & -3\\ 3 & -i\end{bmatrix} \Rightarrow \left(\overline{A^t}\right)=\begin{bmatrix}1-i & -3\\ 3 & i\end{bmatrix}$

$$\overline{A}=\begin{bmatrix}1-i & 3\\ -3 & i\end{bmatrix} \Rightarrow \left(\overline{A}\right)^t=\begin{bmatrix}1-i & -3\\ 3 & i\end{bmatrix}=\left(\overline{A^t}\right)$$

(9) Hermitian Matrix: A square matrix A is called hermitian matrix if it is equal to its conjugate transpose (i.e. $A = A^*$).

Examples: (1) $A=\begin{bmatrix}4 & 2i\\ -2i & 1\end{bmatrix}$, $A^t=\begin{bmatrix}4 & -2i\\ 2i & 1\end{bmatrix}$ and $A^*=\overline{A^t}=\begin{bmatrix}4 & 2i\\ -2i & 1\end{bmatrix}$ so A is hermitian matrix since $A=A^*$.

(2) $B=\begin{bmatrix}2 & 1-2i & 2\\ 1-2i & 1 & -3i\\ 2 & 3i & 3\end{bmatrix}$, $B^t=\begin{bmatrix}2 & 1-2i & 2\\ 1-2i & 1 & 3i\\ 2 & -3i & 3\end{bmatrix}$ and

$B^*=\overline{B^t}=\begin{bmatrix}2 & 1+2i & 2\\ 1+2i & 1 & -3i\\ 2 & 3i & 3\end{bmatrix}$ so B is not hermitian matrix since $B\neq B^*$.

(10) Skew Hermitian Matrix: A square matrix A is called skew hermitian matrix if it is equal to its negative conjugate transpose (i.e. $A=-A^*$).

Note: All the elements of the main diagonal in the skew hermitian matrix are zero or pure imaginary numbers (complex number the real part of it must be equal to zero).

Examples :(1) $A=\begin{bmatrix}0 & -3i & 2\\ 3i & 0 & -i\\ 2 & i & 0\end{bmatrix}$, $\overline{A}=\begin{bmatrix}0 & 3i & 2\\ -3i & 0 & i\\ 2 & -i & 0\end{bmatrix}$ and $(\overline{A})^t=\begin{bmatrix}0 & -3i & 2\\ 3i & 0 & -i\\ 2 & i & 0\end{bmatrix}=A$

so A is not skew hermitian matrix since $A\neq -A^*$.

(2) $B=\begin{bmatrix}2i & 0\\ 0 & -3i\end{bmatrix}$, $\overline{B}=\begin{bmatrix}-2i & 0\\ 0 & 3i\end{bmatrix}$, $(\overline{B})^t=\begin{bmatrix}-2i & 0\\ 0 & 3i\end{bmatrix}$ and

$-(\overline{B})^t=\begin{bmatrix}2i & 0\\ 0 & -3i\end{bmatrix}=B$ so B is skew hermitian matrix since $B=-B^*$.

(11) Triangular Matrix: It is two kinds:

(*a*) Lower Triangular Matrix: A square matrix all its elements above the main diagonal are zeros, that is $a_{ij}=0$ for each i < j.

Examples: **(1)** $\begin{bmatrix}4 & 0 & 0\\ 2 & -1 & 0\\ 3 & 4 & 5\end{bmatrix}$ square lower triangular matrix of order 3×3.

(2) $B=\begin{bmatrix}5 & 0\\ 4 & -7\end{bmatrix}$ square lower triangular matrix of order 2×2.

(b) Upper Triangular Matrix: A square matrix all its elements below the main diagonal are zeros, that is $a_{ij} = 0$ for each i > j.

Examples: **(1)** $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{bmatrix}$ square upper triangular matrix of order 3×3.

(2) $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ square upper triangular matrix of order 2×2.

Note: the identity matrix is upper and lower triangular matrix.

(12) Diagonal Matrix: a square matrix A is called diagonal matrix if all elements are zero except the elements in the main diagonal (i.e. $a_{ij} = 0$ if $i \neq j$).

Other Definition: A square matrix which is upper and lower triangular matrix.

Examples: **(1)** $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ square matrix of order 3×3, diagonal matrix, lower and upper triangular matrix, identity matrix.

(2) $B = \begin{bmatrix} 2 & 0 \\ 0 & -i \end{bmatrix}$ square matrix of order 2×2, diagonal matrix, lower and upper triangular matrix.

(13) Scalar Matrix: A diagonal matrix is called scalar matrix if all main diagonal elements are equal.

Other Definition: A diagonal matrix $A = [a_{ij}]_{n\times n}$ is called scalar matrix if $a_{11} = a_{22} = \ldots = a_{nn} = k$.

Examples: **(1)** $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ **(2)** $\begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$

(14) Row Matrix: An $1\times n$ matrix has one row $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$.

Examples: **(1)** $A = [\,3 \;\; 2 \;\; 1 \;\; 4\,]_{1\times 4}$ **(2)** $A = [\,7 \;\; -5 \;\; 2 \;\; 3 \;\; 1\,]_{1\times 5}$

(15) Column Matrix: An $m \times 1$ matrix has one column $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$.

Examples: **(1)** $B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}_{3\times 1}$ **(2)** $B = \begin{bmatrix} -3 \\ 7 \\ 9 \\ 0 \end{bmatrix}_{4\times 1}$

Exercises: Classify the following matrices according to their types

(1) $\begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & -2 \\ 4 & -2 & 1 \end{bmatrix}$ **(7)** $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ **(13)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 7 & 2 & 1 & 0 \end{bmatrix}$

(2) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ **(8)** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **(14)** $\begin{bmatrix} 0 & -2 & -3 & -4 \\ 2 & 0 & -5 & -6 \\ 3 & 5 & 0 & -7 \\ 4 & 6 & 7 & 0 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & -4 \\ 0 & 4 & 5 \end{bmatrix}$ **(9)** $\begin{bmatrix} -3 & 1 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ **(15)** $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

(4) $\begin{bmatrix} 0 & 1-2i & 5i \\ -1-2i & 0 & 3 \\ 5i & 3 & i \end{bmatrix}$ **(10)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **(16)** $\begin{bmatrix} -1 \\ 6 \\ 8 \\ 0 \end{bmatrix}$

(5) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 0 & 1 \end{bmatrix}$ **(11)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(6) $\begin{bmatrix} i & i \\ i & i \end{bmatrix}$ **(12)** $\begin{bmatrix} 2 & 3 & -4 & 1 \end{bmatrix}$

Operations on Matrices

Addition of Matrices: Matrices are said to be compatible with addition if and only if they have the same degree.

Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ be two matrices. The addition of A and B is denoted by A + B is also an m × n matrix, and

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$$

$$= [a_{ij} + b_{ij}]_{m\times n}$$

$$= [c_{ij}]_{m\times n} \quad \text{where} \quad a_{ij} + b_{ij} = c_{ij} \quad \text{for all possible values of i, j}$$

Examples:

(1) $$\begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix}_{2\times 2} + \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 1+2 & 5+0 \\ 4+(-1) & 6+0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 3 & 6 \end{bmatrix}_{2\times 2}$$

(2) $$A = \begin{bmatrix} 1 & 5 \\ 4 & 6 \end{bmatrix}_{2\times 2}, \; B = \begin{bmatrix} 1 & 5 \\ 4 & 6 \\ 2 & -1 \end{bmatrix}_{3\times 2}$$

A + B not define since A and B have different size. Thus A and B cannot be added.

(3) $$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 & 3 \\ 2 & -3 & 5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & -2 \end{bmatrix}_{2\times 3} + \begin{bmatrix} -1 & 1 & 3 \\ 2 & -3 & 5 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} 2+(-1) & 0+1 & (-1)+3 \\ 3+2 & 5+(-3) & (-2)+5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

$$B + A = \begin{bmatrix} -1 & 1 & 3 \\ 2 & -3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 0 & -1 \\ 3 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)+2 & 1+0 & 3+(-1) \\ 2+3 & (-3)+5 & 5+(-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}$$

We note that A + B = B + A. So Addition of matrices is commutative.

Properties of the Addition of Matrices

Theorem: Let $M_{m\times n}(F)$ be the set of all m×n matrices over F, where $F = \mathbb{R}$ or $\mathbb{C}$. Then:

(1) $A + B = B + A$ (The addition of matrices is commutative).

(2) $(A + B) + C = A + (B + C)$ (The addition of matrices is associative).

(3) $A + O = O + A = A$.

Proof (1): Let $A, B \in M_{m\times n}(F)$ such that $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, $a_{ij}, b_{ij} \in F$

$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [b_{ij} + a_{ij}]_{m\times n}$ (the addition of numbers is commutative)

$= [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$ (definition the addition of matrices)

$= B + A$

Proof (2): Let $A, B, C\in M_{m\times n}(F)$ such that $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$, $a_{ij}, b_{ij}, c_{ij} \in F$

$(A + B) + C = ([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}) + [c_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [d_{ij}]_{m\times n} + [c_{ij}]_{m\times n}$ where $d_{ij} = a_{ij} + b_{ij}$

$= [d_{ij} + c_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}$ $d_{ij} = a_{ij} + b_{ij}$

$= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n}$ (the addition of numbers is associative)

$= [a_{ij}]_{m\times n} + [b_{ij} + c_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [a_{ij}]_{m\times n} + ([b_{ij}]_{m\times n} + [c_{ij}]_{m\times n})$ (definition the addition of matrices)

$= A + (B+ C)$

Proof (3): Let $A, O\in M_{m\times n}(F)$ such that $A=[a_{ij}]_{m\times n}$, $O= [b_{ij}]_{m\times n}$, $a_{ij}, \in F$, $b_{ij} = 0\ \forall$ i, j

$A + O = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [a_{ij} + 0]_{m\times n}$ $b_{ij} = 0\ \forall$ i, j

$= [a_{ij}]_{m\times n}$

$= A$

$O + A = [b_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$

$= [b_{ij} + a_{ij}]_{m\times n}$ (definition the addition of matrices)

$= [0 + a_{ij}]_{m\times n}$ $b_{ij} = 0\ \forall$ i, j

$= [a_{ij}]_{m\times n}$

$= A$

Multiplication of Matrix by Scalar:

If $A = [a_{ij}]_{m\times n}$ matrix, c is a scalar then the scalar multiple c A is the m×n matrix obtained from A by multiplying each entry of A (which is a scalar too) by c. Thus $cA = [c\, a_{ij}]_{m\times n}$.

Examples:

(1) $2\begin{bmatrix} -1 & 0 & i \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2i \\ 6 & 4 & 0 \end{bmatrix}$

(2) $\frac{1}{4}\begin{bmatrix} 4 & -8 & 0 \\ 2 & 12 & 4 \\ 0 & 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ \frac{1}{2} & 3 & 1 \\ 0 & \frac{3}{4} & 2 \end{bmatrix}$

(3) $i\begin{bmatrix} -1 & 0 & i \\ 3 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 & -1 \\ 3i & 2i & 0 \end{bmatrix}$

Remark: If $A = [a_{ij}]_{m\times n}$, then $(-1)\,A = [(-1)\,a_{ij}]_{m\times n} = [-\,a_{ij}]_{m\times n}$.
The matrix $(-1)\,A$ is denoted by $-A$ and it is called the negative of A.
Also, $A + (-A) = O_{m\times n}$, i.e. $-A$ is the additive inverse of A.

Proposition: Given $A \in M_{m\times n}(F)$, there exists $B \in M_{m\times n}(F)$ such that
$A + B = O_{m\times n} = B + A$
In fact A determines B uniquely and $B = -A$.

Proof: Let $A = [a_{ij}]_{m\times n}$, since $B = -A$, so $B = [-\,a_{ij}]_{m\times n}$

$A + B = [a_{ij}]_{m\times n} + [-\,a_{ij}]_{m\times n}$
$= [a_{ij} + (-\,a_{ij})]_{m\times n}$ (definition the addition of matrices)
$= [a_{ij} - a_{ij}]_{m\times n}$
$= O_{m\times n}$

$B + A = [-\,a_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$
$= [(-\,a_{ij}) + a_{ij}]_{m\times n}$ (definition the addition of matrices)
$= [-\,a_{ij} + a_{ij}]_{m\times n}$
$= O_{m\times n}$

Theorem: Let A, B and C are three matrices of the same degree, then

(1) $A + B = A + C \Leftrightarrow B = C$

(2) $B + A = C + A \Leftrightarrow B = C$

Proof (1):

$A + B = A + C \Leftrightarrow -A + (A + B) = -A + (A + C)$ (add (–A) to each side from left)

$\Leftrightarrow (-A + A) + B = (-A + A) + C$ (the addition of matrices is associative)

$\Leftrightarrow O + B = O + C$

$\Leftrightarrow B = C$

Proof (2) (Home Work)

Subtraction of Matrices: Matrices are said to be compatible with subtraction if and only if they have the same degree.

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices. The subtraction of A and B is denoted by A – B is also an m × n matrix, and

$$\begin{aligned} A - B &= A + (-B) \\ &= [a_{ij}]_{m\times n} + [-b_{ij}]_{m\times n} \\ &= [a_{ij} - b_{ij}]_{m\times n} \\ &= [c_{ij}]_{m\times n} \quad \text{where} \quad a_{ij} - b_{ij} = c_{ij} \text{ for all possible values of i, j} \end{aligned}$$

Examples:

(1) $\begin{bmatrix} 8 & 6 & -4 \\ 1 & 10 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 8-2 & 6-1 & -4-(-1) \\ 1-3 & 10-0 & -1-(-2) \end{bmatrix} = \begin{bmatrix} 6 & 5 & -3 \\ -2 & 10 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 2 & -1 \\ 0 & 3 \end{bmatrix}$ not define since the matrices have different size.

(3) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & \frac{1}{3} & -1 \\ \frac{1}{2} & 0 & -7 \end{bmatrix}$, find A + 2 B , A – 2B

$$2B = 2\begin{bmatrix} 2 & \frac{1}{3} & -1 \\ \frac{1}{2} & 0 & -7 \end{bmatrix} = \begin{bmatrix} 4 & \frac{2}{3} & -2 \\ 1 & 0 & -14 \end{bmatrix}$$

$$A+2B=\begin{bmatrix}1 & 0 & 1\\ 2 & -1 & 3\end{bmatrix}+\begin{bmatrix}4 & \frac{2}{3} & -2\\ 1 & 0 & -14\end{bmatrix}=\begin{bmatrix}5 & \frac{2}{3} & -1\\ 3 & -1 & -11\end{bmatrix}$$

$$A-2B=\begin{bmatrix}1 & 0 & 1\\ 2 & -1 & 3\end{bmatrix}-\begin{bmatrix}4 & \frac{2}{3} & -2\\ 1 & 0 & -14\end{bmatrix}=\begin{bmatrix}-3 & -\frac{2}{3} & 3\\ 1 & -1 & 17\end{bmatrix}$$

Remarks:

(1) The subtraction of matrices is not commutative.

Example: If $A=\begin{bmatrix}3 & -1\\ 2 & 4\end{bmatrix}$ and $B=\begin{bmatrix}0 & 4\\ -5 & 2\end{bmatrix}$

$$A-B=\begin{bmatrix}3 & -1\\ 2 & 4\end{bmatrix}-\begin{bmatrix}0 & 4\\ -5 & 2\end{bmatrix}=\begin{bmatrix}3 & -5\\ 7 & 2\end{bmatrix}$$

$$B-A=\begin{bmatrix}0 & 4\\ -5 & 2\end{bmatrix}-\begin{bmatrix}3 & -1\\ 2 & 4\end{bmatrix}=\begin{bmatrix}-3 & 5\\ -7 & -2\end{bmatrix}$$

So we get that $A-B\neq B-A$

(2) The subtraction of matrices is not associative.

Example: If $A=\begin{bmatrix}1 & 2\\ -3 & 0\end{bmatrix}$, $B=\begin{bmatrix}4 & -1\\ 2 & 5\end{bmatrix}$ and $C=\begin{bmatrix}1 & 2\\ 1 & -3\end{bmatrix}$, then

$(A-B)-C\neq A-(B-C)$ apply that. **(Home work)**

Theorem: For any two matrices of the same degree $B-A=-(A-B)$.

Proof: Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$

$-(A-B)=-([a_{ij}]_{m\times n}-[b_{ij}]_{m\times n})$

$=-[a_{ij}-b_{ij}]_{m\times n}$ (definition the subtraction of matrices)

$=-[c_{ij}]_{m\times n}$ where $c_{ij}=a_{ij}-b_{ij}$

$=[-c_{ij}]_{m\times n}$

$=[-(a_{ij}-b_{ij})]_{m\times n}$ (replaced)

$=[-a_{ij}+b_{ij}]_{m\times n}$

$=[b_{ij}-a_{ij}]_{m\times n}$ (the addition of numbers is commutative)

$=[b_{ij}]_{m\times n}-[a_{ij}]_{m\times n}$ (definition the subtraction of matrices)

$=B-A$

Theorem: Let $M_{m\times n}(F)$ be the set of all m×n matrices over F, where $F = \mathbb{R}$ or $\mathbb{C}$. Then for any scalars r, s and any $A, B \in M_{m\times n}(F)$

(1) $r(A + B) = rA + rB$

(2) $(r + s)A = rA + sA$

(3) $r(sA) = (rs)A = s(rA)$

(4) $1A = A$

(5) $0A = O$

(6) $rA = Ar$

Proof (1): Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $r \in F$

$r(A + B) = r([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n})$

$= r[a_{ij} + b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= r[c_{ij}]_{m\times n}$ where $c_{ij} = a_{ij} + b_{ij}$

$= [r\,c_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r(a_{ij} + b_{ij})]_{m\times n}$ (replaced)

$= [r\,a_{ij} + r\,b_{ij}]_{m\times n}$ (distribution of multiplication over the addition in numbers)

$= [r\,a_{ij}]_{m\times n} + [r\,b_{ij}]_{m\times n}$ (definition the addition of matrices)

$= r[a_{ij}]_{m\times n} + r[b_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= rA + rB$

Proof (2): Let $A = [a_{ij}]_{m\times n}$ and $r, s \in F$

$(r + s)A = (r + s)[a_{ij}]_{m\times n}$

$= [(r + s)a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r\,a_{ij} + s\,a_{ij}]_{m\times n}$ (distribution of multiplication over the addition in numbers)

$= [r\,a_{ij}] + [s\,a_{ij}]_{m\times n}$ (definition the addition of matrices)

$= r[a_{ij}] + s[a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= rA + sA$

Proof (3): Let $A = [a_{ij}]_{m\times n}$ and $r, s \in F$

$r(sA) = r(s[a_{ij}])_{m\times n}$

$= r[s\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r(s\,a_{ij})]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [(rs)\,a_{ij}]_{m\times n}$ (the multiplication of numbers is associative)

$= (rs)[a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= (rs)A$

$r\,(sA) = r\,(s[a_{ij}])_{m\times n}$

$= r\,[sa_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [r\,(sa_{ij})]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [(rs)a_{ij})]_{m\times n}$ (the multiplication of numbers is associative)

$= [(sr)\,a_{ij})]_{m\times n}$ (the multiplication of numbers is commutative)

$= [s\,(r\,a_{ij})]_{m\times n}$ (the multiplication of numbers is associative)

$= s\,[r\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= s\,(r\,[a_{ij}]_{m\times n})$ (definition the multiplication of matrix by scalar)

$= s\,(rA)$

Proof (4): Let $A = [a_{ij}]_{m\times n}$

$1\,A = 1\,[a_{ij}]_{m\times n} = [1\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [a_{ij}]_{m\times n}$

$= A$

Proof (5): Let $A = [a_{ij}]_{m\times n}$

$0\,A = 0\,[a_{ij}]_{m\times n} = [0\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= O_{m\times n}$

Proof (6): Let $A = [a_{ij}]_{m\times n}$, $r \in F$

$r\,A = r\,[a_{ij}]_{m\times n} = [r\,a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$= [a_{ij}\,r]_{m\times n}$ (the multiplication of numbers is commutative)

$= A\,r$

Multiplication of Matrices:

Two matrices are said to be compatible with multiplication if the number of the columns of the first matrix is equal to the number of the rows of the second matrix, i.e.

If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ then $AB = C = [a_{ij}]_{m\times n}\,[b_{ij}]_{n\times p} = \left[\sum_{k=1}^{n} a_{ik}\,b_{kj}\right]_{m\times p}$

Examples: **(1)** Let $A=\begin{bmatrix}1 & -2 & -1\\ 3 & 0 & -3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ -1 & 1\\ 0 & -2\end{bmatrix}$, find AB and BA?

Solution:

$$AB=\begin{bmatrix}1 & -2 & -1\\ 3 & 0 & -3\end{bmatrix}_{2\times 3}\begin{bmatrix}2 & 0\\ -1 & 1\\ 0 & -2\end{bmatrix}_{3\times 2}$$

$$=\begin{bmatrix}1(2)+(-2)(-1)+(-1)(0) & 1(0)+(-2)(1)+(-1)(-2)\\ 3(2)+0(-1)+(-3)(0) & 3(0)+0(1)+(-3)(-2)\end{bmatrix}=\begin{bmatrix}4 & 0\\ 6 & 6\end{bmatrix}_{2\times 2}$$

$$BA=\begin{bmatrix}2 & 0\\ -1 & 1\\ 0 & -2\end{bmatrix}_{3\times 2}\begin{bmatrix}1 & -2 & -1\\ 3 & 0 & -3\end{bmatrix}_{2\times 3}$$

$$=\begin{bmatrix}(2)(1)+0(3) & 2(-2)+0(0) & 2(-1)+0(-3)\\ (-1)(1)+1(3) & (-1)(-2)+1(0) & (-1)(-1)+1(-3)\\ 0(1)+(-2)(3) & 0(-2)+(-2)(0) & 0(-1)+(-2)(-3)\end{bmatrix}=\begin{bmatrix}2 & -4 & -2\\ 2 & 2 & -2\\ -6 & 0 & 6\end{bmatrix}_{3\times 3}$$

Note that $AB\neq BA$.

(2) $\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}_{2\times 2}\begin{bmatrix}1 & 0\\ -1 & 3\end{bmatrix}_{2\times 2}=\begin{bmatrix}1(1)+1(-1) & 1(0)+1(3)\\ 2(1)+1(-1) & 2(0)+1(3)\end{bmatrix}=\begin{bmatrix}0 & 3\\ 1 & 3\end{bmatrix}_{2\times 2}$

(3) Let $A=\begin{bmatrix}1 & 2\\ -1 & -2\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & 3\\ -1 & 5 & -2\end{bmatrix}$, find AB and BA if exist? **(Home work)**

(4) Let $A=\begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 2\\ 1 & -2 & 1\end{bmatrix}$, find AB and BA if exist?

Solution:

$$AB=\begin{bmatrix}1 & -2\\ 2 & -1\end{bmatrix}_{2\times 2}\begin{bmatrix}2 & -1 & 2\\ 1 & -2 & 1\end{bmatrix}_{2\times 3}=\begin{bmatrix}1(2)+(-2)(1) & 1(-1)+(-2)(-2) & 1(2)+(-2)(1)\\ 2(2)+(-1)(1) & 2(-1)+(-1)(-2) & 2(2)+(-1)(1)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3 & 0\\ 3 & 0 & 3\end{bmatrix}_{2\times 3}$$

BA not exists since the number of B columns is not equal to the number of A rows.

Remarks: (1) Two matrices A and B are said to be commutative if AB = BA.

Example: Is the matrix $A=\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}$ commutative with the matrix $B=\begin{bmatrix}7 & 8\\ 2 & 1\end{bmatrix}$?

Solution:

$$A\,B=\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}_{2\times 2}\begin{bmatrix}7 & 8\\ 2 & 1\end{bmatrix}_{2\times 2}$$

$$=\begin{bmatrix}2(7)+4(2) & 2(8)+4(1)\\ 1(7)+(-1)(2) & 1(8)+(-1)(1)\end{bmatrix}=\begin{bmatrix}22 & 20\\ 5 & 7\end{bmatrix}_{2\times 2}$$

$$B\,A=\begin{bmatrix}7 & 8\\ 2 & 1\end{bmatrix}_{3\times 2}\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}_{2\times 2}$$

$$=\begin{bmatrix}7(2)+8(1) & 7(4)+8(-1)\\ 2(2)+1(1) & 2(4)+1(-1)\end{bmatrix}=\begin{bmatrix}22 & 20\\ 5 & 7\end{bmatrix}_{2\times 2}$$

We get that AB = BA. So A and B are commutative matrices.

(2) The product of two matrices may be equal to zero matrix and each matrix is not a zero matrix.

Example: $AB=\begin{bmatrix}-1 & 1\\ -2 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}-1(2)+1(2) & -1(1)+1(1)\\ -2(2)+2(2) & -2(1)+2(1)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$

(3) The cancellation law not satisfies in matrices multiplication, i.e.

AB = AC $\not\Rightarrow$ B = C

Example: Let $A=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}$ and $C=\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix}$

$$AB=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 3 & 4\end{bmatrix}=\begin{bmatrix}0(1)+1(3) & 0(1)+1(4)\\ 0(1)+2(3) & 0(1)+2(4)\end{bmatrix}=\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix}$$

$$AC=\begin{bmatrix}0 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix}=\begin{bmatrix}0(2)+1(3) & 0(5)+1(4)\\ 0(2)+2(3) & 0(5)+2(4)\end{bmatrix}=\begin{bmatrix}3 & 4\\ 6 & 8\end{bmatrix}$$

AB = AC while B $\neq$ C.

(4) If both matrices A and B are square matrix of the same order with real entries, then it is not necessary that $(AB)^2 = A^2 B^2$.

Example: Let $A=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 0\\ 1 & 1\end{bmatrix}$

$AB = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}-2 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}-2 & 0\\ -2 & -2\end{bmatrix}$

$(AB)^2 = (AB)\,(AB) = \begin{bmatrix}-2 & 0\\ -2 & -2\end{bmatrix}\begin{bmatrix}-2 & 0\\ -2 & -2\end{bmatrix} = \begin{bmatrix}4 & 0\\ 8 & 4\end{bmatrix}$

$A^2 = AA = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$

$B^2 = BB = \begin{bmatrix}-2 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}4 & 0\\ -1 & 1\end{bmatrix}$

$A^2 B^2 = \begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}4 & 0\\ -1 & 1\end{bmatrix} = \begin{bmatrix}4 & 0\\ -4 & 4\end{bmatrix}$

We get that $(AB)^2 \neq A^2 B^2$.

Note that this relation is true when the matrices are commutative, for example consider $A=\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$ and $B=\begin{bmatrix}0 & -3\\ -2 & 2\end{bmatrix}$ apply that $(AB)^2 = A^2 B^2$ **(Home work)**

Theorem: Let A be a matrix of degree m×n, then

(1) $A\,I_n = A$

(2) $I_m A = A$

(3) $AO = O$, $OA = O$

Proof (1): Let $A = [a_{ij}]_{m\times n}$, $I_n = [s_{ij}]_{n\times n}$ such that $s_{ij} = \begin{cases}1 & \text{if } i=j\\ 0 & \text{if } i\neq j\end{cases}$

(i,j) element of $AI_n = \sum_{k=1}^{n} a_{ik}s_{kj}$

$= a_{i1}s_{1j} + a_{i2}s_{2j} + \ldots + a_{ij}s_{jj} + \ldots + a_{in}s_{nj}$

$= a_{i1}(0) + a_{i2}(0) + \ldots + a_{ij}(1) + \ldots + a_{in}(0)$

$= a_{ij}(1)$

$= a_{ij}$

= (i,j) element of A

Degree of the matrix $A = m \times n$ = Degree of the matrix $A_{m\times n}I_{n\times n}$, so $A\,I_n = A$.

Proof (2): **(Home work)**

Proof (3): Let $A = [a_{ij}]_{m\times n}$, $O = [f_{ij}]_{n\times p}$ such that $f_{ij} = 0$ for all i and j

$$A_{m\times n} O_{n\times p} = \left[\sum_{k=1}^{n} a_{ik} f_{kj} \right]_{m\times p}$$

Since $f_{ij} = 0$ for all i and j , then $a_{ik} f_{kj} = 0$. So

$A_{m\times n} O_{n\times p} = O_{m\times p}$

Degree of the matrix A O = m×p = Degree of the matrix O

$\therefore$ AO = O

In the same way we can prove that OA = O.

Theorem: Associative law of multiplication

Let A, B and C matrices compatible with multiplication, then

(1) (AB)C = A(BC)

(2) r(AB) = (rA)B = A(rB), where r is a real number and A, B $\in M_{n\times n}(F)$, F = $\mathbb{R}$ or $\mathbb{C}$.

Proof (1): Let $A = [a_{ij}]_{m\times p}$, $B = [b_{jk}]_{p\times q}$ and $C = [c_{ks}]_{q\times n}$

$(AB)\,C = ([a_{ij}]_{m\times p}\ [b_{jk}]_{p\times q})\ [c_{ks}]_{q\times n}$

$$= \left[\sum_{j=1}^{p} a_{ij} b_{jk} \right]_{m\times q} [c_{ks}]_{q\times n}$$ (definition the multiplication of matrices)

$$= \left[\sum_{k=1}^{q} \sum_{j=1}^{p} (a_{ij} b_{jk}) c_{ks} \right]_{m\times n}$$ (definition the multiplication of matrices)

$$= \left[\sum_{j=1}^{p} \sum_{k=1}^{q} a_{ij} (b_{jk} c_{ks}) \right]_{m\times n}$$ (the multiplication of numbers is associative)

$A(BC) = [a_{ij}]_{m\times p}\ ([b_{jk}]_{p\times q}\ [c_{ks}]_{q\times n})$

$$= [a_{ij}]_{m\times p} \left[\sum_{k=1}^{q} b_{jk} c_{ks} \right]_{p\times n}$$ (definition the multiplication of matrices)

$$= \left[\sum_{j=1}^{p} \sum_{k=1}^{q} a_{ij} (b_{jk} c_{ks}) \right]_{m\times n}$$ (definition the multiplication of matrices)

$$\therefore (AB)\,C = A(BC)$$

Proof (2): Let A = $[a_{ij}]_{n\times n}$, B = $[b_{ij}]_{n\times n}$ and C = $[c_{ij}]_{n\times n}$, r is a real number

r(AB) = r $([a_{ij}]_{n\times n}[b_{ij}]_{n\times n})$

$= r\,[c_{ij}]_{n\times n}$ (definition the multiplication of matrices)

$= [r\,c_{ij}]_{n\times n}$ (definition the multiplication of matrix by scalar)

$= \left[r\sum_{k=1}^{n} a_{ik}b_{kj} \right]_{n\times n}$ (definition the multiplication of matrices)

$= \left[\sum_{k=1}^{n} r(a_{ik}b_{kj}) \right]_{n\times n}$

$= \left[\sum_{k=1}^{n} (ra_{ik})b_{kj} \right]_{n\times n}$ (the multiplication of numbers is associative)

$= [r\,a_{ij}]_{n\times n}\,[b_{ij}]_{n\times n}$ (definition the multiplication of matrices)

$= (r\,[a_{ij}]_{n\times n})\,[b_{ij}]_{n\times n}$ (definition the multiplication of matrix by scalar)

= (rA)B

As the same way we can prove

r(AB) = A(rB) and (rA)B = A(rB) **(Home work)**

Theorem: Distributive law of multiplication over addition

Let A, B and C matrices compatible with multiplication, then

(1) A(B + C) = AB + AC

(2) (B + C) A = BA + CA

Proof (1): Let A = $[a_{ij}]_{m\times n}$, B = $[b_{jk}]_{n\times p}$, C = $[c_{ks}]_{n\times p}$.

Suppose that B + C = D such that $[b_{jk}]_{n\times p} + [c_{ks}]_{n\times p} = [b_{jk} + c_{ks}]_{n\times p} = [d_{ij}]_{n\times p}$

A (B + C) = A · D

$= \left[\sum_{k=1}^{n} a_{ik}d_{kj} \right]_{m\times p}$ (definition the multiplication of matrices)

$= \left[\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \right]_{m\times p}$ $d_{kj} = b_{kj} + c_{kj}$

$= \left[\sum_{k=1}^{n} (a_{ik}b_{kj} + a_{ik}c_{kj}) \right]_{m\times p}$ (distribution of multiplication over the addition in numbers)

$$A\,(B + C) = \left[\sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj} \right]_{m\times p}$$

$$= \left[\sum_{k=1}^{n} a_{ik} b_{kj} \right]_{m\times p} + \left[\sum_{k=1}^{n} a_{ik} c_{kj} \right]_{m\times p} \qquad \text{(definition the addition of matrices)}$$

$$= AB + AC$$

$\therefore\ A\,(B + C) = AB + AC$

Degree of the matrix $B + C = n \times p$

Degree of the matrix $A(B + C) = m \times p$

Degree of the matrix $AB = m \times p$

Degree of the matrix $AC = m \times p$ equal

Degree of the matrix $AB + AC = m \times p$

Proof (2): (Home work)

Definition: If A is any square matrix, then we can define

$$A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \text{ times}}$$

Where k is any positive integer number.

Note that $A^0 = I$.

Theorem: If A is any square matrix, then for any positive integer numbers s and t

(1) $A^s \cdot A^t = A^{s+t}$

(2) $(A^s)^t = A^{st}$

We use the mathematical induction method in the proof.

Proof (1):

When $t = 1 \Rightarrow A^s \cdot A^1 = A^{s+1}$

Suppose that the statement is true when $t = k \Rightarrow A^s \cdot A^k = A^{s+k}$

Is the statement still true when $t = k + 1$?

i.e. $A^s \cdot A^{k+1} = A^{s+k+1}$

$$A^s \cdot A^{k+1} = A^s\,(A^k \cdot A)$$

$$= (A^s \cdot A^k) \cdot A \qquad \text{(the multiplication of matrices is associative)}$$

$$= A^{s+k} \cdot A$$

$$= A^{s+k+1}$$

Proof (2):

When $t = 1 \Rightarrow (A^s)^1 = A^s$

Suppose that the statement is true when $t = k \Rightarrow (A^s)^k = A^{sk}$

Is the statement still true when $t = k + 1$?

i.e. $(A^s)^{k+1} = A^{s(k+1)}$

$$\begin{aligned}(A^s)^{k+1} &= (A^s)^k \cdot (A^s)^1 \\ &= A^{sk} \cdot A^s \\ &= A^{sk+s} \\ &= A^{s(k+1)}\end{aligned}$$

Example: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$, compute A^2, A^3 and find A^k for any positive integer number.

Solution:

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^4 = A^3 \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^5 = A^4 \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\vdots \quad \vdots \qquad\qquad\qquad\qquad \vdots$$

Note that:

When $k = 2n$ (even positive integer number)

$$A^2 = A^4 = \ldots = A^{k=2n} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

When $k = 2n - 1$ (odd positive integer number)

$$A = A^3 = \ldots = A^{k=2n-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Theorem: For any matrices A, B and C,

(1) $(A^t)^t = A$ (the transpose of transpose matrix is equal to the matrix itself)

(2) $(A \pm B)^t = A^t \pm B^t$

(3) $(\alpha A)^t = \alpha A^t$, where α is standard number

(4) $O^t_{m\times n} = O_{n\times m}$

(5) $I^t_n = I_n$ (the transpose of identity matrix is equal to the identity matrix itself)

Proof (1): Let $A = [a_{ij}]_{m\times n}$

$A^t = [a_{ji}]_{n\times m}$ (definition of matrix transpose)

$(A^t)^t = [a_{ij}]_{m\times n}$ (definition of matrix transpose)

$\therefore (A^t)^t = A$

Proof (2): Let $A = [a_{ij}]_{m\times n,}$ $B = [b_{ij}]_{m\times n}$

Suppose $A \pm B = C = [c_{ij}]_{m\times n}$

$(A \pm B)^t = C^t = [c_{ji}]_{n\times m}$ (definition of matrix transpose)

$= [a_{ji} \pm b_{ji}]_{n\times m}$ $c_{ji} = a_{ji} \pm b_{ji}$

$= [a_{ji}]_{n\times m} \pm [b_{ji}]_{n\times m}$ (definition the addition and subtraction of matrices)

$\therefore (A \pm B)^t = A^t \pm B^t$

Generalization: For any compatible matrices with addition $A_1, A_2, \ldots, A_n$

$$(A_1 + A_2 + \ldots + A_n)^t = A_1^t + A_2^t + \ldots + A_n^t$$

Proof (3): $A = [a_{ij}]_{m\times n}$

$\alpha A = \alpha\, [a_{ij}]_{m\times n}$

$= [\alpha\, a_{ij}]_{m\times n}$ (definition the multiplication of matrix by scalar)

$(\alpha A)^t = [\alpha\, a_{ji}]_{n\times m}$ (definition the matrix transpose)

$= \alpha\, [a_{ji}]_{n\times m}$ (definition the multiplication of matrix by scalar)

$= \alpha A^t$

$\therefore (\alpha A)^t = \alpha A^t$

Proof (4): Let $O = [f_{ij}]_{m\times n}$, where $f_{ij} = 0$ for all values of i and j

$O_{m\times n} = [f_{ij}]_{m\times n}$

By definition of matrix transpose, we get

$O^t_{m\times n} = [f_{ji}]_{n\times m}$, where $f_{ji} = 0$ for all values of i and j

$\therefore O^t_{m\times n} = O_{n\times m}$

Proof (5): Let $I_n = [s_{ij}]_{n\times n}$, such that $s_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$I_n = [s_{ij}]_{n\times n}$

$I_n^t = [s_{ji}]_{n\times n}$ (definition of matrix transpose)

$s_{ij} = s_{ji} = 1$ if $i = j$

$s_{ij} = s_{ji} = 0$ if $i \neq j$

$$\therefore\ I_n^t = I_n$$

Theorem: Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, then $(AB)^t = B^tA^t$

Proof:

Let $A^t = C = [c_{ij}]_{n\times m} \longrightarrow (c_{ij} = a_{ji})$

$B^t = D = [d_{ij}]_{p\times n} \longrightarrow (d_{ij} = b_{ji})$

$AB = E = [e_{ij}]_{m\times p} \longrightarrow \left(e_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} \right)$

(i,j) element of $(AB)^t$ = (j,i) element of $AB = e_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki}$

(i,j) element of B^tA^t = (i,j) element of DC

$$= \sum_{k=1}^{n} d_{ik}c_{kj}$$

$$= \sum_{k=1}^{n} b_{ki}a_{jk} \qquad \text{(replaced)}$$

$$= \sum_{k=1}^{n} a_{jk}b_{ki} \qquad \text{(the multiplication of numbers is commutative)}$$

Degree of the matrix $AB = m \times p$

Degree of the matrix $(AB)^t = p \times m$

equal

Degree of the matrix $B^tA^t = p \times m$

$$\therefore (AB)^t = B^tA^t$$

Problems:

(1) Is the matrix A equal to the zero matrix if $A^3 = O$, where A is a matrix of order 3×3?

Solution: $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \neq O_{3\times 3}$, but

$$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O_{3\times 3}$$

(2) For any n×n matrices A, B and C. Prove that

(a) $-(-A) = A$

(b) $A(B - C) = AB - AC$

(c) $(A - B)C = AC - BC$

Proof (a): Let $A = [a_{ij}]_{n\times n} \Rightarrow -A = [-a_{ij}]_{n\times n} \Rightarrow -(-A) = -[-a_{ij}]_{n\times n} = [a_{ij}]_{n\times n} = A$

Proof (b): Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$

$A(B - C) = [a_{ij}]_{n\times n}\left([b_{ij}]_{n\times n} - [c_{ij}]_{n\times n}\right)$

$= [a_{ij}]_{n\times n}[d_{ij}]_{n\times n}$ (definition the subtraction of matrices) and $d_{ij} = b_{ij} - c_{ij}$

$= \sum_{k=1}^{n} a_{ik} d_{kj}$ (definition the multiplication of matrices)

$= \sum_{k=1}^{n} a_{ik}(b_{kj} - c_{kj})$ $d_{kj} = b_{kj} - c_{kj}$

$= \sum_{k=1}^{n} (a_{ik}b_{kj} - a_{ik}c_{kj})$ (distribution the multiplication over the addition in numbers)

$= \sum_{k=1}^{n} a_{ik}b_{kj} - \sum_{k=1}^{n} a_{ik}c_{kj}$

$= [a_{ij}]_{n\times n}[b_{ij}]_{n\times n} - [a_{ij}]_{n\times n}[c_{ij}]_{n\times n}$ (definition the multiplication of matrices)

$= AB - AC$

Proof (c): Home work

(3) Let A and B be n×n matrices such that AB = BA. Prove that

(a) For any positive integer k, $AB^k = B^kA$.

(b) $(A + B)^2 = A^2 + 2AB + B^2$

Proof (a): If k = 1, then AB = BA by hypothesis

Suppose that the statement true for k = n, i.e. $AB^n = B^nA$

To prove the statement true when k = n + 1, i.e. to prove $AB^{n+1} = B^{n+1}A$

$AB^{n+1} = A\,(B^nB)$ (by pervious theorem $A^s A^t = A^{s+t}$)

$= (AB^n)\,B$ (the multiplication of matrices is associative)

$= (B^nA)\,B$ ($AB^n = B^nA$)

$= B^n\,(AB)$ (the multiplication of matrices is associative)

$= B^n\,(BA)$ ($AB = BA$)

$= (B^nB)A$ (the multiplication of matrices is associative)

$= B^{n+1}A$ (by pervious theorem $A^s A^t = A^{s+t}$)

Proof (B):

$(A + B)^2 = (A + B)\,(A + B)$ (definition the power of the matrices)

$= AA + AB + BA + BB$

$= A^2 + AB + AB + B^2$ $\quad AB = BA$

$= A^2 + 2AB + B^2$

(4) Find all matrices $B \in M_{2\times 2}(\mathbb{R})$ such that B commutes (commutative) with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Solution: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a, b, c, d \in \mathbb{R}$

$AB = BA \;\Rightarrow\; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} \;\Rightarrow\; b = 0\;,\; a = d$

$\left\{\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}, a, c \in \mathbb{R}\right\}$.

(5) Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ and the polynomials $f(x) = x^2 + 3x - 10$ and $g(x) = x^2 + 2x - 11$.

Find the values of each polynomial? Is the matrix A is a root of each polynomial?

Solution:

$$f(A) = \left(\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\right)^2 + 3\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 12 & -9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 12 & -9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 4 & 8 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & -2 \end{bmatrix}$$

Hence, A is not a root of the polynomial $f(x)$.

$$g(A) = \left(\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\right)^2 + 2\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 8 & -6 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, A is a root of the polynomial $g(x)$.

Exercises:

(1) Let $A = [3\ 2\ 1]_{1\times 3}$ and $B = \begin{bmatrix} 2 & 5 \\ 2 & -8 \\ -10 & 1 \end{bmatrix}_{3\times 2}$, find AB and BA if exists ?

What do we conclude from this question ?

(2) Let $A = \begin{bmatrix} 6 & -2 & 0 \\ 4 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 & 3 \\ 0 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -3 & 0 \\ 0 & -4 & 0 \end{bmatrix}$, find A^2 , 3C , A + 2 B , A + B , B + A , AC , AB, then find the matrix D if $\frac{1}{2}$ (A – 2 C) + D = 3 B?

(3) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$, find

(a) $(A + B)^2$

(b) $A^2 + 2AB + B^2$

(c) $A^2 - B^2$

(d) $(A + B)(A - B)$

What is the condition make the relation (a) = the relation (b) and
the relation (c) = the relation (d)

(4) Let $A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Prove that $A^2 = I_3$.

(5) Construct a matrices $A = [a_{ij}]_{2\times 3}$, $B = [b_{ij}]_{3\times 4}$ and find AB, where $a_{ij} = i - 3j$ and $b_{ij} = i^2 - j^2$.

(6) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, show that $A^2 = 4A + I_2$.

(7) Compute AB and BA if exists

(a) $A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 4 & -5 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 \\ 1 & -1 \\ 2 & 1 \\ -4 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}$

(8) If $\begin{bmatrix} 3 & a+b \\ -c+d & 4 \end{bmatrix} + \begin{bmatrix} a & 5 \\ -4 & 6d \end{bmatrix} = \begin{bmatrix} 2 & a-b \\ c & 2c+d \end{bmatrix}$, find the values of a, b, c, d?

(9) Let $A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 3 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 2 & -2 \end{bmatrix}$. Prove that

A (B + C) = AB + AC?

(10) Let $A = \begin{bmatrix} 4 & -2 & 3 \\ 5 & 0 & 2 \end{bmatrix}$, then I_2 A = A.

(11) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$. Prove that AB = O?

(12) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & -2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 4 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -4 & -6 & 1 \\ 2 & 3 & 0 \end{bmatrix}$, show that

(a) A + B = B + A **(b)** A + (B + C) = (A + B) + C

(13) If AB = BA, p integer number not negative, prove that $(AB)^p = A^pB^p$.

(14) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ and the polynomials $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 3x - 10$.

Find the values of each polynomial? Is the matrix A is a root of each polynomial?

(15) Let $A = \begin{bmatrix} 0 & 1 & -3 \\ 2 & -2 & 0 \\ 3 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -3 & 5 \\ 2 & 1 & 0 \end{bmatrix}$, find

(a) Third column from AB.
(b) Second column from BA.
(c) Third row from AB.
(d) The elements C_{33} , C_{32} , C_{22} from AB.

Theorems:

(1) If A is a matrix of degree n×n, then $A + A^t$ is a symmetric matrix.

(2) If A is a matrix of degree n×n, then $A - A^t$ is a skew symmetric matrix.

(3) If A is a diagonal matrix, then A^t is also diagonal matrix.

(4) If A is a symmetric matrix, then αA is a symmetric matrix where α is any standard number.

(5) If A is a skew symmetric matrix, then αA is a skew symmetric matrix where α is any standard number.

(6) Any square matrix can be written as a sum of two matrices one of them symmetric matrix and the other is skew symmetric matrix, i.e. $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$

Proof (1): We must prove that $A + A^t = (A + A^t)^t$

We take the right hand side

$(A + A^t)^t = A^t + (A^t)^t$ (by previous theorem $(A \pm B)^t = A^t \pm B^t$))

$= A^t + A$ (the transpose of transpose matrix is equal to the matrix itself)

$= A + A^t$ (the addition of matrices is commutative)

Thus $(A + A^t)$ is symmetric matrix

Proof (2): We must prove that $A - A^t = -(A - A^t)^t$

We take the right hand side

$-(A - A^t)^t = -(A^t - (A^t)^t)$ (by previous theorem $(A \pm B)^t = A^t \pm B^t$))

$= -(A^t - A)$ (the transpose of transpose matrix is equal to the matrix itself)

$= -A^t + A$

$= A + (-A^t)$ (the addition of matrices is commutative)

$= A - A^t$

Thus $(A - A^t)$ is skew symmetric matrix

Proof (3), Proof (4) and Proof (5) (Home work)

Proof (6): Let $A = R + Q$, where

A is square matrix, R symmetric matrix and Q is skew symmetric matrix.

$A = R + Q$ …(1) (from hypothesis)

$A^t = (R + Q)^t$ (by taking the transpose for each side)

$A^t = R^t + Q^t$ (from previous theorem $(A \pm B)^t = A^t \pm B^t$)

$A^t = R - Q$ …(2) (since R is symmetric matrix from hypothesis, and Q is skew symmetric, so $R = R^t$ and $Q = -Q^t$)

By adding the equations (1) and (2), we get

$A + A^t = 2R \;\Rightarrow\; R = \frac{1}{2}(A + A^t)$

Since $\frac{1}{2}$ $(A + A^t)$ is symmetric matrix, so R is symmetric matrix (proved earlier).

Subtraction equation (2) from equation (1), we get

$A - A^t = 2Q \;\; \Rightarrow \;\; Q = \frac{1}{2}\,(A - A^t)$

Since $\frac{1}{2}(A - A^t)$ is skew symmetric matrix, so Q is skew symmetric matrix (proved earlier).

$A = \frac{1}{2}\,(A + A^t) + \frac{1}{2}\,(A - A^t)$ (replaced in (1))

$A = R + Q$

Example: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ -2 & 2 & 1 \end{bmatrix}$, find $\frac{1}{2}\,(A + A^t)$, $\frac{1}{2}\,(A - A^t)$, and show that

$A = \frac{1}{2}(A + A^t) + \frac{1}{2}\,(A - A^t)$.

Solution:

$$A + A^t = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ -2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & 2 \\ 0 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 0 & 8 \\ -2 & 8 & 2 \end{bmatrix}$$

$$\frac{1}{2}(A + A^t) = \frac{1}{2}\begin{bmatrix} 2 & 6 & -2 \\ 6 & 0 & 8 \\ -2 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 0 & 4 \\ -1 & 4 & 1 \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ -2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 4 & -2 \\ 2 & 0 & 2 \\ 0 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix}$$

$$\frac{1}{2}(A - A^t) = \frac{1}{2}\begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & 4 \\ -2 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

$$A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t) = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 0 & 4 \\ -1 & 4 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 6 \\ -2 & 2 & 1 \end{bmatrix} = A$$

Example: Is there exist a symmetric and skew symmetric matrix?

Solution: Let A be a symmetric and skew symmetric matrix.

Since A is symmetric matrix in hypothesis $\Rightarrow$ $A = A^t$ (definition of symmetric matrix)

Since A is skew symmetric matrix in hypothesis $\Rightarrow A = -A^t$ (definition of skew symmetric matrix)

By addition

$$A + A = A^t + (-A^t)$$

$$2A = O_{n\times n} \qquad \text{by divided each side by 2}$$

$$\therefore\ A = O_{n\times n}$$

Theorem: Let A be symmetric matrix. Prove that A^n is symmetric matrix for any positive integer number n

Proof: We must prove that $A^n = (A^n)^t$. By using the mathematical induction method

When $n = 1 \Rightarrow A = A^t$

When $n = 2 \Rightarrow A^2 = (A^2)^t$ since $(A^2)^t = (A\cdot A)^t = A^t\cdot A^t = A\cdot A = A^2$

Suppose the statement is true for $n = k$, i.e. $A^k = (A^k)^t$

To prove the statement is true when $n = k + 1$, i.e. to prove $A^{k+1} = (A^{k+1})^t$

$$\begin{aligned}(A^{k+1})^t &= (A^k\cdot A)^t \\ &= A^t\,(A^k)^t \qquad \text{(from previous theorem } (AB)^t = B^tA^t\text{)} \\ &= A\cdot A^k \\ &= A^{k+1}\end{aligned}$$

$\therefore$ The statement is true for all positive integer values of n

Definition: The sum of the main diagonal elements of the square matrix say the trace of the matrix and denoted by Tr, i.e. $\mathrm{Tr} = \sum_{i=1}^{n} a_{ii}$, where $A = [a_{ij}]_{n\times n}$.

Example: If $A = \begin{bmatrix} 4 & 3 & 2 \\ 0 & 5 & -1 \\ 4 & -3 & 3 \end{bmatrix}$, then the trace of A is

$$\mathrm{Tr}(A) = \sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = 4 + 5 + 3 = 12$$

Example: If the trace of the matrix $B = \begin{bmatrix} x^2 & 2 & 3 & -1 \\ 1 & 2x & 1 & 6 \\ 1 & 1 & -5 & 0 \\ -2 & 3 & 7 & 2x \end{bmatrix}$ equal to zero, find the value of x

Solution: $\mathrm{Tr}(B) = x^2 + 2x - 5 + 2x = x^2 + 4x - 5$

Since $\mathrm{Tr}(B) = 0 \Rightarrow x^2 + 4x - 5 = 0 \Rightarrow (x + 5)(x - 1) = 0 \Rightarrow x = -5\ ,\ x = 1$

Theorem:

1. Tr (A) = Tr (A^t), for a square matrix A.

2. Tr (A + B) = Tr (A) + Tr (B), for any matrices A and B compatible with addition.

3. Tr (kA) = k(Tr A), where k is constant amount and A is a square matrix.

4. Tr (AB) = Tr (BA) , for any matrices A and B compatible with multiplication.

Proof (1):

Let $A = [a_{ij}]_{n\times n}$, so $A^t = [a_{ji}]_{n\times n}$

$$\text{Tr A} = \sum_{i=1}^{n} a_{ii} \qquad \text{(definition the trace of the matrix)}$$

$$\text{Tr } A^t = \sum_{i=1}^{n} a_{ii} \qquad \text{(definition the trace of the matrix)}$$

$$\therefore \text{ Tr A} = \text{Tr } A^t$$

Proof (2): Let A + B = C, where $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $C = [c_{ij}]_{n\times n}$

$A + B = [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n} = [c_{ij}]_{n\times n}$, where $a_{ij} + b_{ij} = c_{ij}$ for all values of i and j

$$\text{Tr (A + B)} = \text{Tr C} = \sum_{i=1}^{n} c_{ii} \qquad \ldots(1)$$

$$\text{Tr A + Tr B} = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$$

$$= \sum_{i=1}^{n} (a_{ii} + b_{ii})$$

$$= \sum_{i=1}^{n} c_{ii} \qquad \ldots(2) \qquad \text{Since } a_{ij} + b_{ij} = c_{ij}$$

From (1) and (2), we get

Tr (A + B) = Tr A + Tr B

Generalization: $Tr(A_1 + A_2 + \ldots + A_k) = TrA_1 + TrA_2 + \ldots + TrA_k$, where A_i matrix of degree n×n and i = 1,2,…,k.

Proof (3): Let $A = [a_{ij}]_{n\times n}$, so $kA = [k\, a_{ij}]_{n\times n}$

$$\text{Tr (kA)} = \sum_{i=1}^{n} k a_{ii} = k \sum_{i=1}^{n} a_{ii} = k \text{ Tr (A)}$$

Proof (4): Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $AB = [c_{ij}]_{n\times n}$, $c_{ij} = \sum_{i=1}^{n} a_{ik} b_{kj}$, $BA = [d_{ij}]_{n\times n}$,

$$d_{ij} = \sum_{i=1}^{n} b_{ik} a_{kj}$$

$$\mathrm{Tr}\,(AB) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} a_{ii}\, b_{ii} \qquad \ldots(1)$$

$$\mathrm{Tr}\,(BA) = \sum_{i=1}^{n} d_{ii} = \sum_{i=1}^{n} b_{ii}\, a_{ii} = \sum_{i=1}^{n} a_{ii}\, b_{ii} \qquad \ldots(2)$$

From (1) and (2), we get Tr (AB) = Tr (BA).

Theorem:

(1) For any matrix A, AA^t symmetric matrix. **(The proof home work)**

(2) If A is a skew symmetric matrix, then $AA^t = A^tA$. **(The proof home work)**

Properties of the Symmetric Matrix:

(1) A is symmetric matrix if and only if A^t is symmetric matrix.

(2) If A and B are symmetric matrices, then (A + B) is symmetric matrix.

(3) If A and B are symmetric matrices, then AB is symmetric matric if and only if AB = BA, i.e. AB symmetric matrix $\Leftrightarrow$ AB = BA.

Proof (1): (Home work)

Proof (2): Let A and B are symmetric matrices, so $A = A^t$, $B = B^t$, we must prove $A + B = (A + B)^t$

$(A + B)^t = A^t + B^t$ (by previous theorem $(A \pm B)^t = A^t \pm B^t$)

$= A + B$ (by hypothesis $A = A^t$, $B = B^t$)

$\therefore$ A + B symmetric matrix

Proof (3): Let A and B are symmetric matrices, so $A = A^t$, $B = B^t$,

$\Rightarrow$ Suppose that AB symmetric matrix, to prove that AB = BA

$AB = (AB)^t$ (definition of symmetric matrix)

$= B^t A^t$ (by previous theorem $(AB)^t = B^tA^t$)

$= BA$ (replaced)

$\therefore$ AB = BA

$\Leftarrow$ Suppose that AB = BA, to prove AB symmetric matrix

AB = BA

$= B^tA^t$ (replaced)

$= (AB)^t$ (by previous theorem $(AB)^t = B^tA^t$)

$\therefore$ AB is symmetric matrix

Remark: The opposite of second property is not true, i.e. if (A + B) is symmetric matrix that is not necessary that the matrices A and B are symmetric matrices.

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ -3 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 & -3 \\ 0 & -1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ are not symmetric matrices, but

$A + B = \begin{bmatrix} 2 & 4 & -3 \\ 4 & -2 & 5 \\ -3 & 5 & 6 \end{bmatrix}$ is symmetric matrix.

Properties of the Skew Symmetric Matrix:

(1) A is a skew symmetric matrix if and only if A^t is a skew symmetric matrix.

(2) If A and B are skew symmetric matrices, then (A + B) is a skew symmetric matrix.

(3) If A and B are skew symmetric matrices, then AB is a skew symmetric matric if and only if AB = – BA, i.e. AB is a skew symmetric matrix $\Leftrightarrow$ AB = – BA.

Proof (1): (Home work)

Proof (2): Let A and B are skew symmetric matrices, so $A = -A^t$, $B = -B^t$, we must prove $A + B = -(A + B)^t$

$-(A + B)^t = -(A^t + B^t)$ (by previous theorem $(A \pm B)^t = A^t \pm B^t$)

$= -(-A + -B)$ (by hypothesis $A = -A^t$, $B = -B^t$)

$= A + B$ (by previous theorem $-(-A) = A$)

$\therefore$ A + B skew symmetric matrix

Proof (3): (Home work)

Remark: The opposite of second property is not true, i.e. if (A + B) is a skew symmetric matrix that is not necessary that the matrices A and B are skew symmetric matrices.

Give example (Home work)

Exercises:

(1) If A is a square matrix of degree n, p is a positive integer number and k real number, then $(kA)^p = k^p A^p$.

(2) Let A and B are matrices such that $AB = A$ and $BA = B$, show that $(A^t)^2 = A^t$ and $(B^t)^2 = B^t$.

(3) Find all matrices A of degree 2×2 with complex entries such that $A^2 = -I_2$.

Answer: $A = \begin{bmatrix} \pm i & 0 \\ 0 & \mp i \end{bmatrix}$.

Chapter two
Determinants

Determinant:

A function whose domain is the set of square matrices and their range (codomain) the set F.

Where: (F is the set of real or complex numbers)

The value of the square matrix function is called "determinant" for that matrix and is written as follows:

$f(A) = \det(A) = |A|$

$f([a]) = |a| = a$

Examples:

(1) $f([-8]) = |-8| = -8$

(2) $f\left(\left[\frac{2}{3}\right]\right) = \left|\frac{2}{3}\right| = \frac{2}{3}$

Determinant of the matrix of degree 2×2

It is the product of the elements of the main diagonal minus the product of the elements of the secondary diagonal. That is:

$$\det\left(\left[a_{ij}\right]_{2\times2}\right) = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}\right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Examples:

(1) $\det\left(\begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}_{2\times2}\right) = \begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} = (4)(1) - (2)(-3) = 4 + 6 = 10$

(2) $\det\left(\begin{bmatrix} 7 & -2 \\ -1 & 3 \end{bmatrix}_{2\times2}\right) = \begin{vmatrix} 7 & -2 \\ -1 & 3 \end{vmatrix} = (7)(3) - (-2)(-1) = 21 - 2 = 19$

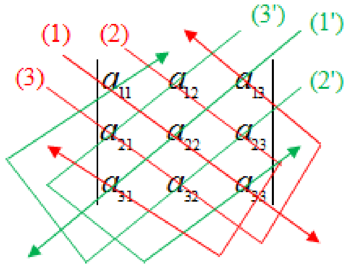
(3) $\det\left(\begin{bmatrix} -8 & 5 \\ 7 & 0 \end{bmatrix}_{2\times2}\right) = \begin{vmatrix} -8 & 5 \\ 7 & 0 \end{vmatrix} = (-8)(0) - (5)(7) = 0 - 35 = -35$

(4) $\det\left(\begin{bmatrix} -\frac{2}{7} & -3 \\ 2 & 6 \end{bmatrix}_{2\times2}\right) = \begin{vmatrix} -\frac{2}{7} & -3 \\ 2 & 6 \end{vmatrix} = (-\frac{2}{7})(6) - (-3)(2) = -\frac{12}{7} - (-6) = \frac{30}{7}$

Determinant of the matrix of degree 3×3

$$\det\left(A=\left[a_{ij}\right]_{3\times3}\right)=\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}_{3\times3}\right)$$

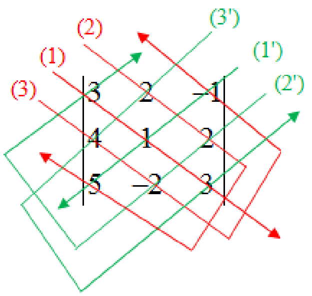
The calculation of the determinant is as follows:



$$= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}$$
$$-a_{13}a_{22}a_{31}-a_{23}a_{32}a_{11}-a_{12}a_{21}a_{33}$$

Example: Find the determinant of the matrix $A=\begin{bmatrix} 3 & 2 & -1\\ 4 & 1 & 2\\ 5 & -2 & 3\end{bmatrix}$

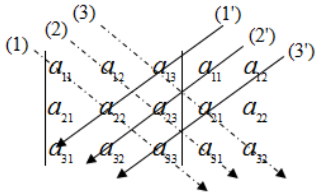
Solution:



$$=(3)(1)(3)+(2)(2)(5)+(4)(-2)(-1)$$
$$-(-1)(1)(5)-(2)(-2)(3)-(2)(4)(3)$$
$$=(9+20+8)+(5+12-24)=54-24=30$$

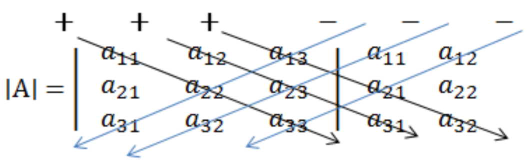
Second method: which is to write the first and second columns after the third column, then we find the multiplication products as follows:

$$\det\left(A=\left[a_{ij}\right]_{3\times3}\right)=\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}_{3\times3}\right)=$$



$$= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}$$
$$-a_{13}a_{22}a_{31}-a_{23}a_{32}a_{11}-a_{12}a_{21}a_{33}$$

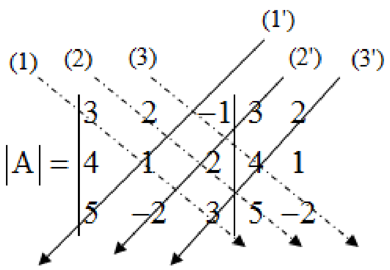
or



Examples: Find the determinant of the following matrices

(1) $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ 5 & -2 & 3 \end{bmatrix}$

Solution:



$= (3)(1)(3) + (2)(2)(5) + (-1)\,(4)(-2) - (-1)(1)(5) - (3)(2)(-2) - (2)(4)(3)$

$= (9 + 20 + 8) + (5 + 12 - 24) = 54 - 24 = 30$

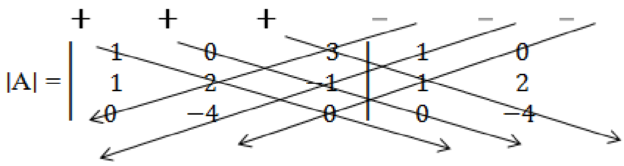
(2) $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & -1 \\ 0 & -4 & 0 \end{bmatrix}$

Solution:

$|A| = (1)\,(2)\,(0) + (0)\,(-1)\,(0) + (3)\,(1)\,(-4)$

$\quad - (3)\,(2)\,(0) - (1)\,(-1)\,(-4) - (0)\,(1)\,(0)$

$\quad = 0 + 0 + (-12) - 0 - 4 - 0 = -16$



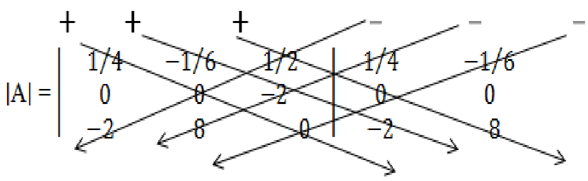
(3) $A = \begin{bmatrix} \frac{1}{4} & -\frac{1}{6} & \frac{1}{4} \\ 0 & 0 & -2 \\ -2 & 8 & 0 \end{bmatrix}$

Solution:

$|A| = (\frac{1}{4})(0)\,(0) + (-\frac{1}{6})(-2)(-2) + (\frac{1}{2})(0)\,(8)$

$\quad - (\frac{1}{2})\,(0)(-2) - (\frac{1}{4})(-2)\,(8) - (-\frac{1}{6})\,(0)\,(0)$

$\quad = 0 + \left(-\frac{2}{3}\right) + 0 - 0 + 4 - 0 = \frac{10}{3}$



Finding the determinant of the matrix using cofactor method

Definition: C_{ij} the cofactor of the element a_{ij} of the elements of the square matrix $A = [a_{ij}]_{n\times n}$ "is a product of $(-1)^{i+j}$ by the determinant of the matrix A after remove the i-th row and j-th column. This determinant is called the miner determinant for a_{ij} and denoted by M_{ij}"

Examples:

(1) The cofactor of the element a_{22} in the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$C_{22} = (-1)^{2+2}M_{22} = (-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

(2) The cofactor of the element a_{23} in the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$C_{23} = (-1)^{2+3}M_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

(3) The cofactor of the element (– 4) in the position (1,1) in the matrix $B = \begin{bmatrix} -4 & 3 & 1 \\ 2 & -1 & -4 \\ 3 & 1 & 5 \end{bmatrix}$ is

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^{1+1}\begin{vmatrix} -1 & -4 \\ 1 & 5 \end{vmatrix} = (-1)(5) - (-4)(1) = -5 + 4 = -1$$

Theorem: Matrix determinant: is the sum of the product of the elements of a row (column) by their cofactors.

Let C_{ij} is the cofactor of the element a_{ij} in the matrix $A = [a_{ij}]_{n\times n}$, so the determinant of the matrix is as follows:

(1) If chose the row i, then $|A| = \det A = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$

(2) If chose the column j, then $|A| = \det A = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$

Remark: When using this method, we choose the row or column that contains the largest number of zeros.

Examples:

(1) If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Find det A?

Solution: chose the row 1

$$|A| = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12}$$

$$= a_{11}(-1)^{1+1}|a_{22}| + a_{12}(-1)^{1+2}|a_{21}| = a_{11}\,a_{22} - a_{12}\,a_{21}$$

(2) If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Find det A?

Solution: chose the row 1

$$|A| = \det A = A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}(a_{21}\,a_{33} - a_{23}\,a_{31}) + a_{13}(a_{21}\,a_{32} - a_{22}\,a_{31})$$

$$= a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}a_{21}\,a_{32} - a_{13}a_{22}\,a_{31}$$

$$= a_{11}a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}a_{21}\,a_{32} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} - a_{13}a_{22}\,a_{31}$$

(3) Find $|A| = \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 2 \\ 5 & -2 & 3 \end{vmatrix}$

Solution: chose the row i = 2

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= 4(-1)^{2+1}\begin{vmatrix} 2 & -1 \\ -2 & 3 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 5 & 3 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 3 & 2 \\ 5 & -2 \end{vmatrix}$$

$$= -4(6-2) + 1(9+5) - 2(-6-10) = -4(4) + 1(14) - 2(-16) = -16 + 14 + 32 = 30$$

(3) Find $|B| = \begin{vmatrix} 3 & 2 & 1 & 4 \\ 0 & 1 & 0 & 0 \\ 4 & -2 & 0 & 1 \\ -1 & 3 & 0 & 2 \end{vmatrix}$

Solution: We can chose the row i = 2 or the column j = 3

Now we chose the row i = 2

$|B| = b_{21}C_{21} + b_{22}C_{22} + b_{23}C_{23} + b_{24}C_{24}$

$$|B| = 0(-1)^{2+1}\begin{vmatrix} 2 & 1 & 4 \\ -2 & 0 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 1(-1)^{2+2}\begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 3 & 2 & 4 \\ 4 & -2 & 1 \\ -1 & 3 & 2 \end{vmatrix} + 0(-1)^{2+4}\begin{vmatrix} 3 & 2 & 1 \\ 4 & -2 & 0 \\ -1 & 3 & 0 \end{vmatrix}$$

$$|B| = \begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix}$$

To find the value of this determinant we can find it directly or by using the cofactor by chose the second column

$$|B| = \begin{vmatrix} 3 & 1 & 4 \\ 4 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1(-1)^{1+2}\begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} + 0(-1)^{2+2}\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} + 0(-1)^{3+2}\begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix} = -(8+1) = -9$$

Exercise: Resolve the previous example using the third column

Example: Prove that $|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos 2\theta$.

Proof:

$$|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

Example: Prove that $|B| = \begin{vmatrix} \cos\theta & 1 & 1 \\ 1 & \cos\theta & 1 \\ \sin\theta & 1 & 1 \end{vmatrix} = (\cos\theta - \sin\theta)(\cos\theta - 1)$.

Proof:

$$|B| = \begin{vmatrix} \cos\theta & 1 & 1 \\ 1 & \cos\theta & 1 \\ \sin\theta & 1 & 1 \end{vmatrix} = \cos^2\theta + 1 + \sin\theta - \sin\theta\cos\theta - \cos\theta - 1$$

$$= \cos^2\theta - \sin\theta\cos\theta - \cos\theta + \sin\theta$$

$$= \cos\theta(\cos\theta - \sin\theta) - 1(\cos\theta - \sin\theta)$$

$$= (\cos\theta - \sin\theta)(\cos\theta - 1)$$

Some Properties of Determinants:

Property (1):

If all elements of row or column of any matrix are zero, then their determinant equal to zero.

Proof: Let the row i all elements of it are zero.

$a_{i1} = a_{i2} = \ldots = a_{in} = 0 \Rightarrow (0,0,\ldots,0)$ the row i

We open the determinant around the row i

$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$

$= 0\,C_{i1} + 0\,C_{i2} + \ldots + 0\,C_{in}$

$= 0 + 0 + \ldots + 0 = 0$

The same method remains if all elements of one of the columns are zeros. (**Home work**)

Examples:

(1) $\begin{vmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 0$ since all elements of the first row are zero

(2) $\begin{vmatrix} -4 & 0 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 0$ since all elements of the second column are zero

Property (2):

The determinant of the square matrix A equal to the determinant of its transpose, i.e. $|A| = |A^t|$

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$, so $A^t = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$

$|A| = a_{11}C_{11} + a_{12}\,C_{12} + \dots + a_{1n}\,C_{1n}$ (by opening the determinant about 1st row)

$|A^t| = a_{11}C_{11} + a_{12}\,C_{12} + \dots + a_{1n}\,C_{1n}$ (by opening the determinant about 1st column)

Thus $|A| = |A^t|$.

Examples:

(1) $|A| = \begin{vmatrix} 3 & -2 \\ 1 & 4 \end{vmatrix} = 12 + 2 = 14$ and $|A^t| = \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} = 12 + 2 = 14$

$$\therefore |A| = |A^t|$$

(2) Show the correctness of $|A| = \begin{vmatrix} 1 & -2 & 2 & 1 \\ 4 & 3 & -1 & 3 \\ 0 & 2 & 0 & 0 \\ -1 & 2 & 3 & 4 \end{vmatrix} = 80$ and $|A^t| = 80$?

Solution: To find the value of the determinant to the matrix A, we choose the third row (why ?)

$|A| = a_{31}C_{31} + a_{32}\,C_{32} + + a_{33}\,C_{33} + a_{34}\,C_{34}$

$$= 0(-1)^{3+1}\begin{vmatrix} -2 & 2 & 1 \\ 3 & -1 & 3 \\ 2 & 3 & 4 \end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix} 1 & 2 & 1 \\ 4 & -1 & 3 \\ -1 & 3 & 4 \end{vmatrix} + 0(-1)^{3+3}\begin{vmatrix} 1 & -2 & 1 \\ 4 & 3 & 3 \\ -1 & 2 & 4 \end{vmatrix} +$$

$$0(-1)^{3+4}\begin{vmatrix} 1 & -2 & 2 \\ 4 & 3 & -1 \\ -1 & 2 & 3 \end{vmatrix}$$

$$|A| = -2\begin{vmatrix} 1 & 2 & 1 \\ 4 & -1 & 3 \\ -1 & 3 & 4 \end{vmatrix} = -2[(-4) + (-6) + (12) - (1) - (9) - (32)]$$

$$= -2\,(2 - 42) = -2\,(-40) = 80$$

To find the value of the determinant to the transpose of the matrix A, we choose the third column.

$|A^t| = a_{13}\,C_{13} + a_{23}\,C_{23} + a_{33}\,C_{33} + a_{43}\,C_{43}$

$$|A^t| = \begin{vmatrix} 1 & 4 & 0 & -1 \\ -2 & 3 & 2 & 2 \\ 2 & -1 & 0 & 3 \\ 1 & 3 & 0 & 4 \end{vmatrix} = 0(-1)^{1+3}\begin{vmatrix} -2 & 3 & 2 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 4 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix} +$$

$$0(-1)^{3+3}\begin{vmatrix} 1 & 4 & -1 \\ -2 & 3 & 2 \\ 1 & 3 & 4 \end{vmatrix} + 0(-1)^{4+3}\begin{vmatrix} 1 & 4 & -1 \\ -2 & 3 & 2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$|A^t| = -2\begin{vmatrix} 1 & 4 & -1 \\ 2 & -1 & 3 \\ 1 & 3 & 4 \end{vmatrix}$$

$= -2[(-4) + (-6) + (12) - (1) - (9) - (32)]$

$= -2\,(2 - 42) = -2\,(-40) = 80$

$$\therefore |A| = |A^t|$$

(3) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 7 & 9 \\ 4 & 5 & 8 \end{bmatrix}$

$|A| = 168 + 0 + 0 - 0 - 135 - 0$

$= 168 - 135$

$= 33$

$$\begin{array}{c} \\ |A| = \end{array}\begin{array}{ccc|cc} + & + & + & - & - \; - \\ 3 & 0 & 0 & 3 & 0 \\ 1 & 7 & 9 & 1 & 7 \\ 4 & 5 & 8 & 4 & 5 \end{array}$$

And

$A^t = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 7 & 5 \\ 0 & 9 & 8 \end{bmatrix}$

$|A^t| = 168 + 0 + 0 - 0 - 135 - 0$

$= 168 - 135$

$= 33$

$$\begin{array}{c} \\ |A^T| = \end{array}\begin{array}{ccc|cc} + & + & + & - & - \; - \\ 3 & 1 & 4 & 3 & 1 \\ 0 & 7 & 5 & 0 & 7 \\ 0 & 9 & 8 & 0 & 9 \end{array}$$

Property (3):

If a matrix B results from interchanging row by row or column by column in matrix A, then det B = – det A

In other words: If replace a row with a row (column by column) in the matrix, then the signal determinant is changed.

Proof: when n = 2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ , so } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

If we replace the second row with the first row $R_1 \leftrightarrow R_2$

$$B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \text{ , so } |B| = \begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix} = a_{21}a_{12} - a_{11}a_{22}$$

$$= -(a_{11}a_{22} - a_{21}a_{12})$$

$$= -|A|$$

when n = 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = \underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}}_{\text{Multiply towards the main diagonal}} - \underbrace{a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11}}_{\text{Multiply towards the secondary diagonal}}$$

If we replace the second row with the first row $R_1 \leftrightarrow R_2$

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|B| = a_{21}a_{12}a_{33} + a_{22}a_{13}a_{31} + a_{11}a_{32}a_{23} - a_{23}a_{12}a_{31} - a_{11}a_{22}a_{33} - a_{13}a_{32}a_{21}$$

$$= -(-a_{21}a_{12}a_{33} - a_{22}a_{13}a_{31} - a_{11}a_{32}a_{23} + a_{23}a_{12}a_{31} + a_{11}a_{22}a_{33} + a_{13}a_{32}a_{21})$$

$$= -(\underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}}_{\text{Multiply towards the main diagonal}} - \underbrace{a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{32}a_{23}a_{11}}_{\text{Multiply towards the secondary diagonal}})$$

$$= -|A|$$

As well as if we replace the first row with the third row $R_1 \leftrightarrow R_3$, we get $|B| = -|A|$

In the same way if the replace was on the columns.

And so in the same way we prove when n = k.

$\therefore$ The property is true for all values of n.

Examples:

(1) $A=\begin{bmatrix}2 & -3\\ 1 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 3\\ 2 & -3\end{bmatrix}$ (replace the first row with the second row $R_1 \leftrightarrow R_2$)

$$|A|=\begin{vmatrix}2 & -3\\ 1 & 3\end{vmatrix}=6-(-3)=9$$

$$|B|=\begin{vmatrix}1 & 3\\ 2 & -3\end{vmatrix}=-3-6=-9$$

$$\therefore\ |B|=-|A|$$

(2) $A=\begin{bmatrix}1 & 2 & 3\\ 1 & -1 & 1\\ 2 & 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 1 & -1 & 1\end{bmatrix}$ (replace the third row with the second row $R_2 \leftrightarrow R_3$)

$$|A|=\begin{vmatrix}1 & 2 & 3\\ 1 & -1 & 1\\ 2 & 3 & 4\end{vmatrix} = (1)(-1)(4) + (2)(1)(2) + (3)(1)(3) - (3)(-1)(2) - (1)(3)(1) - (2)(1)(4)$$

$$= (-4 + 4 + 9) + (6 - 3 - 8) = 9 - 5 = 4$$

$$|B|=\begin{vmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 1 & -1 & 1\end{vmatrix} = (1)(3)(1) + (2)(4)(1) + (3)(2)(-1) - (3)(3)(1) - (1)(4)(-1) - (2)(2)(1)$$

$$= (3 + 8 - 6) - (9 + 4 - 4) = 5 - 9 = -4$$

$$\therefore\ |B|=-|A|$$

Let $C=\begin{bmatrix}1 & 3 & 2\\ 1 & 1 & -1\\ 2 & 4 & 3\end{bmatrix}$ (replace the third column with the second column $C_2 \leftrightarrow C_3$).

Show that $|C|=-|A|$ **(Home work)**

Let $D=\begin{bmatrix}3 & 2 & 1\\ 1 & -1 & 1\\ 4 & 3 & 2\end{bmatrix}$ (replace the third column with the first column $C_1 \leftrightarrow C_3$).

Show that $|D|=-|A|$ **(Home work)**

Property (4):

If the elements of two rows (two columns) are equal in a square matrix then its determinant is equal to zero.

Proof: Suppose in the matrix A the elements of the row (i) equal to the elements of the row (ℓ).

Suppose the matrix B results from replace the row (i) by the row (ℓ).

$|B| = -|A|$ (by Property (3))

But $B = A$, so $|B| = |A|$ (the row (i) = the row (ℓ))

$|A| = -|A|$ (replaced)

$2|A| = 0 \Rightarrow 2 \neq 0$

$\therefore |A| = 0$

Examples:

(1) $\begin{vmatrix} 4 & -2 & 3 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = 0$ (the second and third rows are equal)

(2) $\begin{vmatrix} 3 & 1 & 2 & 2 \\ -2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 2 & 1 & 1 \end{vmatrix} = 0$ (the third and fourth columns are equal)

Property (5):

Determinant of the product of two square matrices of the same degree = product of the determinants of those two matrices. i.e. $|AB| = |A| \cdot |B|$

Proof: We prove this property when $n = 2$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

$|AB| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$

After opening and arranging it, produce:

$|AB| = \boxed{a_{11}b_{11}a_{22}b_{22} - a_{11}a_{22}b_{12}b_{21} - b_{11}b_{22}a_{12}a_{21} + a_{12}a_{21}b_{12}b_{21}}$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

$$|B| = b_{11}b_{22} - b_{12}b_{21}$$

$$|A| \cdot |B| = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

$$= \boxed{a_{11}b_{11}a_{22}b_{22} - a_{11}a_{22}b_{12}b_{21} - b_{11}b_{22}a_{12}a_{21} + a_{12}a_{21}b_{12}b_{21}}$$

$$\therefore \ |AB| = |A| \cdot |B|$$

As the same way when n = 3.
Thus this property can be proved when n = k.

Examples:

(1) Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, show that $|AB| = |A| \cdot |B|$?

Solution: $|AB| = |A| \cdot |B|$

$$\left| \begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \right| = \begin{vmatrix} 2 & -3 \\ -1 & 5 \end{vmatrix} \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$

$$\begin{vmatrix} -4 & -1 \\ 9 & 11 \end{vmatrix} = (10-3) \cdot (3-8)$$

$$-44 + 9 = (7)\,(-5)$$

$$-35 = -35$$

(2) Let $A = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$, show that $|AB| = |A| \cdot |B|$?

Solution: $|AB| = |A| \cdot |B|$

$$\left| \begin{bmatrix} 0 & 2 & 2 \\ 3 & -1 & 0 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix} \right| = \begin{vmatrix} 0 & 2 & 2 \\ 3 & -1 & 0 \\ 1 & 4 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 4 & 2 & 8 \\ 3 & 7 & -1 \\ 3 & 10 & 7 \end{vmatrix} = (20)\,(13)$$

$$260 = 260$$

(3) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 7 & 2 \end{bmatrix}$, find $|A|$, $|B|$, $|AB|$, $|A^3 B|$, $|B^2 A|$? **(Home work)**

(4) If $|A| = 4$, find $|A^3|$?

Solution:

$|A^3| = |A \cdot A^2|$

$= |A| \cdot |A^2|$ (by property (5))

$= |A|\,|A \cdot A|$

$= |A| \cdot |A| \cdot |A|$ (by property (5))

$|A^3| = |A|^3$

$= 4^3$

$|A^3| = 64$

This example leads to the following corollary:

Corollary: If A is a square matrix, then $|A^k| = |A|^k$, where k is a positive integer number. **(The prove home work)**

Generalization: If $A_1, A_2, \ldots, A_n$ are square matrices of the same degree, then $|A_1 A_2 \ldots A_n| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$. **(The prove home work)**

Remark: $|A + B| \neq |A| + |B|$, for example:

$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$

$|A+B| = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 4$, $|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 7$, $|B| = \begin{vmatrix} 0 & 3 \\ -1 & 2 \end{vmatrix} = 3$

$|A + B| \neq |A| + |B|$

$4 \neq 7 + 3$

$4 \neq 10$

Property (6):

If A, B and D are three matrices of degree (n×n) equal in all rows except in the row (i) in the matrix D such that $d_{ij} = a_{ij} + b_{ij}$, j = 1,2,…, n, then det D = det A + det B. The same way if it is a column.

Proof: Let

$$A=\begin{array}{cc}\begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & \cdots & \cdots & a_{in} \\ \vdots & \vdots & & & & \vdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n} & \begin{array}{c} \\ \\ \\ \text{i-th row} \\ \\ \\ \end{array}\end{array}, \quad B=\begin{array}{cc}\begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ \vdots & \vdots & & & & \vdots \\ b_{i1} & b_{i2} & \cdots & \cdots & \cdots & b_{in} \\ \vdots & \vdots & & & & \vdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n} & \begin{array}{c} \\ \\ \\ \text{i-th row} \\ \\ \\ \end{array}\end{array}$$

$$D=\begin{array}{cc}\begin{bmatrix} d_{11} & d_{12} & \cdots & \cdots & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \cdots & \cdots & d_{2n} \\ \vdots & \vdots & & & & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & \cdots & \cdots & a_{in}+b_{in} \\ \vdots & \vdots & & & & \vdots \\ d_{n1} & d_{n2} & \cdots & \cdots & \cdots & d_{nn} \end{bmatrix}_{n\times n} & \begin{array}{c} \\ \\ \\ \text{i-th row} \\ \\ \\ \end{array}\end{array}, \text{ where } d_{ij}=a_{ij}+b_{ij}\text{ , j = 1,2,…,n}$$

We open the determinant of the matrix D about the row i

$$|D|=\sum_{k=1}^{n}((a_{ik}+b_{ik})C_{ik})$$

$$=\sum_{k=1}^{n}a_{ik}\,C_{ik}+\sum_{k=1}^{n}b_{ik}\,C_{ik}$$

$$|D|=\ |A|\ +\ |B|$$

By theorem:
Matrix determinant is the sum of the product of the elements of a row (column) by their cofactors.

In the same way if
The column j from D = The column j from A + The column j from B, i.e.

$$D=\begin{array}{c}\begin{vmatrix} d_{11} & d_{12} & \cdots & a_{1j}+b_{1j} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & a_{2j}+b_{2j} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & a_{nj}+b_{nj} & \cdots & d_{nn} \end{vmatrix} \\ \text{j-th column}\end{array}$$

$$=\begin{array}{c}\text{j-th column} \\ \begin{vmatrix} d_{11} & d_{12} & \cdots & a_{1j} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & a_{2j} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & a_{nj} & \cdots & d_{nn} \end{vmatrix}\end{array}+\begin{array}{c}\text{j-th column} \\ \begin{vmatrix} d_{11} & d_{12} & \cdots & b_{1j} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & b_{2j} & \cdots & d_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ d_{n1} & d_{n2} & \cdots & b_{nj} & \cdots & d_{nn} \end{vmatrix}\end{array}$$

$$|D|=|A|+|B|$$

Examples:

$$\textbf{(1)}\quad |C| = \begin{vmatrix} 1 & 0 & 2 \\ \boxed{4 \quad 6 \quad 8} \\ -2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ \boxed{1 \quad 2 \quad 1} \\ -2 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ \boxed{3 \quad 4 \quad 7} \\ -2 & 1 & 0 \end{vmatrix}$$

$$24 \qquad = 9 + 15$$

$$\textbf{(2)}\quad \begin{vmatrix} \boxed{3} & 1 & 0 \\ \boxed{6} & 2 & -1 \\ \boxed{9} & 1 & 4 \end{vmatrix} = \begin{vmatrix} \boxed{2} & 1 & 0 \\ \boxed{2} & 2 & -1 \\ \boxed{2} & 1 & 4 \end{vmatrix} + \begin{vmatrix} \boxed{1} & 1 & 0 \\ \boxed{4} & 2 & -1 \\ \boxed{7} & 1 & 4 \end{vmatrix}$$

$$24 + (-9) + (0) - (0) - (-3) - (24) = 8 + (-14)$$

$$-6 = -6$$

Property (7):

If all the elements of a row (column) in a square matrix A multiplying by a fixed ($k \neq 0$), then the determinant of it multiply by the fixed k. i.e.

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k\,|A|$$

Proof: Let B is the matrix obtained by multiply the row (i) in the matrix A by the fixed ($k \neq 0$), when we open the determinant of the matrix B about the row i, we get

$$|B| = k\, a_{i1}C_{i1} + k\, a_{i2}C_{i2} + \ldots + k\, a_{in}C_{in}$$

$$= k\,(a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in})$$

$$|B| = k\,|A|$$

The same way when the column of the matrix is multiply.

Examples:

equal

$$\textbf{(1)}\quad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 3 \begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 1 \\ 2 & 8 & 2 \end{vmatrix} = 3(0) = 0$$

(2) $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = \begin{vmatrix} (2)(1) & (2)(3) \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & (3)(1) \\ 1 & (3)(4) \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = (6)(3) = 18$

(3) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, so $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{vmatrix} = (-15)$

Let B is the matrix obtained by multiply the first column of the matrix A by the fixed $k = 10$, then

$$|B| = \begin{vmatrix} 10 & -1 & 2 \\ 20 & 3 & 2 \\ 30 & 2 & 1 \end{vmatrix} = (-150) = 10(-15)$$

$|B| = 10\ |A|$

Corollary:

If $B = [b_{ij}]_{n\times n}$ is a matrix obtained from multiply each entry of $A = [a_{ij}]_{n\times n}$ by c, i.e. $b_{ij} = c\, a_{ij}$, $\forall$ i, j, $1 \le i, j \le n$. Then $|B| = c^n\, |A|$.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $B = \begin{bmatrix} c\,a_{11} & c\,a_{12} & \cdots & c\,a_{1n} \\ c\,a_{21} & c\,a_{22} & \cdots & c\,a_{2n} \\ \vdots & \vdots & & \vdots \\ c\,a_{i1} & c\,a_{i2} & \cdots & c\,a_{in} \\ \vdots & \vdots & & \vdots \\ c\,a_{n1} & c\,a_{n2} & \cdots & c\,a_{nn} \end{bmatrix}$.

Then

$$|B| = c\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ c\,a_{21} & c\,a_{22} & \cdots & c\,a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c\,a_{n1} & c\,a_{n2} & \cdots & c\,a_{nn} \end{vmatrix}$$ Extract the number c from the first row (R_1)

$$= c \cdot c\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c\,a_{n1} & c\,a_{n2} & \cdots & c\,a_{nn} \end{vmatrix}$$ Extract the number c from the second row (R_2)

By repeating the process n-times, we get $|B| = c^n\, |A|$.

Example: Let $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 1 & 2\\ 3 & 4 & 1\end{bmatrix}$, let $c = 3$. Show that $3^3\,|A| = |B|$, where $B = 3A$.

Property (8):

If the corresponding elements of two rows (two columns) in a square matrix are proportional then the determinant of that matrix = zero.

Proof: Let the row (i) proportional with the row (ℓ) in the matrix $A_{n\times n}$, then:

row (ℓ) = row (i) × (fixed k) , where $k \neq 0$ (by property (7))

$$|A|=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ a_{i1} & a_{i1} & \cdots & a_{in}\\ a_{\ell 1} & a_{\ell 2} & \cdots & a_{\ell n}\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}\begin{array}{l}\\ \\ \text{row (i)}\\ \text{row }(\ell)\\ \\ \end{array} \qquad |A|=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ a_{i1} & a_{i1} & \cdots & a_{in}\\ ka_{i1} & ka_{i2} & \cdots & ka_{in}\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}\begin{array}{l}\\ \\ \text{row (i)}\\ \text{row }(\ell)\\ \\ \end{array}$$

replacing the row (ℓ)

$$|A|=k\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix} \quad \text{equal} \qquad \text{(by property (7))}$$

$|A| = k\,(0) = 0$ (by property (4))

Examples:

(1) $|A|=\begin{vmatrix}3 & 6\\ 1 & 2\end{vmatrix}=0$ the first row proportional with the second row where the ratio between them $\frac{3}{1}$

(2) $|B|=\begin{vmatrix}1 & 2 & 3\\ 2 & 0 & 6\\ 2 & 4 & 6\end{vmatrix}=0$ the first row proportional with the third row where the ratio between them $\frac{1}{2}$

or the first column proportional with the third column where the ratio between them $\frac{1}{3}$

(3) $|B|=\begin{vmatrix}4 & 0 & 6\\ 2 & -1 & 3\\ 10 & 3 & 15\end{vmatrix}=0$ the first column proportional with the third column where the ratio between them $\frac{2}{3}$

Property (9):

If add to the elements of a row (column) in a square matrix the multiplying of the corresponding elements in another row (column) by the constant k (k ≠ 0), then the value of its determinant does not change.

Proof: Let

$$A=\begin{array}{cl}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} & \begin{array}{l} \\ \\ \text{row (i)} \\ \\ \text{row (s)} \\ \\ \\ \end{array}\end{array}$$

$$B=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ ka_{i1}+a_{s1} & ka_{i2}+a_{s2} & \cdots & ka_{in}+a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ Adding the row (i) multiplying by k to the row (s)

$$|B|=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$ (by property 6)

two rows are proportional

But the determinant of the second matrix = 0 by property (8), so

$|A| = |B|$.

Example: If $A=\begin{bmatrix}-1 & 2\\ 3 & 5\end{bmatrix}$

(1) $B=\begin{bmatrix}-1 & 2+10(-1)\\ 3 & 5+10(3)\end{bmatrix}$ Adding to the second column ten times of the first column

$$|B|=\begin{vmatrix}-1 & -8\\ 3 & 35\end{vmatrix}=-35+24=-11$$

$$|A|=\begin{vmatrix}-1 & 2\\ 3 & 5\end{vmatrix}=-5-6=-11$$

$\therefore\ |A| = |B|$

(2) Adding to the first row double the second row (multiplying the second row by 2 and adding it to the first row)

$$B=\begin{bmatrix}-1+2(3) & 2+2(5)\\ 3 & 5\end{bmatrix}=\begin{bmatrix}5 & 12\\ 3 & 5\end{bmatrix}$$

Is $|A| = |B|$

$$\begin{vmatrix}-1 & 2\\ 3 & 5\end{vmatrix}=\begin{vmatrix}5 & 12\\ 3 & 5\end{vmatrix}$$

$(-11) = (-11)$

Property (10):

The determinant of a triangular matrix A equal to the product of the elements of the main diagonal, i.e. If $A = [a_{ij}]_{n\times n} \Rightarrow |A| = \det A = a_{11}a_{22} \dots a_{nn}$.

Proof: Let A be an (n×n) upper triangular matrix, to prove $|A| = \det A = a_{11}a_{22} \dots a_{nn}$. We use the method of mathematical induction.

(1) When n = 2, $A=\begin{bmatrix}a_{11} & a_{12}\\ 0 & a_{22}\end{bmatrix} \Rightarrow |A| = a_{11}a_{22}$

$\therefore$ The property true when n = 2.

(2) Suppose the property is true when k = n

$$|A|=\begin{vmatrix}a_{11} & a_{12} & a_{13} & \dots & a_{1k}\\ 0 & a_{22} & a_{23} & \dots & a_{2k}\\ 0 & 0 & a_{33} & \dots & a_{3k}\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \dots & a_{kk}\end{vmatrix}=a_{11}a_{22}a_{33}\dots a_{kk}$$

(3) Is the property stay true when n = k + 1

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & a_{1(k+1)} \\ 0 & a_{22} & a_{23} & \dots & a_{2k} & a_{2(k+1)} \\ 0 & 0 & a_{33} & \dots & a_{3k} & a_{3(k+1)} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \dots & a_{(k+1)k} & a_{(k+1)(k+1)} \end{vmatrix}$$

We open the determinant around the last row (k + 1).

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ 0 & a_{22} & a_{23} & \dots & a_{2k} \\ 0 & 0 & a_{33} & \dots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{kk} \end{vmatrix} a_{(k+1)(k+1)} = a_{11}a_{22}a_{33}\dots a_{kk}a_{(k+1)(k+1)}$$

Apply the same proof if the matrix is lower triangular matrix. **(Home work)**

Examples:

(1) $|A| = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ a & -1 & 5 \end{vmatrix} = (3)(1)(5) = 15$

(2) $|B| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 8 \end{vmatrix} = (1)(4)(8) = 32$

(3) $|A| = \begin{vmatrix} 4 & -7 & 8 \\ 0 & 5 & 4 \\ 0 & 0 & -1 \end{vmatrix} = (4)(5)(-1) = -20$

(4) $|B| = \begin{vmatrix} -7 & 0 & 0 \\ 12 & 8 & 0 \\ 15 & 6 & -2 \end{vmatrix} = (-7)(8)(-2) = 112$

Remark: The determinant of the unity matrix = 1, since it is upper and lower triangular matrix and the elements of its main diagonal $1 \times \dots \times 1 \times 1$.

Property (11):

The sum of the products of multiplying the elements of a row (column) in the square matrix A by the cofactors for the corresponding elements in another row (column) of the matrix A equal to zero.

$$\text{If}\quad A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{s1} & a_{s2} & \cdots & a_{sn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \text{row (i)} \\ \text{row (s)} \\ \\ \\ \end{array}$$

And $C_{i1}, C_{i2}, \ldots, C_{in}$ the cofactors for the row (i), then

$a_{s1} C_{i1} + a_{s2} C_{i2} + \ldots + a_{sn} C_{in} = 0$

The same way if the cofactors for any column multiply by the corresponding elements in another column respectively.

For example, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and C_{11}, C_{12}, C_{13} the cofactors for the first row respectively, if we multiply the cofactors for the first row by the elements of the second row or the third row, then

$a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} = 0$

$a_{31} C_{11} + a_{32} C_{12} + a_{33} C_{13} = 0$

Example: If $|A| = \begin{vmatrix} 2 & -1 & 2 \\ 3 & 4 & 2 \\ 3 & -2 & 1 \end{vmatrix}$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 2 \\ -2 & 1 \end{vmatrix} = 8$ (the cofactors for the element $a_{11} = 2$)

$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} = 3$ (the cofactors for the element $a_{12} = -1$)

$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 4 \\ 3 & -2 \end{vmatrix} = -18$ (the cofactors for the element $a_{13} = 2$)

$a_{21}\, C_{11} + a_{22}\, C_{12} + a_{23}\, C_{13} =$

$3(8) + 4(3) + 2(-18) = 24 + 12 - 36 = 0$

Also,

$a_{31}\, C_{11} + a_{32}\, C_{12} + a_{33}\, C_{13} =$

$3(8) + (-2)(3) + 1(-18) = 24 - 6 - 18 = 0$

Property (12):

If the elements of the matrix is complex number, then

The determinant of the conjugate matrix = conjugate determinant of the matrix, i.e. $|\bar{A}| = \overline{|A|}$

Example: Let $A = \begin{bmatrix} i & -2i \\ 2+i & 3i \end{bmatrix}$, then $\bar{A} = \begin{bmatrix} -i & 2i \\ 2-i & -3i \end{bmatrix}$

$$|\bar{A}| = \begin{vmatrix} -i & 2i \\ 2-i & -3i \end{vmatrix} = 3i^2 - 2i(2-i)$$

$$= 3i^2 - 4i + 2i^2 = 5i^2 - 4i = \boxed{-5 - 4i}$$

$$|A| = \begin{vmatrix} i & -2i \\ 2+i & 3i \end{vmatrix} = 3i^2 - (-2i)(2+i)$$

$$= 3i^2 + 4i + 2i^2 = 5i^2 + 4i = -5 + 4i$$

$$\overline{|A|} = \overline{-5+4i} = \boxed{-5-4i}$$

$$\therefore\ |\bar{A}| = \overline{|A|}$$

Examples:

(1) Find the value of the determinant of each matrix by using the properties of the determinant (without opening it mathematically)

(a) $A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ -1 & -3 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$ Multiply the third column by k= –1 and added it to the second column (Property 9)

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -2 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$ Multiply the first row by k= 1 and added it to the second row (Property 9)

$= -2$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

(b) $B = \begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Solution:

$$|B| = 2\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$ Take out a common factor the number 2 from the third row (Property 7)

$$|B| = 2\begin{vmatrix} 4 & 3 & 2 \\ 4 & 0 & 8 \\ 1 & 2 & 3 \end{vmatrix}$$ Multiply the third row by k= 1 and added it to the second row (Property 9)

$$|B| = (2)(4)\begin{vmatrix} 4 & 3 & 2 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$ Take out a common factor the number 4 from the second row (Property 7)

$$|B| = -8\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$ Replacement the first column with the second column with change the signal (Property 3)

$$|B| = -8\begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$ Multiply the second row by k= – 1 and added it to the third row (Property 9)

$$|B| = (-8)(3)\begin{vmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix}$$ Take out a common factor the number 3 from the first row (Property 7)

$$|B| = -24\begin{vmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 2 \\ 0 & \frac{-8}{3} & \frac{-1}{3} \end{vmatrix}$$ Multiply the first row by k= – 2 and added it to the third row (Property 9)

$$|B| = -24\begin{vmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$ Multiply the second row by k= $\frac{8}{3}$ and added it to the third row (Property 9)

$|B| = -24\,[(1)(1)(5)] = -120$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

(c) $C = \begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix}$

Solution:

$$|C| = 2\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$ Take out a common factor the number 2 from the first row (Property 7)

$$|C| = 2\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}$$ Multiply the first row by k= – 3 and added it to the second row (Property 9)

$|C| = (2)(4) = 8$ The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

(2) Prove the correctness of the following $\begin{vmatrix} 1 & -4 & k \\ 2 & 3 & 2k \\ 3 & 6 & 3k \end{vmatrix} = 0$ without opening it mathematically.

Proof:

$$\begin{vmatrix} 1 & -4 & k \\ 2 & 3 & 2k \\ 3 & 6 & 3k \end{vmatrix} = k\begin{vmatrix} 1 & -4 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$ Take out a common factor k from the third column (Property 7)

$= k(0) = 0$ The first column equal to the third column (Property 4)

Other proof:

Since the elements of the first column proportional with the elements of the third column and the ratio is k : 1, then

$$\begin{vmatrix} 1 & -4 & k \\ 2 & 3 & 2k \\ 3 & 6 & 3k \end{vmatrix} = 0$$ If two columns are proportional then the value of the determinant equal to zero (Property 8)

(3) Prove that $\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = 0$ without opening it mathematically.

Proof:

$$\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix} = \begin{vmatrix} a & 1 & a+b+c \\ b & 1 & a+b+c \\ c & 1 & a+b+c \end{vmatrix}$$ Multiplying the first column by k= 1 and added it to the third column (Property 9)

$$= (a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$ Take out a common factor $(a+b+c)$ from the third column (Property 7)

$$= (a+b+c)\,(0) = 0$$ If the elements of two rows (two columns) are equal in a square matrix then its determinant is equal to zero (Property 4)

(4) Prove that $\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix} = 2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ without opening it mathematically.

Proof: We take the left side

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ 2(a_1+b_1+c_1) & 2(a_2+b_2+c_2) & 2(a_3+b_3+c_3) \end{vmatrix}$$ Multiplying the first and second rows by k= 1 and added it to the third row (Property 9)

$$= 2\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+b_1+c_1 & a_2+b_2+c_2 & a_3+b_3+c_3 \end{vmatrix}$$ Take out a common factor the number 2 from the third row (Property 7)

$$=2\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ Multiplying the second row by k= –1 and added it to the third row (Property 9)

$$=2\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ Multiplying the third row by k= –1 and added it to the first row (Property 9)

$$=2\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$ Multiplying the first row by k= –1 and added it to the second row (Property 9)

$$=-2\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ Replacement the first row with the third row with change the signal (Property 3)

$$=2\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ Replacement the second row with the third row with change the signal (Property 3)

(5) Without opening the determinant. Prove that

$$\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} = -(a_1-a_2)(a_2-a_3)(a_3-a_1)$$

Proof: We take the left side

$$\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1^2-a_2^2 & a_1-a_2 & 0 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$ Multiplying the second row by k= –1 and added it to the first row (Property 9)

$$=(a_1-a_2)\begin{vmatrix} a_1+a_2 & 1 & 0 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$ Take out a common factor ($a_1 - a_2$) from the first row (Property 7)

$$=(a_1-a_2)\begin{vmatrix} a_1+a_2 & 1 & 0 \\ a_2^2-a_3^2 & a_2-a_3 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Multiplying the third row by k= –1 and added it to the second row (Property 9)

$$=(a_1-a_2)(a_2-a_3)\begin{vmatrix} a_1+a_2 & 1 & 0 \\ a_2+a_3 & 1 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Take out a common factor (a_2 – a_3) from the second row (Property 7)

$$=(a_1-a_2)(a_2-a_3)\begin{vmatrix} a_1+a_2 & 1 & 0 \\ a_3-a_1 & 0 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Multiplying the first row by k= –1 and added it to the second row (Property 9)

$$=(a_1-a_2)(a_2-a_3)(a_3-a_1)\begin{vmatrix} a_1+a_2 & 1 & 0 \\ 1 & 0 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Take out a common factor (a_3 – a_1) from the second row (Property 7)

$$=-(a_1-a_2)(a_2-a_3)(a_3-a_1)\begin{vmatrix} 1 & 0 & 0 \\ a_1+a_2 & 1 & 0 \\ a_3^2 & a_3 & 1 \end{vmatrix}$$

Replacement the first row with the second row with change the signal (Property 3)

$$=-(a_1-a_2)(a_2-a_3)(a_3-a_1)$$

The determinant of a triangular matrix equal to the product of the elements of the main diagonal = 1 (Property 10)

(6) Find the value of x without opening the determinant mathematically.

$$\begin{vmatrix} 2 & 2-x \\ 1+x & 0 \end{vmatrix}=0$$

Proof:

$$(1+x)\begin{vmatrix} 2 & 2-x \\ 1 & 0 \end{vmatrix}=0$$

Take out a common factor (1 + x) from the second row (Property 7)

$$-(1+x)\begin{vmatrix} 1 & 0 \\ 2 & 2-x \end{vmatrix}=0$$

Replacement the first row with the second row with change the signal (Property 3)

$$-(1+x)(2-x)=0$$

The determinant of a triangular matrix equal to the product of the elements of the main diagonal (Property 10)

$(1 + x)(2 – x) = 0$ Multiply each side by – 1

$1+x=0 \Rightarrow x=-1$ or $2-x=0 \Rightarrow x=2$

(7) Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}0 & 5\\ 4 & 3\end{bmatrix}$. Is $|A+B| = |A| + |B|$?

Solution: $A+B=\begin{bmatrix}1 & 7\\ 7 & 7\end{bmatrix} \Rightarrow |A+B| = 7-49=-42$

$|A| = 4-6=-2$, $|B| = 0-20=-20$.

So we get, $|A+B| \neq |A| + |B|$ since $-42 \neq -2 + -20$ (i.e. $-42 \neq -22$)

Exercises:

(1) Find the value of the determinant of each matrix by using the properties of the determinant or by reduction to triangular matrix (without opening it mathematically)

(a) $\begin{vmatrix}1 & 0 & -3\\ 2 & 2 & -1\\ 4 & 1 & 4\end{vmatrix}$ **(b)** $\begin{vmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 4\end{vmatrix}$ **(c)** $\begin{vmatrix}2 & 3 & 4\\ 0 & 0 & 3\\ 0 & 2 & 1\end{vmatrix}$ **(d)** $\begin{vmatrix}3 & 4 & 2\\ 2 & 5 & 0\\ 3 & 0 & 0\end{vmatrix}$ **(e)** $\begin{vmatrix}4 & -3 & 5\\ 5 & 2 & 0\\ 2 & 0 & 4\end{vmatrix}$

(f) $\begin{vmatrix}2 & 1 & 2\\ -1 & 0 & 3\\ x & y & z\end{vmatrix}$ **(g)** $\begin{vmatrix}3 & 1 & -2\\ 2a & 2b & 2c\\ 2 & 1 & 2\end{vmatrix}$ **(h)** $\begin{vmatrix}1 & 4 & 3 & 1\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 0\\ 0 & 3 & 1 & 4\end{vmatrix}$ **(j)** $\begin{vmatrix}4 & 0 & 0 & 0\\ -1 & 2 & 0 & 0\\ 1 & 2 & -3 & 0\\ 1 & 5 & 3 & 5\end{vmatrix}$

(2) Find the value of a if $\begin{vmatrix}a-1 & 2\\ 3 & a-2\end{vmatrix}=0$ without opening it mathematically?

(3) If $|A| = -5$ and $|B| = 2$, find the value of $|A|^2$, $|A^4|$, $|A^2B^2|$.

(4) Prove that $\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ y+z & z+x & x+y\end{vmatrix}=0$ without opening it mathematically.

(5) Prove that $\begin{vmatrix}2a_1+b_1 & 2b_1+c_1 & 2c_1+a_1\\ 2a_2+b_2 & 2b_2+c_2 & 2c_2+a_2\\ 2a_3+b_3 & 2b_3+c_3 & 2c_3+a_3\end{vmatrix}=9\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}$ without opening it mathematically

(6) Prove that $\begin{vmatrix} yz & x^2 & x^2 \\ y^2 & xz & y^2 \\ z^2 & z^2 & xy \end{vmatrix} = \begin{vmatrix} yz & xy & xz \\ xy & xz & yz \\ xz & yz & xy \end{vmatrix}, xyz \neq 0$ without opening it mathematically.

(7) Without opening the determinant, show that the equation obtaining from the value of the following determinant is of degree two and has the roots a and b

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = 0, a \neq b$$

(8) If the points (x_1,y_1), (x_2,y_2), (x_3,y_3) on a straightened one, then prove that

$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ without opening it mathematically.

(9) Find the value of x for all the following without opening it mathematically

(a) $\begin{vmatrix} x-2 & 7 \\ 3 & x+2 \end{vmatrix} = 0$ **(b)** $\begin{vmatrix} x-1 & 0 & 1 \\ 0 & x-1 & 0 \\ 1 & 0 & x-1 \end{vmatrix} = 0$

(10) If $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix}$, show that

(a) $AB \neq BA$.
(b) $|AB| = |A| \cdot |B|$.
(c) $|AB| = |BA|$.

(11) Prove that $|AB| = |BA|$ for any matrices A and B?

(12) Let k be any real number and A is a matrix of degree $n \times n$, prove that $|kA| = k^n |A|$

(13) Without opening the determinant prove that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}, a \neq 0, b \neq 0, c \neq 0.$$

Note: Multiply and divided by *abc*.

(14) If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -2$, find $\begin{vmatrix} a_1 & -\frac{1}{2}a_3 & a_2 & a_3 \\ b_1 & -\frac{1}{2}b_3 & b_2 & b_3 \\ c_1 & -\frac{1}{2}c_3 & c_2 & c_3 \end{vmatrix}$?

Chapter three
Inverse of Matrix

Definition: The inverse of the square matrix A denoted by A^{-1} it is a matrix has the same degree of the matrix A such that $A A^{-1} = A^{-1}A = I$.

Or A is called **inverse matrix** or **invertible** if there exists a matrix B such that AB = BA = I, in this case the B is called the inverse of the matrix A and written $B = A^{-1}$.

Remark: The degree of matrix B (the inverse of the matrix A) must be equal to the degree of the matrix A and the rectangular (non-square) matrix has no inverse.

Examples:

(1) Find the inverse of the matrix $A = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$?

Solution: Let $B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A^{-1}$ is the inverse of the matrix A.

It must be prove that $AB = BA = I_2$.

$$\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2a+b & -2c+d \\ 3a-b & 3c-d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$-2a + b = 1 \quad \ldots(1)$ $\qquad\qquad$ $-2c + d = 0 \quad \ldots(3)$

$3a - b = 0 \quad \ldots(2)$ $\qquad\qquad$ $3c - d = 1 \quad \ldots(4)$

——————— by addition $\qquad$ ——————— by addition

$a = 1$ $\qquad\qquad$ $c = 1$

so, $b = 3$ $\qquad\qquad$ so, $d = 2c \Rightarrow d = 2(1) = 2$

$\therefore\ B = A^{-1} = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$ which is the inverse of the matrix A.

Investigation:

It must be $AB = I_2 \Rightarrow AB = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

It must be $BA = I_2 \Rightarrow BA = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

$\therefore\ B = A^{-1}$

(2) Find the inverse of the matrix $A=\begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix}$ if exists?

Solution: Let $B=\begin{bmatrix}a & c\\ b & d\end{bmatrix}=A^{-1}$ is the inverse of the matrix A.

It must be prove that $AB = BA = I_2$.

$$AB=\begin{bmatrix}2 & -4\\ 1 & -2\end{bmatrix}\begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}2a-4b & 2c-4d\\ a-2b & c-2d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$2a-4b=1 \quad \ldots(1)$ $\qquad$ $2c-4d=0 \quad \ldots(3)$

$a-2b=0 \quad \ldots(2)$ $\qquad$ $c-2d=1 \quad \ldots(4)$

by multiply the equations (2) and (4) by 2

$2a-4b=1 \quad \ldots(1)$ $\qquad$ $2c-4d=0$

$2a-4b=0 \quad \ldots(2)$ $\qquad$ $2c-4d=2$

by subtraction $\qquad$ by subtraction

$0=1$ $\qquad$ $0=-2$

This is not possible. So there is no solution to the system.

$\therefore$ We hypothesized that there is an inverse of the matrix A is not true.

$\therefore$ There is no inverse to the matrix A.

Properties:

(1) The square matrix A is called not invertible if $\det(A) = |A| = 0$.

(2) The square matrix A is called invertible if $\det(A) = |A| \neq 0$ and $|A|=\dfrac{1}{|A^{-1}|}$.

Proof:

Suppose the matrix A invertible $\Leftrightarrow$ there exists a matrix B, such that

$AB = BA = I_n \;\Rightarrow\; |AB| = |I_n|$ (taking determinant for each side)

$\Rightarrow\; |A|\,|B| = 1$ (previous theorems $|AB| = |A|\,|B|$ and $\det(I_n) = 1$)

$\Rightarrow\; |A| \neq 0$ and $|B| \neq 0$

So we get $|A|=\dfrac{1}{|B|} \;\Rightarrow\; |A|=\dfrac{1}{|A^{-1}|}$ or $|A^{-1}|=\dfrac{1}{|A|}$

Remark: Sometimes invertible matrix is called (inverse matrix) or non-singular or has an inverse.
And not invertible matrix is called (non-inverse matrix), singular, or has no inverse.

Examples:

(1) $A = \begin{bmatrix} 3 & 1 \\ 12 & 4 \end{bmatrix}$ not invertible matrix since $|A| = 0$.

(2) $B = \begin{bmatrix} 5 & -3 \\ 2 & 7 \end{bmatrix}$ invertible matrix since $|B| \neq 0$.

(3) Is the zero matrix has inverse (invertible)?

Solution: The zero matrix does not have an inverse because its determinant = zero.

Or If we suppose the matrix B is the inverse of the zero matrix O_n, then

$O_n B = O_n \neq I_n$ which is a contradiction

$\therefore$ The zero matrix does not have an inverse

(4) Is the identity matrix has inverse (invertible)?

Solution: Yes, the identity matrix has inverse which is the same matrix.

Suppose the matrix B is the inverse of the identity matrix I_n

$I_nB = BI_n = I_n$ (the definition of an inverse matrix)

$B = B = I_n$ (previous theorem $I_nA = A$)

$\therefore$ $B = I_n$

Therefore, the inverse of the identity matrix is the identity matrix itself.

(5) If $A = A^{-1}$. Show that $\det(A) = \pm 1$.

Solution: $A = A^{-1} \Rightarrow A\,A = I_n$ (the definition of an inverse matrix)

$\Rightarrow (\det(A))^2 = \det(I_n)$ (taking determinant for each side)

$\Rightarrow (\det(A))^2 = 1$ ($\det(I_n) = 1$)

$\Rightarrow \det(A) = \pm 1$

(6) If A is nonsingular matrix such that $A^2 = A$. Find the value of det(A) ?

Solution: Since A is nonsingular, so A^{-1} exists.

$A^2 = A$

$A^2 A^{-1} = A A^{-1}$ (multiply each side by A^{-1})

$A A A^{-1} = A A^{-1}$ ($A^2 = A A$)

$A I_n = I_n$ ($A A^{-1} = I_n$)

$A = I_n$ (previous theorem $A I_n = A$)

$\det(A) = \det(I_n)$ (taking determinant for each side)

$\det(A) = 1$ ($\det(I_n) = 1$)

Theorem: If A is an invertible matrix, then A^t is also an invertible matrix and $(A^t)^{-1} = (A^{-1})^t$.

Proof: Since A is invertible matrix, $\exists\ A^{-1}$ (n×n) matrix) such that

$A A^{-1} = A^{-1} A = I_n$

$(A A^{-1})^t = (A^{-1} A)^t = (I_n)^t$ (taking transpose for each side)

$(A^{-1})^t A^t = A^t (A^{-1})^t = I_n$ (by previous theorems, $(I_n)^t = I_n$ and $(AB)^t = B^tA^t$)

$\therefore$ A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Remark: The previous method of finding the inverse of a matrix (by definition) is impractical to find the inverse of a matrix of degrees higher than (2×2). But there are other methods to do this.

Finding the inverse of a matrix (by the adjoint matrix method)

The Method of Adjoint Matrix

If $A = [a_{ij}]_{n\times n}$ such that $|A| \neq 0$ and $C = [c_{ij}]_{n\times n}$ represents the matrix of coefficients cofactor for the matrix A.

Theorem: without proof

The adjoint matrix for the matrix A is transposed matrix of the coefficients cofactor for the matrix A and denoted by adj(A), i.e. $\text{adj}(A) = C^t(A) = [c_{ij}]^t_{n\times n}$

The inverse of the matrix A is $A^{-1} = \dfrac{\text{adj}(A)}{|A|} = \left[\dfrac{c_{ij}}{|A|}\right]^t = \dfrac{[c_{ij}]^t}{|A|}$.

Examples:

(1) Find the inverse of the matrix $A=\begin{bmatrix}4 & 2\\ 3 & 2\end{bmatrix}$ by adjoint matrix method?

Solution:

$$|A|=\begin{vmatrix}4 & 2\\ 3 & 2\end{vmatrix}=2$$

$c_{11}=(-1)^{1+1}|2|=2$, $c_{12}=(-1)^{1+2}|3|=-3$

$c_{21}=(-1)^{2+1}|2|=-2$, $c_{22}=(-1)^{2+2}|4|=4$

$$C(A)=\begin{bmatrix}2 & -3\\ -2 & 4\end{bmatrix} \Rightarrow adj(A)=C^t(A)=\begin{bmatrix}2 & -2\\ -3 & 4\end{bmatrix}$$

The inverse of the matrix A is

$$A^{-1}=\frac{adj(A)}{|A|}=\frac{\begin{bmatrix}2 & -2\\ -3 & 4\end{bmatrix}}{2}=\begin{bmatrix}1 & -1\\ -\frac{3}{2} & 2\end{bmatrix}$$

Investigation:

$$AA^{-1}=\begin{bmatrix}4 & 2\\ 3 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -\frac{3}{2} & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \text{ and } A^{-1}A=\begin{bmatrix}1 & -1\\ -\frac{3}{2} & 2\end{bmatrix}\begin{bmatrix}4 & 2\\ 3 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

(2) Find the inverse of the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}$ by adjoint matrix method ?

Solution:

$$|A|=\begin{vmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{vmatrix}=(12+6+6)-(9+8+6)=24-23=1$$

$$c_{11}=(-1)^{1+1}\begin{vmatrix}3 & 3\\ 2 & 4\end{vmatrix}=6 \ , \ c_{12}=(-1)^{1+2}\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}=-1 \ , \ c_{13}=(-1)^{1+3}\begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix}=-1$$

$$c_{21}=(-1)^{2+1}\begin{vmatrix}2 & 3\\ 2 & 4\end{vmatrix}=-2 \ , \ c_{22}=(-1)^{2+2}\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}=1 \ , \ c_{23}=(-1)^{2+3}\begin{vmatrix}1 & 2\\ 1 & 2\end{vmatrix}=0$$

$c_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} = -3$, $c_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$, $c_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$

$$C(A) = \begin{bmatrix} 6 & -1 & -1 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \ \text{adj}(A) = C^t(A) = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The inverse of the matrix A is

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 6 & -2 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Theorem: without proof

If $A = [a_{ij}]_{n\times n}$ then $A(\text{adj}(A)) = (\text{adj}(A))A = (\det(A))\, I_n$.

Exercises:

(1) Find the inverse for each of the following matrices (if exists)

(a) $A = \begin{bmatrix} -2 & 1 \\ 3 & 1 \end{bmatrix}$ **(b)** $A = \begin{bmatrix} 1 & 3 \\ 5 & 9 \end{bmatrix}$ **(c)** $B = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ **(d)** $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

(2) For any square matrix of degree $(n\times n)$ show that $\text{adj}(A^t) = (\text{adj } A)^t$.

(3) If A is a square matrix and $|A| \neq 0$. Prove that A has inverse and $A^{-1} = \frac{1}{|A|}\text{adj}(A)$.

Theorem: If A is invertible matrix, then αA invertible matrix for any scalar number $\alpha \neq 0$ and $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.

Proof: We must prove $(\alpha A)(\frac{1}{\alpha}A^{-1}) = (\frac{1}{\alpha}A^{-1})(\alpha A) = I_n$

$(\alpha A)(\frac{1}{\alpha}A^{-1}) = (\alpha\frac{1}{\alpha})(AA^{-1})$ (by previous theorem (rA)(sB) = (rs)(AB))

$= 1 \cdot I_n = I_n$ (A is invertible matrix so $A A^{-1} = I_n$)

In the same way we prove that $(\frac{1}{\alpha}A^{-1})(\alpha A) = I_n$.

$\therefore\ (\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$

Theorem: If the matrix A has inverse, then this inverse is unique.
Proof: Let B and C are the inverses for the matrix A.
By the definition of an inverse matrix, we have

$AB = BA = I_n$ …(1)

$AC = CA = I_n$ …(2)

$B = B\,I_n$ (by previous theorem $A = AI_n$)

$B = B(AC)$ (by (2))

$B = (BA)C$ (the multiplication of matrices is associative)

$B = I_nC$ (by (1))

$B = C$ (by previous theorem $AI_n = A$)

$\therefore$ The inverse matrix is unique.

Theorem: If A is invertible matrix, then A^{-1} is invertible matrix. **(Home work)**

Theorem: If A is invertible matrix, then $(A^{-1})^{-1} = A$.
Or The inverse of the inverse matrix is equal to the matrix itself.
Proof:

$A^{-1}(A^{-1})^{-1} = (A^{-1})^{-1}A^{-1} = I_n$ (the definition of an inverse matrix)

But $A^{-1}A = A\,A^{-1} = I_n$

So each of A and $(A^{-1})^{-1}$ are inverse for the matrix A^{-1}

Since the inverse of the matrix is unique, so we get $(A^{-1})^{-1} = A$

Theorem: If A and B are invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$.
Proof: We must prove that $(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I_n$

$(AB)(B^{-1}A^{-1}) = ((AB)B^{-1})A^{-1}$ (the multiplication of matrices is associative)

$= (A(BB^{-1}))A^{-1}$ (the multiplication of matrices is associative)

$= (AI_n)A^{-1}$ (B^{-1} is the inverse matrix of the matrix B)

$= AA^{-1}$ (by previous theorem $AI_n = A$)

$= I_n$ (the definition of an inverse matrix)

In the same way we prove $(B^{-1}A^{-1})(AB) = I_n$.

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, prove that $(AB)^{-1} = B^{-1}A^{-1}$

Proof: $A^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$, $AB = \begin{bmatrix} 2 & -1 \\ 4 & -1 \end{bmatrix}$, $B^{-1}A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$,

$(AB)^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -2 & 1 \end{bmatrix}$. So that $(AB)^{-1} = B^{-1}A^{-1}$

Corollary (1): If A, B and C are invertible matrices, then ABC invertible matrix. i.e. $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$.

Proof:

$(ABC)^{-1} = ((AB)C)^{-1}$ (the multiplication of matrices is associative)

$= C^{-1} (AB)^{-1}$ (by previous theorem $(AB)^{-1} = B^{-1} A^{-1}$)

$= C^{-1} B^{-1} A^{-1}$ (by previous theorem $(AB)^{-1} = B^{-1} A^{-1}$)

Corollary (2): If $A_1, A_2, \ldots A_n$ are invertible matrices of the same degree, then $A_1A_2\ldots A_n$ is invertible matrix where n is positive integer number and $(A_1A_2\ldots A_n)^{-1} = A_n^{-1}\ldots A_2^{-1} A_1^{-1}$ **(Home work)**

Corollary (3): If A is invertible matrix then A^n is invertible matrix for any positive integer number n and $(A^n)^{-1} = (A^{-1})^n$.

Proof: By using corollary (2)

Let $A_1 = A_2 = \ldots = A_n = A$

Since $(A_1A_2\ldots A_n)^{-1} = A_n^{-1}\ldots A_2^{-1} A_1^{-1}$

So $(AA\ldots A)^{-1} = A^{-1}\ldots A^{-1} A^{-1}$

$(A^n)^{-1} = (A^{-1})^n$

Remark: We can prove Corollary (3) by using mathematical induction method

Definition: If A is invertible matrix, we define

$A^n = (A^{-1})^{-n}$ where n is a negative integer number

$= \underbrace{A^{-1}A^{-1}...A^{-1}}_{(-n)-\text{times}}$

$A^0 = I$

Examples:

(1) Let A square matrix of degree (n×n) such that $A^2 + 2A + I_n = O_n$. Show that A is invertible matrix and find its inverse?

Solution:

$A^2 + 2A + I_n = O_n$ (given)

$A^2 + 2A = O_n - I_n$

$A^2 + 2A = -I_n$

$A^2 + 2A I_n = -I_n$ (by previous theorem $AI_n = I_nA = A$)

$-A^2 - 2A I_n = I_n$

$A(-A - 2I_n) = I_n$

$A(-A - 2I_n) = (-A - 2I_n)A = I_n$

From that we have A is invertible matrix and $A^{-1} = -A - 2I_n$

(2) If $A = \lambda I_n$ where λ is scalar number. Prove that

A is invertible matrix $\Leftrightarrow$ $\lambda \neq 0$ and $A^{-1} = \frac{1}{\lambda} I_n$

Proof: $\Rightarrow$ Let A is invertible matrix. To prove $\lambda \neq 0$

Suppose $\lambda = 0$ $\Rightarrow$ $A = \lambda I_n$

$\Rightarrow$ $A = O_n$

But O_n not invertible matrix (since their determinant equal to zero)

$\therefore$ A is not invertible matrix which is a contradiction.

$\therefore$ Our hypothesis $\lambda = 0$ not true, so $\lambda \neq 0$.

$\Leftarrow$ Let $\lambda \neq 0$. To prove A is invertible matrix

$A = \lambda I_n$ (given)

$A^{-1} = \frac{1}{\lambda} I_n^{-1}$ (by previous theorem (if A invertible matrix then αA is invertible matrix and $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$))

$A^{-1} = \frac{1}{\lambda} I_n$ (The inverse matrix of identity matrix is the identity matrix itself)

(3) Let A be a square matrix such that $A^k = O$ for some positive integer values, this matrix is called nilpotent. Show that A is not invertible matrix.

Solution: Suppose A is invertible matrix

$\Rightarrow$ A^k is invertible matrix (by If A is invertible matrix then A^n is invertible matrix for any positive integer number n)

$\Rightarrow$ O is invertible matrix which is a contradiction

$\therefore$ A is not invertible matrix.

Exercise: Give example for nilpotent matrix of degree (2×2).

(4) If A and B are invertible matrices of any degree. Is A + B invertible matrix? If it is that, is $(A + B)^{-1} = A^{-1} + B^{-1}$?

Solution: If A and B are invertible matrices it is not necessary A + B invertible matrix.

For example: let $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ are invertible matrices, but

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ not invertible matrix.

It may be A, B and A + B are invertible matrices, but not necessary that $(A + B)^{-1} = A^{-1} + B^{-1}$.

For example: $A=\begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$, $A+B=\begin{bmatrix}1 & 0\\ 0 & 6\end{bmatrix}$

$A^{-1}=\begin{bmatrix}-1 & 0\\ 0 & \frac{1}{3}\end{bmatrix}$, $B^{-1}=\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix}$, $(A+B)^{-1}=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{6}\end{bmatrix}$

But $(A + B)^{-1} \neq A^{-1} + B^{-1}$, since

$$\begin{bmatrix}1 & 0\\ 0 & \frac{1}{6}\end{bmatrix} \neq \begin{bmatrix}-1 & 0\\ 0 & \frac{1}{3}\end{bmatrix} + \begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{3}\end{bmatrix} = \begin{bmatrix}-\frac{1}{2} & 0\\ 0 & \frac{2}{3}\end{bmatrix}$$

(5) Let A + I invertible matrix. Show that (A – I) and $(A + I)^{-1}$ commute matrices?
i.e. $(A - I)(A + I)^{-1} = (A + I)^{-1}(A - I)$ **(Home work)**

Remark: The matrix A which satisfy $A^2 = I$ is called involutory matrix.
(It is the matrix which is if it is multiply by itself the value equal to the identity matrix)
The involutory matrix is the inverse of itself.
The identity matrix is involutory matrix.

Example:

The matrix $A=\begin{bmatrix}-1 & -2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$ is involutory matrix, since $A^2 = I$.

Exercises:

(1) Let A be a square invertible matrix. Prove that $|A^k| = |A|^k$, where k is a negative integer number.

(2) Let A be a square matrix and $A^2 = O$. Prove that (I – A) is invertible matrix.

(3) If A, B, C are square matrices of degree (n×n) and A is invertible matrix, then $AB = AC \Rightarrow B = C$

Definition: We say for the matrix of degree m×n is of **reduced echelon form** denoted by (r.e.f.) if satisfy the following conditions:

(a) Rows consisting entirely of zeros, if exists, are appear at the bottom of the matrix.

(b) The first non-zero entry in every row that is not completely composed of zeros is equal to 1 and it is called "the leading entry for that row".

(c) If the sequential rows i and i + 1 not completely composed of zeros, the leading entry for the row i + 1 is appears on the right of the leading entry for the row i.

(d) If there exist a column has a leading entry for a row, then all other entries for that column equal to zero.

Remark: The matrix of the reduced echelon form may be not containing entirely rows of zeros.

Examples:

(1) $A = \begin{bmatrix} 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Matrix of echelon form (e.f.) but not of the reduced echelon form (r.e.f.) since the leading entry for the first row =2 ≠ 1 (the condition (b) not satisfy).

(2) $B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Matrix not of the reduced echelon form (r.e.f.) since the leading entry for the second row not on the right of the leading entry for the first row (the condition (c) not satisfy).

(3) $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Matrix of the reduced echelon form (r.e.f.)

(4) $D = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{bmatrix}$ Matrix of the reduced echelon form (r.e.f.)

(5) $E = \begin{bmatrix} 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ Matrix of the reduced echelon form (r.e.f.)

(6) $F = \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Matrix of the reduced echelon form (r.e.f.)

(7) $\begin{bmatrix} 1 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ Matrix not of the reduced echelon form (r.e.f.) since the condition (a) not satisfies.

(8) $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ Matrix not of the reduced echelon form (r.e.f.) since the condition (b) not satisfies.

(9) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & -5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Matrix not of the reduced echelon form (r.e.f.) since the condition (c) not satisfies.

(10) $\begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Matrix not of the reduced echelon form (r.e.f.) since the condition (d) not satisfies.

Linear Systems and Gaussian Elimination Method

Definition of the linear system: It is a set of (m) of linear equations and (n) of unknowns (variables), and it is expressed in the following form:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

It is written briefly as follows AX = B, where

A is the coefficients matrix (coefficients of the unknowns)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

X is the column of the unknowns $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$

B is the absolute quantities column $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

Examples:

(1) Consider the following linear system

$$\left.\begin{aligned} 2x_1 - 3x_2 &= 8 \\ 3x_1 + x_2 &= 1 \end{aligned}\right\} \qquad \ldots(1)$$

The absolute quantities column is $B = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$, the column of the unknowns is $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, the coefficients matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}$, so we can write the system as follows:

$$\begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

When we compute the multiplication operator we get the system of the equations (1).

(2) $x + y + z = 1$

$-y + 2z = 0$

$x + 2y \quad = 2$

The absolute quantities column is $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, the column of the unknowns is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

the coefficients matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, so we can write the system as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Definition the solution of the linear system: It is a set of the value of the unknowns which satisfy each equation in the system.

Remark: Each set of linear equation (AX=B) can be represent it by the matrix [A:B], this matrix called augumented matrix.

Examples:

(1) Consider the following linear system

$3x_1 + 5x_2 - 2x_3 = 5$

$4x_1 - x_2 + 3x_3 = 14$

$x_1 + x_2 + x_3 = 7$

The augumented matrix is $\begin{bmatrix} 3 & 5 & -2 & \vdots & 5 \\ 4 & -1 & 3 & \vdots & 14 \\ 1 & 1 & 1 & \vdots & 7 \end{bmatrix}$

(2) If the augumented matrix is $\begin{bmatrix} 1 & 1 & 1 & \vdots & 2 \\ 0 & -2 & 1 & \vdots & 3 \\ 3 & 0 & 4 & \vdots & 1 \end{bmatrix}$, so the set of the equations of the system

$x_1 + x_2 + x_3 = 2$

$-2x_2 + x_3 = 3$

$3x_1 \quad + 4x_3 = 1$

Remark: From above we note that the system of linear equations transfer to a system simple than it (equivalent system) by using some operations on equations. Also, the representation of the system by an augumented matrix means that each row in the matrix represents an equation, and for this the system can be solved by using the augumented matrix and these operations on the augumented matrix correspond to (similarity) operations on the equations. These operations on an augumented matrix are called **elementary row operations** and denoted by (e.r.o.) which are:

(1) We multiply the row (i) by a constant number α not equal to zero. It is denoted by $\{R_i = \alpha r_i\}$.

(2) We multiply the row (i) by a constant number α not equal to zero and add it to the row (j) where $(i \neq j)$ is denoted by $\{R_j = r_j + \alpha r_i\}$.

(3) Replace row (i) with row (j) or vice versa and denote it by $\{r_j \leftrightarrow r_i\}$.

The above method which is used to solve any linear system is called Gaussian elemination method.

Gauss-Jordan Reduction Method

Solving the system AX = B in this method depends on converting the augumented matrix to the reduced echelon form.

Examples: (1) Solve the following linear equations systems by Gauss-Jordan reduction method

(a) $x - 4y = 11$
$x - 2y = 7$

$$[A:B]=\left[\begin{array}{cc:c}1 & -4 & 11\\ 1 & -2 & 7\end{array}\right]\xrightarrow{R_2=r_2-r_1}\left[\begin{array}{cc:c}1 & -4 & 11\\ 0 & 2 & -4\end{array}\right]\xrightarrow{R_2=\frac{1}{2}r_2}\left[\begin{array}{cc:c}1 & -4 & 11\\ 0 & 1 & -2\end{array}\right]$$

$$\xrightarrow{R_1=r_1+4r_2}\left[\begin{array}{cc:c}1 & 0 & 3\\ 0 & 1 & -2\end{array}\right]$$. The solution is $x = 3$ and $y = -2$.

(b) $2x + 3y = 0$
$4x - y = 0$

$$[A:B]=\left[\begin{array}{cc:c}2 & 3 & 0\\ 4 & -1 & 0\end{array}\right]\xrightarrow{R_2=r_2-2r_1}\left[\begin{array}{cc:c}2 & 3 & 0\\ 0 & -7 & 0\end{array}\right]\xrightarrow{R_2=\frac{-1}{7}r_2}\left[\begin{array}{cc:c}2 & 3 & 0\\ 0 & 1 & 0\end{array}\right]$$

$$\xrightarrow{R_1=r_1-3r_2}\left[\begin{array}{cc:c}2 & 0 & 0\\ 0 & 1 & 0\end{array}\right]\xrightarrow{R_1=\frac{1}{2}r_1}\left[\begin{array}{cc:c}1 & 0 & 0\\ 0 & 1 & 0\end{array}\right]$$

The solution is $x = 0$ and $y = 0$

(2) Solve (if possible) the following system
$x_1 + x_2 = 2$
$2x_1 + 4x_2 = -1$

Solution:

$$[A:B]=\left[\begin{array}{cc:c}1 & 1 & 2\\ 2 & 4 & -1\end{array}\right]\xrightarrow{R_2=r_2-2r_1}\left[\begin{array}{cc:c}1 & 1 & 2\\ 0 & 2 & -5\end{array}\right]\xrightarrow{R_2=\frac{1}{2}r_2}\left[\begin{array}{cc:c}1 & 1 & 2\\ 0 & 1 & -5/2\end{array}\right]$$

$$\xrightarrow{R_1=r_1-r_2}\left[\begin{array}{cc:c}1 & 0 & 9/2\\ 0 & 1 & -5/2\end{array}\right]\Rightarrow X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}9/2\\ -5/2\end{bmatrix}$$ or $x_1 = 9/2, x_2 = -5/2$

(3) Solve (if possible) the following system

$2x - 3y = 8$

$3x + y = 1$ **(Home work)**

(4) Solve (if possible) the following system

$x_1 + x_2 + x_3 = 1$

$2x_1 + 3x_2 + 3x_3 = 3$

$x_1 + 2x_2 + 2x_3 = 5$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 3 & 3 & \vdots & 3 \\ 1 & 2 & 2 & \vdots & 5 \end{bmatrix} \xrightarrow[R_3 = r_3 - r_1]{R_2 = r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 4 \end{bmatrix} \xrightarrow{R_3 = r_3 - r_2} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 0 & 0 & \vdots & 3 \end{bmatrix}$$

The third row means $0x_1 + 0x_2 + 0x_3 = 3$ which implies that $0 = 3$ which is impossible so the system has no solution.

Remark: When solving this system by elimination method in solving equations, you find the same answer (that the system is inconsistent) that is, it has no solution.

Example: Solve (if possible) the following system

$x + y + z = 1$

$2x + y + z = 1$

$3x + y + z = 1$

Solution:

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 2 & 1 & 1 & \vdots & 1 \\ 3 & 1 & 1 & \vdots & 1 \end{bmatrix} \xrightarrow[R_3 = r_3 - 3r_1]{R_2 = r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -1 & \vdots & -1 \\ 0 & -2 & -2 & \vdots & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 = r_3 - 2r_2} \begin{bmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & -1 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \xrightarrow{R_1 = r_1 + r_2} \begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & -1 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

This mean $x = 0$

$-y - z = -1$

$y + z = 1$

Which mean when we take value for y we can find the value of z.
This mean the system has infinite solutions.
The solution is $x = 0$, $z = 1 - y$, y = any real number.

Or the solution $\{(0,a,1-a): a \in \mathbb{R}\}$.
If $a = 1$, so (0,1,0) is solution, or if $a = 2$, so (0,2, –1) is solution, ….and so on.

Equivalent Matrices

If the two matrices A and B are of the same degree, then A is a row equivalent with B If B can be obtained from A with an operation or (Elementary Row Operations) (e.r.o.), the equivalence is symbolized by (~) we say (A ~ B).

We can note that:

(1) For any matrix A, then A ~ A.

(2) For any matrices A and B, if A ~ B, then B ~ A.

(3) For any matrices A B and C, if A ~ B and B ~ C, then A ~ C.

Examples for the equivalent matrices:

(1) If $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 = r_3 - r_2} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 = 2r_2} \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 2 \\ -1 & 0 & 0 \end{bmatrix} = B$$

So A~B (row equivalent).

(2) Show that $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \sim I_2$

Solution:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \xrightarrow{R_2 = r_2 + r_1} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 = r_1 + 2r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$\therefore A \sim I_2$

Exercise: Show that the matrices $A=\begin{bmatrix}1 & 2 & 1\\ 2 & 3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0 & -3\\ 0 & -1 & -2\end{bmatrix}$ are row equivalent?

Theorem: (without proof)

The square matrix of degree (n×n) has inverse if it is row equivalent for the identity matrix.

For example the matrix $A=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$ in example (2) above has inverse since it is row equivalent to I_2.

Theorem: (without proof)

If A is a square matrix of degree n×n, then the linear system $A_{n\times n}\cdot X_{n\times 1}=B_{n\times 1}$ has unique solution if and only if $|A|\neq 0$.

Remark: If A is a square matrix of degree n×n, then the linear system $A_{n\times n}\cdot X_{n\times 1}=B_{n\times 1}$ has infinite number of the solutions or has no solution if and only if $|A|=0$.

Examples:

(1) Let $A=\begin{bmatrix}2 & -3\\ 2 & -3\end{bmatrix}\Rightarrow |A|=0$, so the linear system

$2x-3y=8$
$2x-3y=3$
———————— by subtraction
$0=6$
Has no solution

(2) Let $A=\begin{bmatrix}2 & -3\\ 4 & -6\end{bmatrix}\Rightarrow |A|=0$, so the linear system

$2x-3y=8$
$4x-6y=16$
———————— multiply the first equation by (2)

$$4x - 6y = 16$$
$$4x - 6y = 16$$
———————— by subtraction
$$0 = 0$$

Has infinite number of solutions.

Exercises: Solve the following linear systems

(1) $x + y + z = 1$
$x + 2y + 3z = -1$
$x + 4y + 4z = -9$

(2) $x + 2y + 2z = 1$
$x + 5y + 2z = 4$
$x + 8y + 2z = 8$

(3) $x + y = 3$
$2x - y = 1$

(4) Find the value of a which make the following linear systems have no solution

(a) $x - 2y = 5$
$3x + a\,y = 1$

(b) $x - y + 2z = 3$
$2x + ay + 3z = 1$
$-3x + 3y + z = 4$

Gramer's Rule

This method using to find the solutions of the linear system which its coefficients matrix is square matrix and its determinant $\neq 0$.

Theorem: (without proof)

Let $A_{n\times n} \cdot X_{n\times 1} = B_{n\times 1}$ and $|A| \neq 0$, then $x_j = \dfrac{|A_j|}{|A|}$, $j = 1,2,\ldots,n$, where

A_j is the matrix obtained it by replace the column j for the matrix A by the absolute quantities column B.

Examples: Using the Gramer's rule to find the solution for the following systems:

(1) $x_1 - 2x_2 = 8$
$5x_1 + 2x_2 = 4$

Solution:

$$|A| = \begin{vmatrix} 1 & -2 \\ 5 & 2 \end{vmatrix} = 12 \quad , \quad |A_1| = \begin{vmatrix} \boxed{\begin{matrix} 8 \\ 4 \end{matrix}} & \begin{matrix} -2 \\ 2 \end{matrix} \end{vmatrix} = 24 \quad , \quad |A_2| = \begin{vmatrix} \begin{matrix} 1 \\ 5 \end{matrix} & \boxed{\begin{matrix} 8 \\ 4 \end{matrix}} \end{vmatrix} = -36$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{24}{12} = 2 \quad , \quad x_2 = \frac{|A_2|}{|A|} = \frac{-36}{12} = -3$$

The solution is $x_1 = 2$, $x_2 = -3$

(2) $-2x_1 + 3x_2 - x_3 = 1$

$x_1 + 2x_2 - x_3 = 4$

$-2x_1 - x_2 + x_3 = -3$

Solution:

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2 \quad , \quad x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{-2} = \frac{-4}{-2} = 2 \quad ,$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3 \quad , \quad x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{-2} = \frac{-8}{-2} = 4$$

Exercises: Solve the following linear systems by Gramer's rule

(1) $2x + y + z = 6$

$3x + 2y - 2z = -2$

$x + y + 2z = -4$

(2) $2x + 4y + 6z = 2$

$x \quad + 2z = 0$

$2x + 3y - z = -5$

Homogeneous Linear Systems

We say the linear system of the formula

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= 0 \end{aligned}$$

It is a homogeneous system that can be written in the matrix form AX = O.
The solution $x_1 = x_2 = \ldots\ x_n = 0$ is called the trivial solution for the homogeneous system.

If $x_1, x_2, \ldots, x_n$ solution for the linear system where $x_i \neq 0$ for some values of i , then this solution is called non trivial.

$\therefore$ The homogeneous linear system always consistent (if it has some solutions), the trivial solution is one of their solutions.

Theorem: If A is a square matrix of degree n×n, then the homogeneous system A X = O has trivial solution if and only if $|A| \neq 0$.

Example:

$3x + y = 0$

$2x + 4y = 0$

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \Rightarrow |A| = 10 \neq 0.$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_1 = r_1 - r_2} \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} \xrightarrow{R_2 = r_2 - 2r_1} \begin{bmatrix} 1 & -3 \\ 0 & 10 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{10} r_2} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 = r_1 + 3r_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution is $x = y = 0$

$\therefore$ The trivial solution is the only solution for this system.

Remark: If A is a square matrix of degree n×n, then the homogeneous system A X = O has infinite number of the solutions if and only if $|A| = 0$.

Examples:

(1) $x - 5y = 0$

$2x - 10y = 0$

$$A = \begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \Rightarrow |A| = 0$$

$$\begin{bmatrix} 1 & -5 \\ 2 & -10 \end{bmatrix} \xrightarrow{R_2 = r_2 - 2r_1} \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

$x - 5y = 0 \Rightarrow x = 5y$

$0x + 0y = 0$

The solution is the set $\{(5a, a) : a \in \mathbb{R}\}$.

So we get this system has infinite number of the solution.

(2) Find the value of α which make the linear system $(\alpha I - A)X = O$ has non trivial solution if $A = \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix}$.

Solution: The homogeneous system $(\alpha I - A)X = O$ has non trivial solution if and only if $|\alpha I - A| = 0$

$$\alpha I - A = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \alpha - 2 & -6 \\ -2 & \alpha - 3 \end{bmatrix}$$

Since $|\alpha I - A| = 0$

$$\therefore \begin{vmatrix} \alpha - 2 & -6 \\ -2 & \alpha - 3 \end{vmatrix} = 0$$

$(\alpha - 2)(\alpha - 3) - (-2)(-6) = 0$

$\alpha^2 - 5\alpha + 6 - 12 = 0$

$\alpha^2 - 5\alpha - 6 = 0$

$(\alpha - 6)(\alpha + 1) = 0 \;\Rightarrow\; \alpha = 6, \alpha = -1$

Examples: Solve each of the following linear systems by Gauss-Jordan method

(1) $x + 2y + 3z = 0$

$-x + 3y + 2z = 0$

$2x + y - 2z = 0$

Note: since the absolute quantities column is zeros, so we can write the coefficients matrix only in augumented matrix.

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 0 \\ -1 & 3 & 2 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \xrightarrow[R_3=r_3-2r_1]{R_2=r_2+r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & -3 & -8 \end{bmatrix} \xrightarrow{R_2=\frac{1}{5}r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -8 \end{bmatrix}$$

$$\xrightarrow[R_3=r_3+3r_2]{R_1=r_1-2r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow{R_3=-\frac{1}{5}r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2=r_2-r_3]{R_1=r_1-r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The solution is $x = y = z = 0$

$\therefore$ The trivial solution is the only solution for this system.

2. $x + 2y - z = 0$
$x + 3y + 2z = 0$
$3x + 8y + 3z = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ 3 & 8 & 3 \end{bmatrix} \xrightarrow[R_3=r_3-3r_1]{R_2=r_2-r_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow[R_3=r_3-2r_2]{R_1=r_1-2r_2} \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$x - 7z = 0 \Rightarrow x = 7z$
$y + 3z = 0 \Rightarrow y = -3z$
z = any real number
$\therefore$ This system has infinite number of solutions.
$\therefore$ The solution is the set $\{(7a,-3a,a): a \in \mathbb{R}\}$.

Exercises: Solve the following linear systems by Gauss-Jordan method

(1) $2x - 2y + 2z = 0$
$4x - 7y + 3z = 0$
$2x - y + 2z = 0$

(2) $x + 3y - 3z = 0$
$x + 3y - 2z = 0$
$2x + 6y - 3z = 0$

(3) $x + y + z + w = 0$
$x + w = 0$
$x + 2y + z = 0$

(4) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Prove that the homogeneous system AX = O has the only trivial solution if and only if $ab - bc \neq 0$.

Abstract for the method of transformations on rows (Gauss-Jordan method)

To find the inverse of the matrix A by the method (Gauss - Jordan), we write the matrix A with the identity matrix in the following form: $[A: I_n]$ then transformed the matrix A by the transformations of rows into I_n and thus transforms I_n to A^{-1}, (performed the transformations of rows operator in both matrices at the same time)

Examples: Find the inverse (if exists) for each of the following matrices

(1) $A=\begin{bmatrix}1&2&3\\1&3&3\\1&2&4\end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\1&3&3&0&1&0\\1&2&4&0&0&1\end{array}\right]\xrightarrow[R_3=r_3-r_1]{R_2=r_2-r_1}\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&0&-1&1&0\\0&0&1&-1&0&1\end{array}\right]$$

$$\xrightarrow{R_1=r_1-2r_2}\left[\begin{array}{ccc|ccc}1&0&3&3&-2&0\\0&1&0&-1&1&0\\0&0&1&-1&0&1\end{array}\right]\xrightarrow{R_1=r_1-3r_3}\left[\begin{array}{ccc|ccc}1&0&0&6&-2&-3\\0&1&0&-1&1&0\\0&0&1&-1&0&1\end{array}\right]$$

So the inverse of the matrix A is $A^{-1}=\begin{bmatrix}6&-2&-3\\-1&1&0\\-1&0&1\end{bmatrix}$.

(2) $A=\begin{bmatrix}1&2&-3\\1&-2&1\\5&-2&-3\end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc}1&2&-3&1&0&0\\1&-2&1&0&1&0\\5&-2&-3&0&0&1\end{array}\right]\xrightarrow[R_3=r_3-5r_1]{R_2=r_2-r_1}\left[\begin{array}{ccc|ccc}1&2&-3&1&0&0\\0&-4&4&-1&1&0\\0&-12&12&-5&0&1\end{array}\right]$$

$$\xrightarrow{R_3=r_3-3r_2}\left[\begin{array}{ccc|ccc}1&2&-3&1&0&0\\0&-4&4&-1&1&0\\0&0&0&-2&-3&1\end{array}\right]$$

At this point the matrix A is row equivalent to $B=\begin{bmatrix}1&2&-3\\0&-4&4\\0&0&0\end{bmatrix}$.

So the matrix A is singular (A has no inverse).

(3) Find the value(s) of a which make the inverse of the matrix $A=\begin{bmatrix}1&1&0\\1&0&0\\1&2&a\end{bmatrix}$ exists.

What is A^{-1}?

Solution:

$$\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\1&0&0&0&1&0\\1&2&a&0&0&1\end{array}\right]\xrightarrow[R_3=r_3-r_1]{R_2=r_2-r_1}\left[\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&-1&0&-1&1&0\\0&1&a&-1&0&1\end{array}\right]$$

$$\xrightarrow[R_2=r_2+r_3]{R_1=r_1-r_3}\left[\begin{array}{ccc|ccc}1&0&-a&2&0&-1\\0&0&a&-2&1&1\\0&1&a&-1&0&1\end{array}\right]\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1&0&-a&2&0&-1\\0&1&a&-1&0&1\\0&0&a&-2&1&1\end{array}\right]$$

To be the third row not equal to zero, it must $a\neq 0$.

$$\xrightarrow{\frac{1}{a}R_3}\left[\begin{array}{ccc|ccc}1&0&-a&2&0&0\\0&1&a&-1&0&0\\0&0&1&\frac{-2}{a}&\frac{1}{a}&\frac{1}{a}\end{array}\right]\xrightarrow[R_2=r_2-a\,r_3]{R_1=r_1+a\,r_3}\left[\begin{array}{ccc|ccc}1&0&0&0&1&1\\0&1&0&1&-1&-1\\0&0&1&\frac{-2}{a}&\frac{1}{a}&\frac{1}{a}\end{array}\right]$$

So $A^{-1}=\begin{bmatrix}0&1&1\\1&-1&-1\\\frac{-2}{a}&\frac{1}{a}&\frac{1}{a}\end{bmatrix}$.

Exercises: Find the inverse (if exists) for each of the following matrices

(1) $A=\begin{bmatrix}1&1\\0&2\end{bmatrix}$

(2) $B=\begin{bmatrix}1&2&1\\2&4&2\\0&-1&4\end{bmatrix}$

(3) $C=\begin{bmatrix}2&3&4\\0&-1&1\\1&2&2\end{bmatrix}$

(4) $D=\begin{bmatrix}1&1&1\\0&2&5\\5&5&1\end{bmatrix}$

Solving Linear Systems By Using The Inverse

Consider the linear system $A_{n\times n}\, X_{n\times 1} = B_{n\times 1}$, where the matrix $A_{n\times n}$ has inverse, then $X_{n\times 1} = A^{-1}_{n\times n} B_{n\times 1}$

Proof:

$A\,X = B$

$A^{-1}(A\,X) = A^{-1}B$ (multiply each side by A^{-1})

$(A^{-1}A)\,X = A^{-1}B$ (the multiplication of the matrices is associative)

$I_n\,X = A^{-1}B$ (the definition of the inverse ($A^{-1}A = I$))

$X = A^{-1}B$

Remark: We use this method when the matrix is square and has inverse.

Examples:

(1) Solve the following linear system by using the inverse of the matrix

(a) $x + 2y = 4$
$3x + 4y = 5$

Solution: The system can be written as follows $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

We compute the inverse for the coefficients matrix as follows

$$\left[A \;\vdots\; I_n\right] \to \left[I_n \;\vdots\; A^{-1}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = r_2 - 3r_1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = -\frac{1}{2}r_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] \xrightarrow{R_1 = r_1 - 2r_2} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{array}\right] = \left[I_2 \;\vdots\; A^{-1}\right]$$

The inverse of the coefficients matrix is $A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$

$$X = A^{-1}B \;\Rightarrow\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ \frac{7}{2} \end{bmatrix}.$$

The solution is $x = -3$ and $y = \frac{7}{2}$.

(b) $3x - 4y = -5$
$-2x + 3y = 4$

Solution: The system can be written as follows $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

We compute the inverse for the coefficients matrix as follows

$\left[A \;\vdots\; I_n \right] \to \left[I_n \;\vdots\; A^{-1} \right]$

$$\left[\begin{array}{cc:cc} 3 & -4 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = r_1 + r_2} \left[\begin{array}{cc:cc} 1 & -1 & 1 & 1 \\ -2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = r_2 + 2r_1} \left[\begin{array}{cc:cc} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 = r_1 + r_2} \left[\begin{array}{cc:cc} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

The inverse of the coefficients matrix is $A^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$

$X = A^{-1}B$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The solution is $x = 1$ and $y = 2$.

(3) Solve the linear system
$$-10x_1 + 5x_2 + 3x_3 = 1$$
$$7x_1 - 3x_2 - 2x_3 = -2$$
$$-4x_1 + 2x_2 + x_3 = 0$$

Where $\begin{bmatrix} -10 & 5 & 3 \\ 7 & -3 & -2 \\ -4 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & -5 \end{bmatrix}$

Solution: The system can be written as follows $\begin{bmatrix} -10 & 5 & 3 \\ 7 & -3 & -2 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

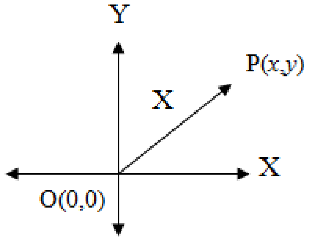
Since the inverse of the coefficients matrix given, then

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 & 5 & 3 \\ 7 & -3 & -2 \\ -4 & 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

The solution is $x_1 = -1$, $x_2 = -3$ and $x_3 = 2$.

CHAPTER FOUR
VECTORS AND VECTOR SPACES

Vectors: Consider the matrix $X=\begin{bmatrix} x \\ y \end{bmatrix}$ of degree (2×1) paired with X the line segment has a tail O(0,0) Its vertex is P(x,y) and vice versa with the directed line segment $\overrightarrow{OP}$.

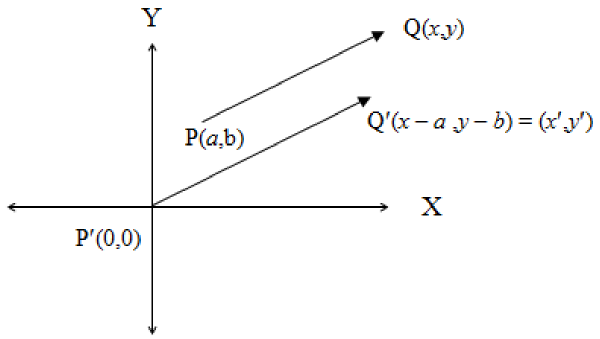


The vector is in the plane: It is the matrix $\vec{X}=\begin{bmatrix} x \\ y \end{bmatrix}$ of degree (2×1), where x and y are real numbers called components of $\vec{X}$.

Equal vectors: The vectors $\vec{X}=\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\vec{Y}=\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are equal if and only if the corresponding elements are equal, that is $x_1 = x_2$ and $y_1 = y_2$.

Example: The vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ are not equal since the corresponding elements in the second row are not equal

Remark: The beginning of the vector may not be the origin point, so its beginning may be the point (a,b). The line vector $\overrightarrow{PQ}$ beginning from P(a,b) (not the origin point) and ending with the point Q(x,y), so this vector can be represented by the vector P'O'(x',y') whose beginning is at O and its vertex is the point ($x - a$,$y - b$).

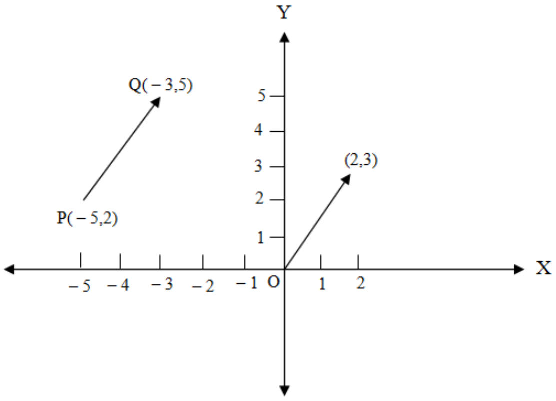


Examples:

(1) If Q(s,t) is the vertex of the vector $\overrightarrow{PQ}$, where $\overrightarrow{P'Q'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ whose beginning P(–5,2), we can find the values of s and t as follows:

$x - a = x' \Rightarrow s - (-5) = 2 \Rightarrow s = -3$

$y - b = y' \Rightarrow t - 2 = 3 \Rightarrow t = 5$

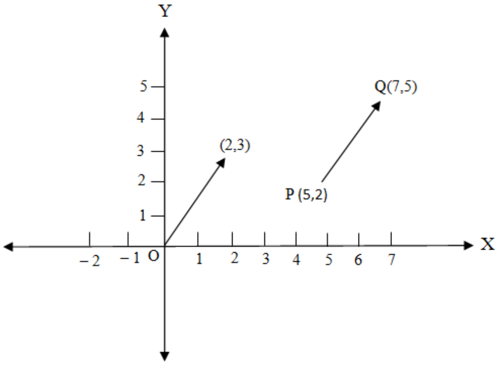


(2) If P(a,b) is the beginning of the vector $\overrightarrow{PQ}$, where $\overrightarrow{P'Q'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and the vertex is Q(7,5), find the value of each a, b ?

Solution:

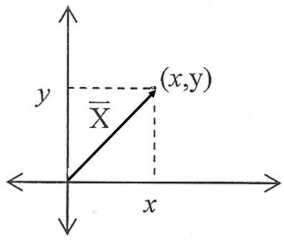
$7 - a = 2 \Rightarrow a = 5$ and

$5 - b = 3 \Rightarrow b = 2$



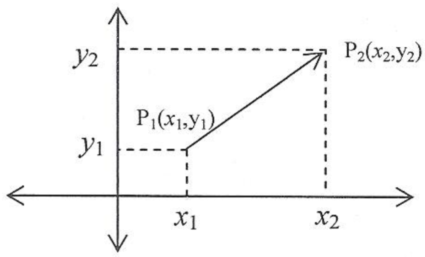
Definitions:

(1) The length of the vector $\overrightarrow{X}(x,y)$ is $\left\|\overrightarrow{X}\right\| = \sqrt{x^2 + y^2}$.



(2) The length of the straight line segment $\overrightarrow{P_1P_2}$ it is the distance between the two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ which is equal to

$\left\|\overrightarrow{P_1P_2}\right\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Examples:

(1) Find the length of the vector $\overrightarrow{X} = (6, -8)$?

Solution:

$$\left\|\overrightarrow{X}\right\| = \sqrt{x^2 + y^2} = \sqrt{(6)^2 + (-8)^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

(2) Find the distance between the two points P(2,3) and Q(5,–1) (The length of the straight line segment $\overrightarrow{PQ}$)?

Solution:

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{(5-2)^2 + (-1-3)^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Remark: The two vectors $\overrightarrow{X_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\overrightarrow{X_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are parallel if $x_1\, y_2 = x_2\, y_1$, that is, if and only if they are located on vertical or straight lines with the same slope,

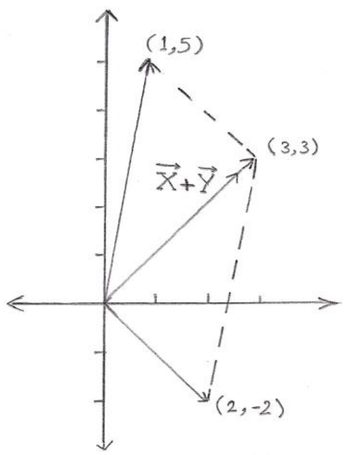
If $m_1 = \dfrac{y_1}{x_1}$, $m_2 = \dfrac{y_2}{x_2}$ $\Rightarrow$ $m_1 = m_2 \Rightarrow \dfrac{y_1}{x_1} = \dfrac{y_2}{x_2}$ $\Rightarrow$ $x_1\, y_2 = x_2 y_1$.

Operations on vectors:

Definition: Let both $\overrightarrow{X} = (x_1, y_1)$ and $\overrightarrow{Y} = (x_2, y_2)$ be vectors in the plane, so their sum is $\overrightarrow{X} + \overrightarrow{Y} = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Example: Let $\overrightarrow{X} = (1,5)$ and $\overrightarrow{Y} = (2,-2)$, then

$\overrightarrow{X} + \overrightarrow{Y} = (1,5) + (2,-2) = (3,3)$.

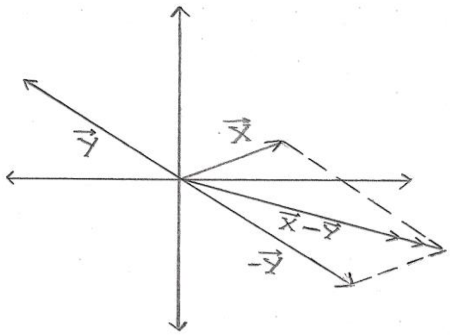


Definition: Let $\vec{X}=(x,y)$ and k is any real number, then $k\vec{X}=k(x,y)=(kx,ky)$

If $k>0$, then $k\vec{X}$ has the same direction of $\vec{X}$.

If $k<0$, then $k\vec{X}$ has opposite direction of $\vec{X}$.

Remark: The vector $\vec{O}=(0,0)$ is called the zero vector and $\vec{X}+\vec{O}=\vec{X}$. Also $\vec{X}+(-1)\vec{X}=\vec{O}$ and writes $(-1)\vec{X}$ as the form $-\vec{X}$ and called minus $\vec{X}$, and $\vec{X}-\vec{Y}=\vec{X}+(-\vec{Y})$ called the difference between $\vec{X}$ and $\vec{Y}$.



Note that adding two vectors represents one of the diagonals of a parallelogram and subtracting two vectors representing the other.

The angle between two vectors: the angle between two non-zero vectors $\vec{X}=(x_1,y_1)$ and $\vec{Y}=(x_2,y_2)$ is the angle θ and $0\le\theta\le 180^o$

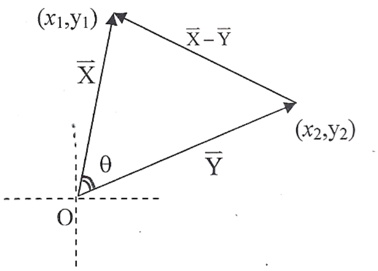
$$\left\|\vec{X}-\vec{Y}\right\|^2=\left\|\vec{X}\right\|^2+\left\|\vec{Y}\right\|^2-2\left\|\vec{X}\right\|\left\|\vec{Y}\right\|\cos\theta \qquad \text{(The law of the cosine)} \qquad \dots(1)$$

$$\vec{X}-\vec{Y}=(x_1-x_2)+(y_1-y_2)$$

$$\left\|\vec{X}-\vec{Y}\right\|^2=(x_1-x_2)^2+(y_1-y_2)^2$$

$$=\underbrace{x_1^2+y_1^2}+\underbrace{x_2^2+y_2^2}-2(x_1x_2+y_1y_2)$$

$$=\left\|\vec{X}\right\|^2+\left\|\vec{Y}\right\|^2-2(x_1x_2+y_1y_2)$$



Substituting in (1) we get that

$$\boxed{\cos\theta=\frac{x_1x_2+y_1y_2}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}}$$, where $\left\|\vec{X}\right\|\neq 0,\left\|\vec{Y}\right\|\neq 0$.

Inner Product

Definition: Let $\vec{X}=(x_1,y_1)$ and $\vec{Y}=(x_2,y_2)$ be two vectors, the inner product of the two vectors $\vec{X}$ and $\vec{Y}$ or dot product is defined as

$\vec{X}\cdot\vec{Y}= x_1x_2+y_1y_2$

Accordingly, the previous law will be

$$\boxed{\cos\theta=\frac{\vec{X}\cdot\vec{Y}}{\|\vec{X}\|\|\vec{Y}\|}}\ ,\ 0\le\theta\le\pi$$

Example: Let $\vec{X}=(2,4)$ and $\vec{Y}=(-1,2)$, then

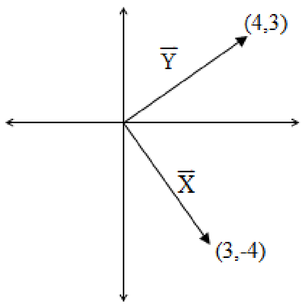
$\vec{X}\cdot\vec{Y}$ = (2)(–1) +(4)(2) = 6 , $\|\vec{X}\|=\sqrt{2^2+4^2}=\sqrt{20}$, $\|\vec{Y}\|=\sqrt{(-1)^2+2^2}=\sqrt{5}$

$\cos\theta=\dfrac{6}{\sqrt{20}\sqrt{5}}=0.6\Rightarrow\theta=53.2^\circ$ approximately

Remark: If $\vec{X}\cdot\vec{Y}$ = 0, then cos θ = 0 and the vectors are orthogonal if and only if $\vec{X}\cdot\vec{Y}=0$

Example: The two vectors $\vec{X}=(3,-4)$ and $\vec{Y}=(4,3)$ are orthogonal since

$\vec{X}\cdot\vec{Y}$ = (3)(4) +(–4)(3) = 0.



Theorem: Let $\vec{X}$, $\vec{Y}$ and $\vec{Z}$ are vectors, k is any number, then

(1) $\vec{X}\cdot\vec{X}=\|\vec{X}\|^2\ge 0$ satisfy the equality if and only if $\vec{X}=\vec{O}$.

(2) $\vec{X}\cdot\vec{Y}=\vec{Y}\cdot\vec{X}$ (Commutative property)

(3) $(\vec{X}+\vec{Y})\cdot\vec{Z}=\vec{X}\cdot\vec{Z}+\vec{Y}\cdot\vec{Z}$ (The property of distributing multiplication on the addition)

(4) $(k\vec{X})\cdot\vec{Y}=\vec{X}\cdot(k\vec{Y})=k(\vec{X}\cdot\vec{Y})$.

Unit Vector: It is the vector whose length is equal to one unit.

If $\vec{X}$ a non-zero vector, the unit vector is the vector $\vec{U} = \dfrac{1}{\|\vec{X}\|}\vec{X}$

Example: Let $\vec{X} = (-4,3)$ be a vector, then

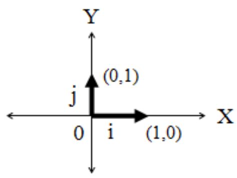
$\|\vec{X}\| = \sqrt{(-4)^2 + 3^2} = \sqrt{16+9} + \sqrt{25} = 5$

$\vec{U} = \frac{1}{5}(-4,3) = (\frac{-4}{5}, \frac{3}{5})$ it is the unit vector because

$\|\vec{U}\| = \sqrt{(\frac{-4}{5})^2 + (\frac{3}{5})^2} = \sqrt{\frac{16+9}{25}} = 1$

Remark:

(1) i = (1,0) and j = (0,1) unit vector in $\mathbb{R}^2$ they are orthogonal to where i lies toward the positive X-axis and j toward the positive Y-axis.



(2) The vector $\vec{X} = (x, y)$ in $\mathbb{R}^2$ we writes in terms of i and j in $\mathbb{R}^2$ as follows X = x i + y j.

Example: Let $\vec{X} = (-3,7)$, then X = – 3 i + 7j.

Vector of type n

Definition: The matrix $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ is said to be a vector of type n and $x_1, x_2, \dots, x_n$ called the components of the vector $\vec{X}$.

Remarks: The two vectors $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ and $\vec{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$ of the type n is equal if $(1 \le i \le n)$ and $(x_i = y_i)$.

Example: The two vectors $\overrightarrow{X} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 5 \end{bmatrix}$ and $\overrightarrow{Y} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ of type 4, $\overrightarrow{X} \neq \overrightarrow{Y}$ since the third component of them not equal $(4 \neq -2)$.

Remark: The vector can also be written in a row. For example, in the previous example the vectors can be written as $\overrightarrow{X} = [1,3,-2,5]$, $\overrightarrow{Y} = [1,3,4,5]$.

Operations on vectors:

Let $\overrightarrow{X} = [x_1, x_2, \ldots, x_n]$ and $\overrightarrow{Y} = [y_1, y_2, \ldots, y_n]$ be any vectors, k any number, then

(1) $k\overrightarrow{X} = k\,[x_1, x_2, \ldots, x_n] = [k\,x_1, k\,x_2, \ldots, k\,x_n]$

(2) $\overrightarrow{X} + \overrightarrow{Y} = [x_1, x_2, \ldots, x_n] + [y_1, y_2, \ldots, y_n] = [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$

(3) $\overrightarrow{X} - \overrightarrow{Y} = \overrightarrow{X} + (-\overrightarrow{Y}) = [x_1, x_2, \ldots, x_n] - [y_1, y_2, \ldots, y_n] = [x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n]$

Meaning multiplying a vector by any number, it is the same to the law of multiplying a matrix by a number, as well as addition and subtraction.

Example: Let $\overrightarrow{X_1} = [2,3,-4]$, $\overrightarrow{X_2} = [-1,2,6]$, $\overrightarrow{X_3} = \left[\frac{1}{2}, \frac{3}{4}, \frac{-5}{8}\right]$, find the value of

(1) $2\overrightarrow{X_1} + \overrightarrow{X_2} - 8\overrightarrow{X_3}$ **(2)** $\frac{1}{2}(\overrightarrow{X_1} - \overrightarrow{X_2})$

Solution:

(1) $2\overrightarrow{X_1} + \overrightarrow{X_2} - 8\overrightarrow{X_3} = 2\,[2,3,-4] + [-1,2,6] - 8\left[\frac{1}{2}, \frac{3}{4}, \frac{-5}{8}\right]$

$= [4,6,-8] + [-1,2,6] - [4,6,-5] = [-1,2,3]$

(2) $\frac{1}{2}(\overrightarrow{X_1} - \overrightarrow{X_2}) = \frac{1}{2}\,([2,3,-4] - [-1,2,6]) = \frac{1}{2}\,[3,1,-10] = \left[\frac{3}{2}, \frac{1}{2}, -5\right]$

Exercises:

(1) Let $\overrightarrow{X_1} = [3,1,-4]$, $\overrightarrow{X_2} = [2,2,-3]$, $\overrightarrow{X_3} = [0,-4,1]$, $\overrightarrow{X_4} = [-4,-4,6]$, prove that

(a) $2\overrightarrow{X_1} - 5\overrightarrow{X_2} = [-4,-8,7]$

(b) $2\overrightarrow{X_2} + \overrightarrow{X_4} = \vec{O}$

(c) $2\overrightarrow{X_1} - 3\overrightarrow{X_2} - \overrightarrow{X_3} = \vec{O}$

(d) $2\overrightarrow{X_1} - \overrightarrow{X_2} - \overrightarrow{X_3} + \overrightarrow{X_4} = \vec{O}$

(2) Draw a diagram of a straight segment directed at $\mathbb{R}^2$ which represents the following

(a) $\overrightarrow{X_1} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **(b)** $\overrightarrow{X_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **(c)** $\overrightarrow{X_3} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ **(d)** $\overrightarrow{X_4} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

(3) Find the vertex for each of the following vectors and draw a diagram for it

(a) The vector $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and the tail (3,2).

(b) The vector $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and the tail (1,2).

(4) Find $\vec{X} + \vec{Y}$, $\vec{X} - \vec{Y}$ and $3\vec{X} - 2\vec{X}$

(a) $\vec{X} = (2,3)$, $\vec{Y} = (-2,5)$.

(b) $\vec{X} = (0,3)$, $\vec{Y} = (3,2)$.

(5) Let $\vec{X} = (1,2), \vec{Y} = (-3,4), \vec{Z} = (x,4), \vec{U} = (-2,y)$, find x and y such that

(a) $\vec{Z} = 2\vec{X}$ **(b)** $\frac{3}{2}\vec{U} = \vec{Y}$ **(c)** $\vec{Z} + \vec{U} = \vec{X}$

(6) Find the length for each of the following vectors

(a) (1,2) **(b)** (–3,–4) **(c)** (0,2)

(7) Find the distance for each pair of the following points

(a) (3,4),(2,3) **(b)** (3,4),(0,0) **(c)** (2,0),(0,3)

(8) Find the unit vector with the direction of $\vec{X}$

(a) $\vec{X} = (-3,4)$ **(b)** $\vec{X} = (-2,-3)$ **(c)** $\vec{X} = (5,0)$

(9) Find $\vec{X} \cdot \vec{Y}$ for each of the following vectors

(a) $\vec{X} = (1,2), \vec{Y} = (2,-3)$ **(b)** $\vec{X} = (-3,-4), \vec{Y} = (4,-3)$

(10) Prove that

(a) $i \cdot i = j \cdot j = 1$ **(b)** $i \cdot j = 0$

(11) Which of the following vectors $\overrightarrow{X_1} = (1,2)$, $\overrightarrow{X_2} = (0,1)$, $\overrightarrow{X_3} = (-2,-4)$, $\overrightarrow{X_4} = (-2,1)$, $\overrightarrow{X_5} = (2,5)$, $\overrightarrow{X_6} = (-6,3)$ are

(a) orthogonal **(b)** In the same direction

Theorem: Let $\vec{X}, \vec{Y}, \vec{Z}$ be a vectors in $\mathbb{R}^n$ and let c and d numbers, then

(a) $\vec{X} + \vec{Y}$ vector in $\mathbb{R}^n$ ($\mathbb{R}^n$ is closed under the addition of vectors operation)

(1) $\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$

(2) $\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{X} + \vec{Y}) + \vec{Z}$

(3) There exists unique vector $\vec{O}$ which is called zero vector in $\mathbb{R}^n$ such that $\vec{X} + \vec{O} = \vec{O} + \vec{X} = \vec{X}$.

(4) There exists unique vector $-\vec{X}$ in $\mathbb{R}^n$ such that $\vec{X} + (-\vec{X}) = (-\vec{X}) + \vec{X} = \vec{O}$

(b) $c\vec{X}$ vector in $\mathbb{R}^n$

(5) $c(\vec{X} + \vec{Y}) = c\vec{X} + c\vec{Y}$

(6) $(c + d)\vec{X} = c\vec{X} + d\vec{X}$

(7) $c(d\vec{X}) = (cd)\vec{X}$

(8) $1 \cdot \vec{X} = \vec{X}$

Definition: The length of the vector of the norm $\vec{X} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is

$$\boxed{\|\vec{X}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2}} \qquad \ldots(1)$$

Or it is a distance between the point $(x_1, x_2, \ldots, x_n)$ and the original point.

Example: Find the value of the numerical constant k to make the norm of the vector $\vec{A} = (5, 3, k)$ equal to $\sqrt{50}$?

Solution:

$\|\vec{A}\| = \sqrt{5^2 + 3^2 + k^2}$

$\sqrt{50} = \sqrt{25 + 9 + k^2}$

$\sqrt{50} = \sqrt{34 + k^2}$

$50 = 34 + k^2$ Square both sides

$50 - 34 = k^2$

$16 = k^2 \Rightarrow k = \mp 4$

Definition: The distance between the points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ is the length of the vector $\vec{X} - \vec{Y}$ where $\vec{X} = (x_1, x_2, \ldots, x_n)$ and $\vec{Y} = (y_1, y_2, \ldots, y_n)$.

$$\boxed{\|\vec{X} - \vec{Y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}} \qquad \ldots(2)$$

Example: Let $\vec{X} = (2, 3, 2, -1), \vec{Y} = (4, 2, 1, 3)$

$\|\vec{X}\| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2} = \sqrt{18}$

$\|\vec{Y}\| = \sqrt{4^2 + 2^2 + 1^2 + 3^2} = \sqrt{30}$

$\|\vec{X} - \vec{Y}\| = \sqrt{(2-4)^2 + (3-2)^2 + (2-1)^2 + (-1-3)^2} = \sqrt{22}$

Example: Find the value of the numerical constant k to make the distance between the vectors $\vec{A} = (3, -1, 6, 3)$ and $\vec{B} = (2, k, 1, -4)$ equal to 6 units? **(Home work)**

Remarks:

(1) The length of the vector represents the distance between the vector and the original point.

(2) The distance between two vectors in $\mathbb{R}^n$ represents the distance between the vertices points of the vectors.

(3) Prove that $\|\vec{X} - \vec{Y}\| = \|\vec{Y} - \vec{X}\|$.

Inner Product on $\mathbb{R}^n$

Definition: Let $\vec{X} = (x_1, x_2, \ldots, x_n)$ and $\vec{Y} = (y_1, y_2, \ldots, y_n)$ vectors in $\mathbb{R}^n$ then the inner product is defined as the form

$\vec{X} \cdot \vec{Y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

$$\boxed{\vec{X} \cdot \vec{Y} = \sum_{i=1}^{n} x_i y_i}$$

The inner product is also called point product.

Example: Let $\vec{X} = (2, 3, 2, -1)$ and $\vec{Y} = (4, 2, 1, 3)$ two vectors, then

$\vec{X} \cdot \vec{Y} = (2)(4) + (3)(2) + (2)(1) + (-1)(3) = 8 + 6 + 2 - 3 = 13$

Cauchy-Schwarz Inequality

Theorem: Let $\vec{X}, \vec{Y}$ be a vectors in $\mathbb{R}^n$, then $|\vec{X} \cdot \vec{Y}| \leq \|\vec{X}\| \|\vec{Y}\|$.

Proof: If $\vec{Y} = \vec{O}$, then $\|\vec{Y}\| = 0$ and $\vec{X} \cdot \vec{Y} = 0$ and the theorem satisfy.

Let $\vec{X} \neq \vec{O}$, $\vec{Y} \neq \vec{O}$ and r arbitrary fixed, then

$(\vec{X} - r\vec{Y}) \cdot (\vec{X} - r\vec{Y}) \geq 0$ (previous theorem)

$\vec{X} \cdot \vec{X} - 2r\vec{X} \cdot \vec{Y} + r^2 \vec{Y} \cdot \vec{Y} \geq 0$

$\|\vec{X}\|^2 - 2r\vec{X} \cdot \vec{Y} + r^2 \|\vec{Y}\|^2 \geq 0$

Because r represent any constant, the inequality above is true when $r = \dfrac{\vec{X}\cdot\vec{Y}}{\vec{Y}\cdot\vec{Y}}$

$$\|\vec{X}\|^2 - 2\frac{\vec{X}\cdot\vec{Y}}{\vec{Y}\cdot\vec{Y}}\,\vec{X}\cdot\vec{Y} + \left(\frac{\vec{X}\cdot\vec{Y}}{\vec{Y}\cdot\vec{Y}}\right)^2 \|\vec{Y}\|^2 \ge 0$$

$$\|\vec{X}\|^2 - 2\frac{\left(\vec{X}\cdot\vec{Y}\right)^2}{\left\|\vec{Y}\right\|^2} + \left(\frac{\left(\vec{X}\cdot\vec{Y}\right)}{\left\|\vec{Y}\right\|^2}\right)^2 \|\vec{Y}\|^2 \ge 0$$

$$\|\vec{X}\|^2 - 2\frac{\left(\vec{X}\cdot\vec{Y}\right)^2}{\left\|\vec{Y}\right\|^2} + \frac{\left(\vec{X}\cdot\vec{Y}\right)^2}{\left\|\vec{Y}\right\|^2} \ge 0$$

$\|\vec{X}\|^2\|\vec{Y}\|^2 - 2(\vec{X}\cdot\vec{Y})^2 + (\vec{X}\cdot\vec{Y})^2 \ge 0$ multiply by $\|\vec{Y}\|^2$

$$\|\vec{X}\|^2\|\vec{Y}\|^2 - (\vec{X}\cdot\vec{Y})^2 \ge 0$$

$$\|\vec{X}\|^2\|\vec{Y}\|^2 \ge (\vec{X}\cdot\vec{Y})^2$$

$|\vec{X}\cdot\vec{Y}| \le \|\vec{X}\|\|\vec{Y}\|$ $\left(\sqrt{\vec{X}^2} = \left|\vec{X}\right|\right)$, taking the square root of both sides

Example: Let $\vec{X}$ = (2,3) and $\vec{Y}$ = (1,0), then

$\vec{X}\cdot\vec{Y}$ = (2)(1) + (3)(0) = 2

$$\left\|\vec{X}\right\| = \sqrt{2^2+3^2} = \sqrt{13}\ ,\quad \left\|\vec{Y}\right\| = \sqrt{1^2+0^2} = 1,\ \left\|\vec{X}\right\|\left\|\vec{Y}\right\| = \sqrt{13}\,(1) = \sqrt{13}$$

$$\left|\vec{X}\cdot\vec{Y}\right| = 2 < \sqrt{13} = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|$$

Definition: The angle between the two non-zero vectors $\vec{X}$ and $\vec{Y}$ is the unique number θ and $0 \le \theta \le \pi$, where $\cos\theta = \dfrac{\vec{X}\cdot\vec{Y}}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}$

From Cauchy-Schwarz inequality we note that $\left|\dfrac{\vec{X}\cdot\vec{Y}}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|}\right| \le 1$ also we know that $|\cos\theta| \le 1$.

Remark: We note that $\left|\vec{X}\cdot\vec{Y}\right| = \left|\cos\theta\right|\left\|\vec{X}\right\|\left\|\vec{Y}\right\|$. That is, the Cauchy-Schwarz inequality becomes equal if we multiply the right-hand side by the absolute cosine of the angle between the two vectors.

Example: Let $\vec{X} = (1,0,0,1)$ and $\vec{Y} = (0,1,0,1)$, then

$\left\|\vec{X}\right\| = \sqrt{2}$, $\left\|\vec{Y}\right\| = \sqrt{2}$, $\vec{X}\cdot\vec{Y} = 1$.

Therefore,

$$\cos\theta = \frac{\vec{X}\cdot\vec{Y}}{\left\|\vec{X}\right\|\left\|\vec{Y}\right\|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \;\Rightarrow\; \theta = \cos^{-1}\frac{1}{2} \;\Rightarrow\; \theta = 60^\circ.$$

Exercise: Find the angle between each of the following vectors:

(1) $\vec{X} = (0,1)$ and $\vec{Y} = (1,0)$ **Solution:** 90°

(2) $\vec{X} = (0,0,1)$ with itself **Solution:** 0°

(3) $\vec{X} = (0,0,1)$ and $\vec{Y} = (1,0,1)$ **Solution:** 45°

(4) $\vec{X} = (2,3,4)$ with itself **Solution:** 0°

Definition: Let $\vec{X},\vec{Y}$ be a vectors in $\mathbb{R}^n$ we say that

(1) $\vec{X}$ and $\vec{Y}$ are orthogonal if $\vec{X}\cdot\vec{Y} = 0$ **or** if one of the vectors is zero.

(2) $\vec{X}$ and $\vec{Y}$ are parallel if $\left|\vec{X}\cdot\vec{Y}\right| = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|$.

(3) $\vec{X}$ and $\vec{Y}$ are in the same direction if $\vec{X}\cdot\vec{Y} = \left\|\vec{X}\right\|\left\|\vec{Y}\right\|$

Remark: The previous definition can be formulated as follows:

Let $\vec{X},\vec{Y}$ be a vectors in $\mathbb{R}^n$ and θ is the angle between them, then

(1) $\vec{X}$ and $\vec{Y}$ are orthogonal if cos θ = 0.

(2) $\vec{X}$ and $\vec{Y}$ are parallel if cos θ = ± 1.

(3) $\vec{X}$ and $\vec{Y}$ are in the same direction if cos θ = 1.

Example: Let $\vec{X}$ = (1,0,0,1), $\vec{Y}$ = (0,1,1,0) and $\vec{Z}$ = (3,0,0,3).

$\vec{X}\cdot\vec{Y}=0$, $\vec{Y}\cdot\vec{Z}=0$, so $\vec{X}$ and $\vec{Y}$ are orthogonal, also $\vec{Y}$ and $\vec{Z}$ are orthogonal.

$\vec{X}\cdot\vec{Z}=6$, $\|\vec{X}\|=\sqrt{2}$, $\|\vec{Z}\|=\sqrt{18}$ $\Rightarrow$ $\|\vec{X}\|\|\vec{Z}\|=\sqrt{2}\sqrt{18}=\sqrt{36}=6 \Rightarrow \vec{X}\cdot\vec{Z}=\|\vec{X}\|\|\vec{Z}\|$.

Thus $\vec{X}$ and $\vec{Z}$ are parallel and because $\vec{X}\cdot\vec{Z}$ positive then $\vec{X}$ and $\vec{Z}$ are in the same direction.

Exercise: Any pair of the following vectors is parallel and which is orthogonal $\vec{X}_1$ = (–2,3,–1,–1), $\vec{X}_2$ = (–2,–1,–3,4), $\vec{X}_3$ = (1,2,3,–4), if there is a parallel pair are they in the same direction?

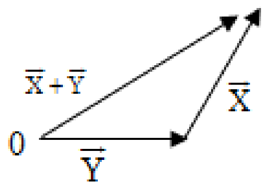
The following theorem is a result of the Cauchy-Schwarz inequality and it is called the triangle inequality

Theorem: Let $\vec{X},\vec{Y}$ be a vectors in $\mathbb{R}^n$, then $\|\vec{X}+\vec{Y}\|\le\|\vec{X}\|+\|\vec{Y}\|$.

Proof:

$\|\vec{X}+\vec{Y}\|^2=(\vec{X}+\vec{Y})\cdot(\vec{X}+\vec{Y})$ ($\vec{X}\cdot\vec{X}=\|\vec{X}\|^2\ge 0$ by previous theorem)

$=\vec{X}\cdot\vec{X}+2(\vec{X}\cdot\vec{Y})+\vec{Y}\cdot\vec{Y}$



$=\|\vec{X}\|^2+2(\vec{X}\cdot\vec{Y})+\|\vec{Y}\|^2$

$\le\|\vec{X}\|^2+2\|\vec{X}\|\|\vec{Y}\|+\|\vec{Y}\|^2$ Cauchy-Schwarz inequality

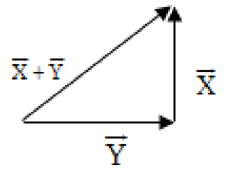
$=(\|\vec{X}\|+\|\vec{Y}\|)^2$

$\|\vec{X}+\vec{Y}\|\le\|\vec{X}\|+\|\vec{Y}\|$ taking the square root of both sides

The following theorem is also an important theorem which is called Pythagorean Theorem.

Theorem: Let $\vec{X}, \vec{Y}$ be a vectors in $\mathbb{R}^n$, then $\|\vec{X}+\vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$ if and only if $\vec{X}$ and $\vec{Y}$ are orthogonal.

Proof: ($\Rightarrow$) By previous theorem $\vec{X}\cdot\vec{X} = \|\vec{X}\|^2 \geq 0$ we get



$\|\vec{X} + \vec{Y}\|^2 = (\vec{X} + \vec{Y})\cdot(\vec{X} + \vec{Y})$

$$= \|\vec{X}\|^2 + 2(\vec{X}\cdot\vec{Y}) + \|\vec{Y}\|^2 \qquad \ldots(1)$$

When $\|\vec{X}+\vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$ then $2(\vec{X}\cdot\vec{Y}) = 0$, then

$\vec{X}\cdot\vec{Y} = 0$ which is mean that $\vec{X}$ and $\vec{Y}$ are orthogonal.

($\Leftarrow$) $\vec{X}$ and $\vec{Y}$ are orthogonal, then $\vec{X}\cdot\vec{Y} = 0$ and so equation (1) become

$\|\vec{X}+\vec{Y}\|^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$

Example: Let $\vec{X}$ = (0,0,0,3) and $\vec{Y}$ = (0,– 4,3,0)

$\|\vec{X}\| = \sqrt{9} = 3$, $\|\vec{Y}\| = \sqrt{25} = 5$, $\|\vec{X}\| + \|\vec{Y}\| = 8$

$\vec{X}+\vec{Y}$ = (0,– 4,3,3) $\Rightarrow \|\vec{X}+\vec{Y}\| = \sqrt{34}$

$\|\vec{X}+\vec{Y}\|^2 = 34 = 3^2 + 5^2 = \|\vec{X}\|^2 + \|\vec{Y}\|^2$

$\vec{X}\cdot\vec{Y} = 0$, so we get that $\vec{X}$ and $\vec{Y}$ are orthogonal

Definition: The unit vector $\vec{U}$ in $\mathbb{R}^n$ is a vector of length one unit.

If $\vec{X}$ is a non zero vector in $\mathbb{R}^n$ then the vector $\vec{U}$ defined as $\vec{U} = \frac{1}{\|\vec{X}\|}\vec{X}$ is a unit vector in the same direction of $\vec{X}$.

Example: Let $\vec{X}$ = (1,2,2) and $\vec{Y}$ = (2,0,0), then $\|\vec{X}\|$ = 3 and $\|\vec{Y}\|$ = 2.

The two vectors $\vec{U_1}$ and $\vec{U_2}$ defined as

$$\vec{U_1} = \frac{\vec{X}}{\|\vec{X}\|} = \frac{1}{3}(1,2,2) \qquad \vec{U_2} = \frac{\vec{Y}}{\|\vec{Y}\|} = \frac{1}{2}(2,0,0)$$

Are unit vectors in the same direction of $\vec{X}$ and $\vec{Y}$ respectively.

In the case $\mathbb{R}^3$ indicates unit vectors in the positive directions of the axes $\overrightarrow{X}$, $\overrightarrow{Y}$ and $\overrightarrow{Z}$ symbols i = (1,0,0), j = (0,1,0) and k = (0,0,1).

Also the vector $\overrightarrow{X} = (x,y,z)$ in $\mathbb{R}^3$ can be written by using the form of the unit vectors i, j and k as the form $\overrightarrow{X} = x\,i + y\,j + z\,k$.

As example the vector $\overrightarrow{X} = (2, -1,3)$ writes as $\overrightarrow{X} = 2i - j + 3k$.

In general: in the case of $\mathbb{R}^n$ the unit vectors in the positive directions of the axes are $\overrightarrow{U_1} = (1,0,0,\ldots,0)$, $\overrightarrow{U_2} = (0,1,0,\ldots,0)$, …, $\overrightarrow{U_n} = (0,0,0,\ldots,1)$ which are mataually orthogonal.

If $\overrightarrow{X} = (x_1,x_2,\ldots,x_n)$ then $\overrightarrow{X} = x_1\overrightarrow{U_1} + x_2\overrightarrow{U_2} + \ldots + x_n\overrightarrow{U_n}$.

Exercises:

(1) Find the unit vector $\overrightarrow{U}$ in the same direction of $\overrightarrow{X}$ for each of the following

(a) $\overrightarrow{X} = (2,-1,3)$

(b) $\overrightarrow{X} = (1,2,3,4)$

(c) $\overrightarrow{X} = (0,1,-1)$

(d) $\overrightarrow{X} = (0,-1,2,-1)$

(2) Write $\overrightarrow{X}$ and $\overrightarrow{Y}$ In terms of unit vectors i, j and k, where $\overrightarrow{X} = (1,2,-3)$ and $\overrightarrow{Y} = (2,3,-1)$.

Vectors Space

Definition: We call V a real vector space (or V vector space over $\mathbb{R}$) if the set of elements has two operations:

$\oplus$: Binary operation on vector space, i.e. $\oplus : V \longrightarrow V$

$\odot$: The multiplication operation $\odot$ is an application from $\mathbb{R} \times V$ to V, and called the multiplication operation by a real number. Meaning $\odot: \mathbb{R} \times V \longrightarrow V$

Satisfy the following axioms:

First: Axioms of Addition

(1) Closure: The effect of vector addition, i.e. if $\vec{X}, \vec{Y} \in V$, then $(\vec{X}+\vec{Y}) \in V$.

(2) Associative: If $\vec{X}, \vec{Y}, \vec{Z} \in V$, then $(\vec{X}+\vec{Y})+\vec{Z} = \vec{X}+(\vec{Y}+\vec{Z})$.

(3) Identity element: For addition there exists an element denoted by $(\vec{O})$ such that,

$\vec{X}+\vec{O} = \vec{O}+\vec{X} = \vec{X}$, $\forall\ \vec{X} \in V$

(4) Addition Inverse: For all $\vec{X} \in V$ there exists $\vec{Y} \in V$ such that

$\vec{X}+\vec{Y} = \vec{Y}+\vec{X} = \vec{O}$, $\vec{Y}$ is called the inverse of $\vec{X}$ and denoted by $(-\vec{X})$.

(5) Commutative: For all $\vec{X}, \vec{Y} \in V$, $\vec{X}+\vec{Y} = \vec{Y}+\vec{X}$.

Second: Axioms of Scalar (Real) Multiplication

(6) For any $\alpha \in \mathbb{R}$ and any $\vec{X} \in V$, then $\alpha\vec{X} \in V$.

(7) If $\alpha, \beta \in \mathbb{R}$ and $\vec{X} \in V$, then $(\alpha\ \beta)\vec{X} = \alpha(\beta\vec{X})$.

(8) For any $\vec{X} \in V$, then $1 \cdot \vec{X} = \vec{X}$.

Third: Axioms of Distributives

(9) If $\alpha, \beta \in \mathbb{R}$ and $\vec{X} \in V$, then $\vec{X} + \beta\vec{X} = \alpha\vec{X}\ (\alpha + \beta)$.

(10) If $\alpha \in \mathbb{R}$ and $\vec{X}, \vec{Y} \in V$, then $\alpha(\vec{X}+\vec{Y}) = \alpha\vec{X} + \alpha\vec{Y}$.

Remark: We call for any element of vector space by a vector.

Example: Show that $\mathbb{R}$ (set of real numbers) with addition and multiplication operations is a vector space

Solution:

First: axioms of addition:

(1) If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$ (closing by the effect of the addition operation)

(2) If $x, y, z \in \mathbb{R}$, then $(x + y) + z = x + (y + z) \in \mathbb{R}$ (the addition operation is associative)

(3) $0 \in \mathbb{R}$ (identity element), $\forall\, x \in \mathbb{R}$ then $x + 0 = 0 + x = x$.

(4) $x \in \mathbb{R}$ and $-x \in \mathbb{R}$, then $x + (-x) = (-x) + x = 0$

$(-x)$ is the additive inverse for x.

(5) $x, y \in \mathbb{R}$, then $x + y = y + x$ (the addition operation is commutative)

Second: Axioms of Scalar Multiplication

(6) If $x, y \in \mathbb{R}$, then $x\,y \in \mathbb{R}$ (closing by the effect of the multiplication operation)

(7) If $x, y, z \in \mathbb{R}$, then $(x\,y)\,z = x\,(y\,z) \in \mathbb{R}$ (the multiplication operation is associative)

(8) $1 \in \mathbb{R}$, $1 \cdot x = x \cdot 1$, $\forall\, x \in \mathbb{R}$.

Third: Axioms of Distributives

(9) If $x, y, z \in \mathbb{R}$, then $(x + y)\,z = x\,z + y\,z \in \mathbb{R}$.

(10) If $x, y, z \in \mathbb{R}$, then $z\,(x + y) = z\,x + z\,y \in \mathbb{R}$.

Thus $(\mathbb{R},+,\cdot)$ is a vector space.

Remark: If V is a vector space on the set of the complex numbers $\mathbb{C}$ then it is called complex vector space.

Example: The set I (set of integer numbers) with addition and multiply be a real number not a vector space, i.e. (I,+,·) is not a vector space.

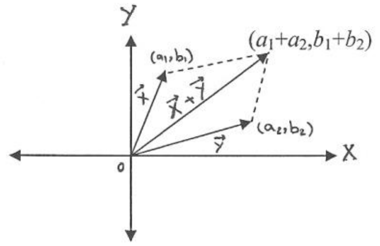
Solution:

The operation of multiplication by a real number does not satisfy or what is called (closed in effect of the multiplication operation).

Take $x = 8 \in \mathrm{I}$, $\alpha = \frac{1}{3}$, then $\alpha x = \frac{1}{3} \cdot 8 = \frac{8}{3} \notin \mathrm{I}$.

So, (I,+,·) is not a vector space.

Example: Check in detail that $\mathbb{R}^2$ vector space, where $\mathbb{R} = \{(a,b): a,b \in \mathbb{R}\}$ if (+) and $(\cdot)$ defined as the formulas



$(a_1,b_1)+(a_2,b_2)=(a_1+a_2,b_1+b_2)$

$k\,(a,b)=(ka,kb)$

Solution:

First: axioms of addition:

(1) Let $\vec{X}=(a_1,b_1),\vec{Y}=(a_2,b_2)\in\mathbb{R}^2, a_1,b_1,a_2,b_2\in\mathbb{R}$

$\vec{X}+\vec{Y}=(a_1,b_1)+(a_2,b_2)$

$=(\underbrace{a_1+a_2}_{\in\mathbb{R}},\underbrace{b_1+b_2}_{\in\mathbb{R}})\in\mathbb{R}^2$ (definition the addition operation of two vectors)

$\therefore$ The closure operation with the effect of vectors addition satisfy

(2) Let $\vec{X}=(a_1,b_1),\vec{Y}=(a_2,b_2),\vec{Z}=(a_3,b_3)\in\mathbb{R}^2, a_1,b_1,a_2,b_2,a_3,b_3\in\mathbb{R}$

$(\vec{X}+\vec{Y})+\vec{Z}=((a_1,b_1)+(a_2,b_2))+(a_3,b_3)$

$=(a_1+a_2,b_1+b_2)+(a_3,b_3)$

$=((a_1+a_2)+a_3,(b_1+b_2)+b_3)$ (definition the addition operation of two vectors)

$=(a_1+(a_2+a_3),b_1+(b_2+b_3))$ (the addition of numbers is associative)

$(\vec{X}+\vec{Y})+\vec{Z}=(a_1,b_1)+(a_2+a_3,b_2+b_3)$

$=(a_1,b_1)+((a_2,b_2)+(a_3,b_3)$

$=\vec{X}+(\vec{Y}+\vec{Z})$

$\therefore$ The associative operation is satisfy

(3) For all $\vec{X}=(a,b)\in\mathbb{R}^2$ there exists $\vec{O}=(0,0)\in\mathbb{R}^2$ such that

$\vec{X}+\vec{O}=(a,b)+(0,0)$

$=(a+0,b+0)$ (definition the addition operation of two vectors)

$=(a,b)=\vec{X}$ (Zero is the additive identity element of numbers)

$\therefore$ The identity element exists

(4) For all $\vec{X}=(a,b)\in\mathbb{R}^2$ there exists $-\vec{X}=(-a,-b)\in\mathbb{R}^2$ such that

$\vec{X}+(-\vec{X})=(a,b)+(-a,-b)$

$=(a+(-a),b+(-b))$ (definition the addition operation of two vectors)

$=(0,0)=\vec{O}$

$\therefore$ The addition Inverse exists

(5) Let $\overrightarrow{X}=(a_1,b_1),\overrightarrow{Y}=(a_2,b_2)\in\mathbb{R}^2, a_1,b_1,a_2,b_2\in\mathbb{R}$

$$\overrightarrow{X}+\overrightarrow{Y}=(a_1,b_1)+(a_2,b_2)$$
$$=(a_1+a_2,b_1+b_2)$$
$$=(a_2+a_1,b_2+b_1)$$
$$=(a_2,b_2)+(a_1,b_1) \qquad \text{(the addition of numbers is commutative)}$$
$$=\overrightarrow{Y}+\overrightarrow{X}$$

$\therefore$ The commutative property satisfy

Second: Axioms of Scalar Multiplication

(6) For any $\overrightarrow{X}=(a,b)\in\mathbb{R}^2$ and for any number $r\in\mathbb{R}$

$$r\overrightarrow{X}=r(a,b)=(\underset{\in\mathbb{R}}{ra},\underset{\in\mathbb{R}}{rb})\in\mathbb{R}^2$$

(7) For any $\overrightarrow{X}=(a,b)\in\mathbb{R}^2$ and for any numbers $r,t\in\mathbb{R}$

$$(rt)\overrightarrow{X}=(rt)(a,b)=((rt)a,(rt)b) \qquad \text{(definition the scalar multiplication)}$$
$$=(r(ta),r(tb)) \qquad \text{(the multiplication of numbers is associative)}$$
$$=r(ta,tb) \qquad \text{(definition the scalar multiplication)}$$
$$=r(t(a,b)) \qquad \text{(definition the scalar multiplication)}$$
$$=r(t\overrightarrow{X})$$

(8) For any $\overrightarrow{X}=(a,b)\in\mathbb{R}^2$ and for any number $1\in\mathbb{R}$

$$1\cdot\overrightarrow{X}=1(a,b)$$
$$=(1a,1b) \qquad \text{(definition the scalar multiplication)}$$
$$=(a,b)=\overrightarrow{X}$$

Third: Axioms of Distributives

(9) For any $\overrightarrow{X}=(a,b)\in\mathbb{R}^2$ and for any numbers $\alpha,\beta\in\mathbb{R}$

$$(\alpha+\beta)\overrightarrow{X}=(\alpha+\beta)(a,b)=((\alpha+\beta)a,(\alpha+\beta)b) \qquad \text{(definition the scalar multiplication)}$$
$$=((\alpha a+\beta a),(\alpha b+\beta b))$$
$$=((\alpha a,\alpha b)+(\beta a,\beta b))$$
$$=(\alpha(a,b)+\beta(a,b))$$
$$=\alpha\overrightarrow{X}+\beta\overrightarrow{X}$$

(10) For any $\overrightarrow{X}=(a_1,b_1),\overrightarrow{Y}=(a_2,b_2)\in\mathbb{R}^2, a_1,b_1,a_2,b_2\in\mathbb{R}$ and any $\alpha\in\mathbb{R}$

$$\begin{aligned}
\alpha(\overrightarrow{X}+\overrightarrow{Y}) &= \alpha((a_1,b_1)+(a_2,b_2))\\
&= \alpha(a_1+a_2,b_1+b_2) && \text{(definition the addition operation of two vectors)}\\
&= (\alpha(a_1+a_2),\alpha(b_1+b_2)) && \text{(definition the scalar multiplication)}\\
&= (\alpha a_1+\alpha a_2,\alpha b_1+\alpha b_2) && \text{(the multiplication distribution over the addition of numbers)}\\
&= ((\alpha a_1,\alpha b_1)+(\alpha a_2,\alpha b_2)) && \text{(definition the addition operation of two vectors)}\\
&= \alpha(a_1,b_1)+\alpha(a_2,b_2) && \text{(definition the scalar multiplication)}\\
&= \alpha\overrightarrow{X}+\alpha\overrightarrow{Y}
\end{aligned}$$

$\therefore$ $\mathbb{R}^2$ is a vector space

Exercise: Prove that $\mathbb{R}^3$ is a vector space where $\mathbb{R}^3=\{(a,b,c):a,b,c\in\mathbb{R}\}$?

Definition: Suppose $\mathbb{R}^n=\{(x_1,x_2,...,x_n):x_i\in\mathbb{R}\}$ for $i=1,2,...,n$ and let $\overrightarrow{X}=(x_1,x_2,...,x_n),\overrightarrow{Y}=(y_1,y_2,...,y_n)\in\mathbb{R}^n$, define addition operation on $\mathbb{R}^n$ as

$$\begin{aligned}
\overrightarrow{X}+\overrightarrow{Y} &= (x_1,x_2,...,x_n)+(y_1,y_2,...,y_n)\\
&= (x_1+y_1,x_2+y_2,...,x_n+y_n)
\end{aligned}$$

$\odot$ Multiplication by a scalar (standard) or real number on $\mathbb{R}^n$

$$r\overrightarrow{X}=r(x_1,x_2,...,x_n)=(rx_1,rx_2,...,rx_n)$$

Exercises:

(1) Show that $(\mathbb{R}^n,\oplus,\odot)$ is a vector space.

(2) Let $W=M_{2\times2}=\left\{\begin{bmatrix}a_1 & a_2\\ a_3 & a_4\end{bmatrix}, a_i\in\mathbb{R}, i=1,2,3,4\right\}$ be a set of all matrices of degree 2×2 with the addition and multiplication by a real number on matrices. Prove that W is a vector space on $\mathbb{R}$.

Example: Let $\mathrm{V}=\{(x_1,x_2):x_1,x_2\in\mathbb{R},x_2=\frac{1}{2}x_1+1\}$. Is (V,+,·) vector space such that (+) is the ordinary addition on $\mathbb{R}^2$ and (·) is the ordinary multiplication of an element in $\mathbb{R}^2$ by a real number k.

Solution: Let $\vec{\mathrm{X}},\vec{\mathrm{Y}}\in\mathrm{V}$ such that

$$\vec{\mathrm{X}}=(a_1,a_2),\vec{\mathrm{Y}}=(b_1,b_2):a_1,a_2,b_1,b_2\in\mathbb{R},a_2=\frac{1}{2}a_1+1,b_2=\frac{1}{2}b_1+1$$

$$\vec{\mathrm{X}}+\vec{\mathrm{Y}}=(a_1,a_2)+(b_1,b_2)=(a_1+b_1,a_2+b_2)$$

$$\begin{aligned}a_2+b_2&=(\frac{1}{2}a_1+1)+(\frac{1}{2}b_1+1)\\&=\frac{1}{2}a_1+1+\frac{1}{2}b_1+1\\&=\frac{1}{2}(a_1+b_1)+2\\&\neq\frac{1}{2}(a_1+b_1)+1\end{aligned}$$

$\therefore\ \vec{\mathrm{X}}+\vec{\mathrm{Y}}\notin\mathrm{V}$

For an example: $\vec{\mathrm{X}},\vec{\mathrm{Y}}\in\mathrm{V}$ such that $\vec{\mathrm{X}}=(0,1)$,$1=\frac{1}{2}(0)+1$ and $\vec{\mathrm{Y}}=(2,2)$, $2=\frac{1}{2}(2)+1$

$\vec{\mathrm{X}}+\vec{\mathrm{Y}}=(0,1)+(2,2)=(2,3)\notin\mathrm{V}$ because $3\neq\frac{1}{2}(2)+1$.

So the addition operation not closed on V.

$\therefore$ (V,+,·) is not vector space.

Example: Let $\mathrm{V}=\{(x_1,x_2,x_3):x_1,x_2,x_3\in\mathbb{R}\}$. Is (V,+,·) vector space if (+) and (·) defined as the formulas $c\vec{\mathrm{X}}=c(x_1,x_2,x_3)=(cx_1,x_2,x_3)$ and

$$\vec{\mathrm{X}}+\vec{\mathrm{Y}}=(x_1,x_2,x_3)+(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$$

Solution:

In this example all the conditions satisfy except the first condition of axioms of distributives in (Third).

Let $\vec{\mathrm{X}}=(a,b,c)\in\mathrm{V}$, $\alpha,\beta\in\mathbb{R}$

$$\begin{aligned}(\alpha+\beta)\vec{\mathrm{X}}&=(\alpha+\beta)(a,b,c)\\&=((\alpha+\beta)a,b,c)\end{aligned}$$

$$\alpha\vec{X}+\beta\vec{X}=\alpha(a,b,c)+\beta(a,b,c)$$
$$=(\alpha a,b,c)+(\beta a,b,c)$$
$$=((\alpha+\beta)a,b+b,c+c)$$
$$=((\alpha+\beta)a,2b,2c)$$

$$((\alpha+\beta)a,2b,2c)\neq((\alpha+\beta)a,b,c)$$
$\therefore\ \alpha\vec{X}+\beta\vec{X}\neq(\alpha+\beta)\vec{X}$

$\therefore$ (V,+,·) is not vector space. (Make sure the rest of the conditions are satisfy)

Example: Is (V,⊕,⊙) vector space where ⊕ and ⊙ are defined as the formulas

$$c\odot\vec{X}=c(x_1,x_2,x_3)=(cx_1,0,0)$$
$$\vec{X}\oplus\vec{Y}=(x_1,x_2,x_3)\oplus(y_1,y_2,y_3)=(x_1+y_1,x_2+y_2,x_3+y_3)$$

Solution:

(V,⊕,⊙) is not vector space, because if $\vec{X}=(x_1,x_2,x_3)\in V \quad x_1,x_2,x_3\in\mathbb{R}$

$1\odot\vec{X}=1(x_1,x_2,x_3)=(x_1,0,0)\neq(x_1,x_2,x_3)\Rightarrow 1\odot\vec{X}\neq\vec{X}$.

Example: Let V be the set of the solutions of the system of linear equations $A_{m\times n}\ X_{n\times 1}=B_{m\times 1}$ such that $B_{m\times 1}\neq O_{m\times 1}$. Is (V,+,·) vector space with the addition of matrix and multiply the matrix by a number.

Solution: Let $\vec{X},\vec{Y}\in V$

$A\vec{X}=B \quad ...(1) \qquad \& \qquad A\vec{Y}=B \quad ...(2)$

$$A\left(\vec{X}+\vec{Y}\right)=A\vec{X}+A\vec{Y}$$
$$=B+B=2B \qquad \text{From (1) and (2)}$$
$$\neq B$$

$\therefore\ \left(\vec{X}+\vec{Y}\right)\notin V$

$\therefore\ \left(\vec{X}+\vec{Y}\right)$ not necessary that the system of linear equations has solution.

$\therefore$ V not closed under the addition operation

$\therefore$ (V,+,·) is not vector space

Exercises:

(1) Let $V=\{(x,y):x,y\in\mathbb{R}, y=2x\}$. Is (V,+,·) vector space?

(2) Let V be a set of real functions defined on $\mathbb{R}$ then $(V,\oplus,\odot)$ is a vector space where $f,g\in V, c\in\mathbb{R}, (f\oplus g)(x)=f(x)+g(x), (c\odot g)(x)=c\cdot g(x)$

(3) Let $V=\{(x_1,x_2,0):x_1,x_2\in\mathbb{R}\}$. Is $(V,\oplus,\odot)$ vector space where defined as the formulas $(x_1,x_2,0)\oplus(y_1,y_2,0)=(x_1+y_1,x_2+y_2,0)$ and $c\odot\vec{X}=c(x_1,x_2,x_3)=(cx_1,cx_2,0)$

(4) Let V be the set of all polynomial vectors of degree (2) or less, $p\in$ V, where for all $x\in\mathbb{R}$, $p(x)=a_0+a_1x+a_2x^2$, $a_0, a_1, a_2\in\mathbb{R}$, for $p, q\in$ V

$p(x)=a_0+a_1x+a_2x^2$, $q(x)=b_0+b_1x+b_2x^2$, $a_0, a_1, a_2, b_0, b_1, b_2\in\mathbb{R}$

$(p\oplus q)(x)=(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2$

$(c\odot p)(x)=ca_0+(ca_1)x+(ca_2)x^2$; $c\in\mathbb{R}$

Prove that $(V,\oplus,\odot)$ is a vector space.

(5) Show that is the following sets with the operations defined below represent a vector space and if not, which conditions are not satisfy?

(a) The set of all ordered triads of real numbers (x,y,z) with the two defined operations $(x_1,y_1,z_1)\oplus(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$ and $c\odot(x,y,z)=(x,1,z)$.

(b) The set of all ordered triads of real numbers in the form (0,0,z) with the two defined operations $(0,0,z_1)\oplus(0,0,z_2)=(0,0,z_1+z_2)$ and $c\odot(0,0,z)=(0,0,cz)$.

(c) The set of all ordered pairs of real numbers (x,y) with the two defined operations $(x_1,y_1)\oplus(x_2,y_2)=(x_1+x_2,y_1+y_2)$ and $c\odot(x,y)=(0,0)$.

(6) Let V = $\{f; f:\mathbb{R}\longrightarrow\mathbb{R}, f(0)=1\}$. Is (V,+,·) vector space?

(7) Which of the following is a vector space with ordinary addition and multiplication over $\mathbb{R}^2$

(i) W = $\{(x,0); x\in\mathbb{R}\}$ **(ii)** W = $\{(x,1); x\in\mathbb{R}\}$ **(iii)** W = $\{(x,x); x\in\mathbb{R}\}$

(8) Which of the following is a vector space with ordinary addition and multiplication over $\mathbb{R}^3$?

(i) $W = \{(a + 3b, a, b);\ a, b \in \mathbb{R}\}$ **(ii)** $W = \{(-a, -2a, a);\ a \in \mathbb{R}\}$

(9) Which of the following is a vector space with addition of matrix and multiply the matrix by a number?

(i) $W = \{[a_{ij}]_{2\times2};\ a_{12} = 0,\ a_{ij} \in \mathbb{R}\}$ **(ii)** $W = \{[a_{ij}]_{2\times2};\ a_{11} = a_{22} = 0\}$

(iii) $W = \left\{ \begin{bmatrix} a & c & 0 \\ b & d & 0 \end{bmatrix}_{2\times3}, a, b, c, d \in \mathbb{R} \right\}$

Subspaces

Definition: Let V be a vector space and W is a non empty subset of V, if W is a vector space for the two operations defined on V we say that W is a subspace of V.

Examples: Let $W = \{(a,b,1);\ a, b \in \mathbb{R}\}$, $W \subset \mathbb{R}^3$, is W a subspace of $\mathbb{R}^3$?

Solution: Let $\vec{X}, \vec{Y} \in W$, where

$$\vec{X} = (a_1, b_1, 1), \vec{Y} = (a_2, b_2, 1)$$

$$\vec{X} + \vec{Y} = (a_1, b_1, 1) + (a_2, b_2, 1)$$

$$= (a_1 + a_2, b_1 + b_2, 2) \neq (a_1 + a_2, b_1 + b_2, 1)$$

$$\therefore \left(\vec{X} + \vec{Y}\right) \notin W$$

Thus W is no a subspace of $\mathbb{R}^3$.

Theorem: Let V be a vector space with the two operations $\oplus$ and $\odot$, $\phi \neq W \subseteq V$. Then W is a subspace of V if and only if the two conditions satisfy

(1) If $\vec{X}, \vec{Y} \in W$, then $\vec{X} \oplus \vec{Y} \in W$

(2) If $t \in \mathbb{R}$ and $\vec{X} \in W$, then $t \odot \vec{X} \in W$

Proof: **(⇒)** Let $W \subseteq V$

$\therefore$ W is a subspace from the definition.

$\therefore$ W is closed for addition and closed for multiplication by a number i.e. properties (1) and (2) are satisfied.

(⇐) If W is closed for addition and closed for multiplication by a number, we must prove that W is a vector space.

First: axioms of addition:

(1) W closed for the addition from hypothesis.

(2) If $\vec{X},\vec{Y},\vec{Z}\in W$, then $\vec{X},\vec{Y},\vec{Z}\in V$ because $W\subseteq V$

But the addition is associative on V (since V is a vector space)

$(\vec{X}\oplus\vec{Y})\oplus\vec{Z}=\vec{X}\oplus(\vec{Y}\oplus\vec{Z})$

$\therefore$ $\oplus$ associative on W

(3) Let $\vec{X}\in W$, because W is closed for multiplication by a number, so $(-1)\vec{X}=-\vec{X}\in W$

$-\vec{X}\oplus\vec{X}=(-\vec{X})\oplus\vec{X}=\vec{O}$ (W closed for the addition)

Thus $-\vec{X}$ the additive inverse for the vector $\vec{X}$.

(4) Because $(-\vec{X})\in W$ and $\vec{X}\oplus(-\vec{X})=(-\vec{X})\oplus\vec{X}=\vec{O}$ (W closed for the addition)

Thus $\vec{O}\in W$. So $\vec{O}$ is the identity element for the addition operation.

(5) If $\vec{X},\vec{Y}\in W$, then $\vec{X},\vec{Y}\in V$ because $W\subseteq V$

But the addition is commutative on V (since V is a vector space)

$\therefore\ \vec{X}\oplus\vec{Y}=\vec{Y}\oplus\vec{X}$

$\therefore$ $\oplus$ commutative on W

Second: Axioms of Scalar Multiplication

(6) W is closed for multiplication by a number by hypothesis.

(7) Let $\alpha,\beta\in\mathbb{R}$ and $\vec{X}\in W$, then $\vec{X}\in V$ because $W\subseteq V$

But V is a vector space, thus $(\alpha\beta)\vec{X}=\alpha(\beta\vec{X})\in W$.

(8) Let $\vec{X}\in W$, then $\vec{X}\in V$ because $W\subseteq V$

But V is a vector space, thus $1\odot\vec{X}=\vec{X}$.

Third: Axioms of Distributives

(9) Let $\alpha,\beta\in\mathbb{R}$ and $\vec{X}\in W$, then $\vec{X}\in V$ because $W\subseteq V$

But V is a vector space, thus $(\alpha+\beta)\vec{X}=\alpha\vec{X}\oplus\beta\vec{X}$.

(10) Let $\alpha\in\mathbb{R}$ and $\vec{X},\vec{Y}\in W$, then $\vec{X},\vec{Y}\in V$ because $W\subseteq V$

But V is a vector space, thus $\alpha(\vec{X}\oplus\vec{Y})=\alpha\vec{X}\oplus\alpha\vec{Y}$.

Therefore (W,⊕,⊙) is a vector space and also is a subspace of (V,⊕,⊙).

Example: Let W = {(a,b,0); a, b $\in \mathbb{R}$ }, W $\subset \mathbb{R}^3$, is W a subspace of $\mathbb{R}^3$?

Solution: Let $\overrightarrow{X}=(a_1,b_1,0),\overrightarrow{Y}=(a_2,b_2,0)\in W$

(1) $\overrightarrow{X}+\overrightarrow{Y}=(a_1,b_1,0)+(a_2,b_2,0)$

$=(a_1+a_2,b_1+b_2,0)$

Since the third component equal zero, so $\overrightarrow{X}+\overrightarrow{Y}\in W$

(2) Let $\overrightarrow{X}=(a_1,b_1,0)\in W,\alpha\in\mathbb{R}$, then $\alpha\overrightarrow{X}=\alpha(a_1,b_1,0)=(\alpha a_1,\alpha b_1,0)\in W$

$\therefore \alpha\overrightarrow{X}\in W$

The two properties are satisfy, thus W is a subspace of $\mathbb{R}^3$.

Exercise: Let W={(a,b); b = 2a,a, $b$$\in\mathbb{R}$ }, W $\subset \mathbb{R}^2$. Is W subspace of $\mathbb{R}^2$?

Remarks: If (V,⊕,⊙) is any vector space, then

(1) (V,⊕,⊙) is a subspace by itself because V $\subseteq$ V and V is a vector space.

(2) $W=\left\{\overrightarrow{O}\right\}$ is a subspace of V.

Therefore for any non zero vector space there exists at least two subspaces of it.

Proof (2): $\overrightarrow{O}\in W$, then $\overrightarrow{O}+\overrightarrow{O}=\overrightarrow{O}\in W$

$\therefore$ W is closed for addition operation

$\overrightarrow{O}\in W,\alpha\in\mathbb{R}$, then $\alpha\cdot\overrightarrow{O}=\overrightarrow{O}\in W$

$\therefore$ W is closed for multiplication by a number

$\therefore$ W is a subspace of V

Exercise: Let $V=\mathbb{R}^2$, W = {(x,y); a x + by = 0, x ,y $\in\mathbb{R}$ } where a and b are real numbers. Prove that W is a subspace of $\mathbb{R}^2$.

Examples:

(1) Let $A\overrightarrow{X}=\overrightarrow{O}$ homogeneous system where A is a matrix of degree m×n and let $W\subseteq\mathbb{R}^n$ contains all the solutions for this homogeneous system, then W is a subspace of $\mathbb{R}^n$.

Solution: Let $\overrightarrow{Y},\overrightarrow{Z}\in W\subseteq\mathbb{R}^n$, then $\overrightarrow{Y}$ and $\overrightarrow{Z}$ two solutions for the homogeneous system, i.e. $A\overrightarrow{Y}=\overrightarrow{O}$ and $A\overrightarrow{Z}=\overrightarrow{O}$

(a) $A(\overrightarrow{Y}+\overrightarrow{Z})=A\overrightarrow{Y}+A\overrightarrow{Z}\ =\overrightarrow{O}+\overrightarrow{O}$

$\therefore\ A(\overrightarrow{Y}+\overrightarrow{Z})=\overrightarrow{O}$

Thus $\overrightarrow{Y}+\overrightarrow{Z}$ solution for the homogeneous system, i.e. $\overrightarrow{Y}+\overrightarrow{Z}\in W$.

(b) Let $t \in \mathbb{R}, \vec{Y} \in W$

$A(t\vec{Y}) = t\,(A\vec{Y})$ (by previous theorem)

$= t\,(\vec{O})$

$A(t\vec{Y}) = \vec{O}$

$\therefore t\,\vec{Y} \in W$

Thus $t\,\vec{Y}$ is a solution for the homogeneous system, i.e. $t\ t\ \vec{Y} \in W$.

Therefore W is a subspace of $\mathbb{R}^n$.

(2) Let V be a vector space and let U and W two subspaces of V. Prove that $G = U \cap W$ is a subspace of V.

Proof: It is clear that $G \neq \phi$ because $\vec{O} \in U$ and $\vec{O} \in W$, so $\vec{O} \in U \cap W$

(a) Let $\vec{X}, \vec{Y} \in G \Rightarrow \vec{X}, \vec{Y} \in U \Rightarrow \vec{X} + \vec{Y} \in U$ (U is a subspace)

Also, $\Rightarrow \vec{X}, \vec{Y} \in W \Rightarrow \vec{X} + \vec{Y} \in W$ (W is a subspace)

Thus $\vec{X} + \vec{Y} \in G$

(b) Let $\alpha \in \mathbb{R}$, $\alpha\vec{X} \in U$ and $\alpha\vec{X} \in W$, thus $\alpha\vec{X} \in G$

$\therefore$ G is a subspace of V

Exercises:

(1) Let $W = \{(x,y,z);\ a\,x + by + cz = 0\}$, $a, b, c \in \mathbb{R}$ not all zero. Show that W is a subspace of $\mathbb{R}^3$.

(2) Let W = {A: A is a square matrix of degree 2×2 and invertible matrix}. Is W subspace of $V = M_{2\times2}(\mathbb{R})\}$.

(3) Let W be a set of the diagonal matrices of degree $n\times n$. Is W subspace of $V = M_{n\times n}(\mathbb{R})\}$.

(4) Let W = {A: A is a square matrix of degree 2×2, $A^2 = I$ }. Is W subspace of $V = M_{2\times2}(\mathbb{R})\}$.

(5) Which of the following sets of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, set of all vectors of the form:

(a) $(a,b,2)$

(b) (a,b,c), where $a + b = c$

(c) (a,b,c), where $c > 0$

(d) (a,b,c), where $a = c = 0$

(e) (a,b,c), where $1+2a = b$

(f) (a,b,c), where $a = -c$

(6) Show that the set of the solutions for the system $A\vec{X} = B$, where A is a matrix of degree m×n not a subspace of $\mathbb{R}^n$ if $B \neq O$.

(7) Let V be a vector space, U and W are two subspaces of V. Prove that
$G = U + W = \{\vec{u} + \vec{w}, \vec{u} \in U \text{ and } \vec{w} \in W\}$ is a subspace.

Linear Combination

Definition: We say that the vector $\vec{X}$ is a linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}$ if we can write it as the form $\vec{X} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_n\overrightarrow{X_n}$, where $c_1, c_2, \ldots, c_n$ are numbers

Examples:

(1) Consider the vectors $\overrightarrow{X_1} = (1,2,1,-1)$, $\overrightarrow{X_2} = (1,0,2,-3)$, $\overrightarrow{X_3} = (1,1,0,-2)$ in $\mathbb{R}^4$, show that the vector $\vec{X} = (2,1,5,-5)$ is a linear combination of $\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}$.

Solution: Suppose that we have c_1, c_2 and c_3 where

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + c_3\overrightarrow{X_3} = \vec{X}$$

$$c_1(1,2,1,-1) + c_2(1,0,2,-3) + c_3(1,1,0,-2) = (2,1,5,-5)$$

$$(c_1,2c_1,c_1,-c_1) + (c_2,0,2c_2,-3c_2) + (c_3,c_3,0,-2c_3) = (2,1,5,-5)$$

$$(c_1 + c_2 + c_3,\ 2c_1 + c_3, c_1 + 2c_2, -c_1 - 3c_2 - 2c_3) = (2,1,5,-5)$$

$$c_1 + c_2 + c_3 = 2 \qquad \ldots(1)$$

$$2c_1 + c_3 = 1 \qquad \ldots(2)$$

$$c_1 + 2c_2 = 5 \qquad \ldots(3)$$

$$-c_1 - 3c_2 - 2c_3 = -5 \qquad \ldots(4)$$

We use Gauss-Jordan reduction method to find the values of c_1, c_2 and c_3

$$\left[\begin{array}{ccccc}1 & 1 & 1 & \vdots & 2\\ 2 & 0 & 1 & \vdots & 1\\ 1 & 2 & 0 & \vdots & 5\\ -1 & -3 & -2 & \vdots & -5\end{array}\right] \begin{array}{c}\xrightarrow{R_2=r_2-2r_1}\\ \xrightarrow{R_3=r_3-r_1}\\ \xrightarrow{R_4=r_4+r_1}\end{array} \left[\begin{array}{ccccc}1 & 1 & 1 & \vdots & 2\\ 0 & -2 & -1 & \vdots & -3\\ 0 & 1 & -1 & \vdots & 3\\ 0 & -2 & -1 & \vdots & -3\end{array}\right]$$

$$\begin{array}{c}\xrightarrow{R_1=r_1-r_3}\\ \xrightarrow{R_2=r_2+2r_3}\\ \xrightarrow{R_4=r_4+2r_3}\end{array} \left[\begin{array}{ccccc}1 & 0 & 2 & \vdots & -1\\ 0 & 0 & -3 & \vdots & -3\\ 0 & 1 & -1 & \vdots & 3\\ 0 & 0 & -3 & \vdots & 3\end{array}\right]$$

$c_1 + 2c_3 = -1$

$-3c_3 = 3 \Rightarrow c_3 = -1$

$c_2 - c_3 = 3$

$-3c_3 = 3 \Rightarrow c_3 = -1$

$$\begin{cases} c_1 - 2 = -1 \Rightarrow c_1 = 1\\ c_2 + 1 = 3 \Rightarrow c_2 = 2\\ c_3 = -1\end{cases}$$

Therefore

$1(1,2,1,-1) + 2(1,0,2,-3) - 1(1,1,0,-2) = (2,1,5,-5)$

$\therefore\ \vec{X}$ is a linear combination of $\vec{X}_1, \vec{X}_2$ and $\vec{X}_3$.

(2) Let $\vec{X}_1 = (1,2,-1)$ and $\vec{X}_2 = (1,0,-1)$ two vectors in $\mathbb{R}^3$. Is the vector $\vec{X} = (1,0,2)$ a linear combination of $\vec{X}_1$ and $\vec{X}_2$.

Solution: The vector $\vec{X}$ to be a linear combination of $\vec{X}_1$ and $\vec{X}_2$ we must find two numbers c_1 and c_2 such that

$c_1\vec{X}_1 + c_2\vec{X}_2 = \vec{X}$

$c_1(1,2,-1) + c_2(1,0,-1) = (1,0,2)$

$(c_1, 2c_1, -c_1) + (c_2, 0, -c_2) = (1,0,2)$

$(c_1 + c_2,\ 2c_1, -c_1 - c_2) = (1,0,2)$

$c_1 + c_2 = 1 \quad \ldots(1)$

$2c_1 = 0 \quad \ldots(2)$

$-c_1 - c_2 = 2 \quad \ldots(3)$

We use Gauss-Jordan reduction method to find the values of c_1 and c_2

$$\begin{bmatrix} 1 & 1 & \vdots & 1 \\ 2 & 0 & \vdots & 0 \\ -1 & -1 & \vdots & 2 \end{bmatrix} \xrightarrow[R_3=r_3+r_1]{R_2=r_2-2r_1} \begin{bmatrix} 1 & 1 & \vdots & 1 \\ 0 & -2 & \vdots & -2 \\ 0 & 0 & \vdots & 3 \end{bmatrix}$$

The third row means $0\,c_1 + 0\,c_2 = 3 \Rightarrow 0 = 3$ which is impossible.
Thus the system has no solution.
$\therefore\ \overrightarrow{X}$ is not a linear combination of $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$.

Exercises:

(1) Which of the following vectors is a linear combination of the vectors
$\overrightarrow{X_1} = (2,1,-2)$, $\overrightarrow{X_2} = (-2,-1,0)$, $\overrightarrow{X_3} = (4,2,-3)$

(a) $\overrightarrow{X} = (1,0,0)$ (b) $\overrightarrow{X} = (0,0,1)$ **(c)** $\overrightarrow{X} = (1,1,1)$ **(d)** $\overrightarrow{X} = (4,2,-6)$

(2) If possible express the vector (1,1,1) as a linear combination of the vectors in $\mathbb{R}^3$
$\overrightarrow{X_1} = (2,1,-3)$, $\overrightarrow{X_2} = (1,2,-2)$, $\overrightarrow{X_3} = (2,-5,-1)$.

(3) If possible express the vector $\overrightarrow{X} = (4,5,1)$ as a linear combination of the vectors in $\mathbb{R}^3$ $\overrightarrow{X_1} = (1,2,1)$, $\overrightarrow{X_2} = (2,3,1)$, $\overrightarrow{X_3} = (1,1,0)$. (the solution $c_1 = 2$, $c_2 = -1$, $c_3 = 4$)

Theorem: Let V be a vector space and $\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n} \in V$ and let W be a set of all linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n}$, i.e.
$W = \{c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_n\overrightarrow{X_n}, c_1, c_2, ..., c_n \in \mathbb{R}\}$, then W is a subspace of V.

Proof: because $\overrightarrow{O} = 0\overrightarrow{X_1} + 0\overrightarrow{X_2} + ... + 0\overrightarrow{X_n}$

$\therefore\ \overrightarrow{O} \in W \Rightarrow W \neq \phi$

$\therefore W \subseteq V$

(1) Let $\overrightarrow{X}, \overrightarrow{Y} \in W$ so there exists $c_1, c_2, ..., c_n \in \mathbb{R}$ such that $\overrightarrow{X} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_n\overrightarrow{X_n}$ and there exists $d_1, d_2, ..., d_n \in \mathbb{R}$ such that $\overrightarrow{Y} = d_1\overrightarrow{Y_1} + d_2\overrightarrow{Y_2} + ... + d_n\overrightarrow{Y_n}$

$$\overrightarrow{X} \oplus \overrightarrow{Y} = (c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_n\overrightarrow{X_n}) \oplus (d_1\overrightarrow{Y_1} + d_2\overrightarrow{Y_2} + ... + d_n\overrightarrow{Y_n})$$

$$= (c_1 + d_1)\overrightarrow{X_1} + (c_2 + d_2)\overrightarrow{X_2} + ... + (c_n + d_n)\overrightarrow{X_n}$$

$\overrightarrow{X} \oplus \overrightarrow{Y} = e_1\overrightarrow{X_1} + e_2\overrightarrow{X_2} + ... + e_n\overrightarrow{X_n}$, where $c_1 + d_1 = e_1, c_2 + d_2 = e_2, ..., c_n + d_n = e_n$

$\therefore\ \overrightarrow{X} \oplus \overrightarrow{Y} \in W$ (W closed for the addition)

(2) Let $\overrightarrow{X}=c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+\ldots+c_n\overrightarrow{X_n}\in W, t\in\mathbb{R}$

$$t\overrightarrow{X}=t(c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+\ldots+c_n\overrightarrow{X_n})$$

$$=(tc_1)\overrightarrow{X_1}+(tc_2)\overrightarrow{X_2}+\ldots+(tc_n)\overrightarrow{X_n} \qquad \text{Axioms of Scalar Multiplication: } (\alpha\beta)\overrightarrow{X}=\alpha(\beta\overrightarrow{X})$$

$$=k_1\overrightarrow{X_1}+k_2\overrightarrow{X_2}+\ldots+k_n\overrightarrow{X_n} \qquad k_i=tc_i \;,\; i=1,2,\ldots,n$$

$\therefore\ t\overrightarrow{X}\in W$ (W is closed for multiplication by a number)

Therefore W is a subspace of V.

Span (Generating) Set

Definition: Let $S=\{\overrightarrow{X_1},\overrightarrow{X_2},\ldots,\overrightarrow{X_n}\}$ be a set of vectors in a vector space V. We say that the set S spans V or V generated by S if any vector in V is a linear combination of the vectors of S.

Examples:

(1) Show that the set S = {(1,0), (0,1)} spans $\mathbb{R}^2$.

Solution: We must prove that every vector in $\mathbb{R}^2$ can be written as a linear combination of the vectors (1,0), (0,1).

Let $(a,b)\in\mathbb{R}^2$, to find the values of the numbers c_1 and c_2 such that

$(a,b) = c_1(1,0) + c_2(0,1)$

$= (c_1,0) + (0,c_2)$

$= (c_1,c_2) \Rightarrow c_1 = a \;,\; c_2 = b$

$\therefore\ (a,b) = a(1,0) + b(0,1)$

$\therefore$ S spans $\mathbb{R}^2$.

(2) Let $\overrightarrow{X_1}=(1,2),\overrightarrow{X_2}=(-1,1)$ show that the space spans by the set $\{\overrightarrow{X_1},\overrightarrow{X_2}\}$ is $\mathbb{R}^2$.

Solution: Let $\overrightarrow{X}=(a,b)\in\mathbb{R}^2, a,b\in\mathbb{R}$, to find the values of the numbers c_1 and c_2 such that

$\overrightarrow{X}=c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}$

$(a,b) = c_1(1,2) + c_2(-1,1)$

$= (c_1,2c_1) + (-c_2,c_2)$

$= (c_1 - c_2, 2c_1 + c_2)$

$c_1 - c_2 = a \quad \ldots(1)$

$2c_1 + c_2 = b \quad \ldots(2)$

------------------------------------- by addition

$3c_1 = a + b \Rightarrow c_1 = \frac{1}{3}(a+b)$ and $c_2 = \frac{1}{3}(b - 2a)$

$\overrightarrow{X} = \frac{1}{3}(a+b)\overrightarrow{X_1} + \frac{1}{3}(b-2a)\overrightarrow{X_2}$

$\therefore \{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ spans $\mathbb{R}^2$.

(3) Find the space spans by the set {(–1,2,1)}.

Solution:

W = {c (–1,2,1); $c \in \mathbb{R}$} = {(–c,2c,c); $c \in \mathbb{R}$}

= {c (–1,2,1); $c \in \mathbb{R}$}

This is equation of a straight line passing through the two points (–1,2,1) and (0,0,0).

(4) Find the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ where $\overrightarrow{X_1} = (-1,2,3), \overrightarrow{X_2} = (-2,4,6)$.

Solution: Let W is the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$, i.e.

W = $\{c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2}; c_1, c_2 \in \mathbb{R}\}$

= $\{c_1(-1,2,3) + c_2(-2,4,6)\ ; c_1, c_2 \in \mathbb{R}\}$

= $\{(-c_1, 2c_1, 3c_1) + (-2c_2, 4c_2, 6c_2)\ ; c_1, c_2 \in \mathbb{R}\}$

= $\{(-c_1 - 2c_2, 2c_1 + 4c_2, 3c_1 + 6c_2)\ ; c_1, c_2 \in \mathbb{R}\}$

= $\{(-1(c_1 + 2c_2), 2(c_1 + 2c_2), 3(c_1 + 2c_2))\ ; c_1, c_2 \in \mathbb{R}\}$

= $\{(c_1 + 2c_2)(-1,2,3)\ ; c_1, c_2 \in \mathbb{R}\}$

= $\{c_3(-1,2,3)\ ; c_3 \in \mathbb{R}\}$, suppose $c_1 + 2c_2 = c_3$

This is equation of a straight line passing through the two points (0,0,0) and (–1,2,3).

Note that $\overrightarrow{X_2} = 2\overrightarrow{X_1}$

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} = c_1\overrightarrow{X_1} + c_2(2\overrightarrow{X_1})$

$= (c_1 + 2c_2)\overrightarrow{X_1}$

$= c_3\overrightarrow{X_1}$, suppose $c_1 + 2c_2 = c_3$

$\therefore$ The space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is the same space spans by the set $\{\overrightarrow{X_1}\}$

or **Note that** $\overrightarrow{X_1} = \frac{1}{2}\overrightarrow{X_2}$

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} = c_1(\frac{1}{2}\overrightarrow{X_2}) + c_2\overrightarrow{X_2}$$
$$= (\frac{1}{2}c_1 + c_2)\overrightarrow{X_2}$$
$$= c_4\overrightarrow{X_2} \ , \qquad \text{suppose } \tfrac{1}{2}c_1 + c_2 = c_4$$

$\therefore$ The space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is the same space spans by the set $\{\overrightarrow{X_2}\}$

Remark: Every line passing through the origin point is a subspace of $\mathbb{R}^2$.

(5) Express the zero vector as a linear combination of the two vectors $\overrightarrow{X_1} = (2,3), \overrightarrow{X_2} = (-3,1)$ in $\mathbb{R}^2$

Solution: To find the values of the numbers c_1 and c_2 such that

$\overrightarrow{O} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2}$

$(0,0) = c_1(2,3) + c_2(-3,1)$

$\quad = (2c_1, 3c_1) + (-3c_2, c_2) = (2c_1 - 3c_2,\ 3c_1 + c_2)$

$2c_1 - 3c_2 = 0$

$3c_1 + c_2 = 0 \Rightarrow c_2 = -3c_1$

$\therefore 2c_1 + 9c_1 = 0 \Rightarrow 11c_1 = 0 \Rightarrow c_1 = 0$

$\qquad \Rightarrow c_2 = -3(0) = 0$

$(0,0) = 0\cdot\overrightarrow{X_1} + 0\cdot\overrightarrow{X_2} = 0\cdot(2,3) + 0\cdot(-3,1)$

(6) Let W = $\{(x,y,z): x - 2y + z = 0\}$

(a) Prove that W is a subspace of $\mathbb{R}^3$. **(Home work)**

(b) Find the two vectors $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$ such that W is generated by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$

Solution (b):

W = $\{(x,y,z): x - 2y + z = 0\}$

$\quad = \{(x, y, 2y - x): x, y \in \mathbb{R}\}$

$\quad = \{(0, y, 2y) + (x, 0, -x);\ x, y \in \mathbb{R}\}$

$\quad = \{y(0,1,2) + x(1,0,-1);\ x, y \in \mathbb{R}\}$

This is a set of linear combination of the two vectors (0,1,2) and (1,0,–1).

Let $\overrightarrow{X_1} = (1,0,-1)$ and $\overrightarrow{X_2} = (0,1,2)$

$\therefore$ W is generated by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$, i.e. $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ spans W.

Exercises:

(1) Let $\overrightarrow{X_1} = (1,0)$ and $\overrightarrow{X_2} = (1,1)$ show that the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ is $\mathbb{R}^2$.

(2) Find the space spans by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}\}$ in $\mathbb{R}^3$, where $\overrightarrow{X_1} = (1,-1,2)$ and $\overrightarrow{X_2} = (0,1,1)$.

(3) Which of the following sets spans $\mathbb{R}^2$

(a) (–1,1), (1,2) **(b)** (1,0), (0,1) **(c)** (0,0), (1,1), (–2,–2) **(d)** (2,4), (–1,2)

(4) Which of the following sets spans $\mathbb{R}^3$

(a) (1,0,0), (0,1,0), (0,0,1) **(b)** (1,–1,2), (0,1,1)

(c) (1,1,0), (1,0,1), (0,1,1) **(d)** (2,2,3), (–1,–2,1), (0,1,0)

Linear Independence

Definition: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_k}\}$ be a set of distinct vectors in a vector space V. We say the set S is non linearly independent (dependent) if there exists a constants (c_1, c_2, …,c_k) not all zero, such that

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_k\overrightarrow{X_k} = \overrightarrow{O} \qquad \ldots(1)$$

Conversely, the set S is said to be linearly independent. That is, S is linearly independent if it satisfy (1) only when $c_1 = c_2 = \ldots = c_k = 0$.

Remark: If the set $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_k}\}$, then S is non linearly independent if there exists a linear combination of vectors of the set S equal to zero, i.e.

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_k\overrightarrow{X_k} = \overrightarrow{O}$, such that at least one of $c_i \neq 0$, $1 \leq i \leq k$.

Example: Let S = {(1,0),(0,1)}. Show that S is linearly independent set.

Solution: Let c_1, c_2 be a constants

c_1 (1,0) + c_2 (0,1) = (0,0)

(c_1,0) + (0,c_2) = (0,0)

(c_1,c_2) = (0,0) $\Rightarrow$ $c_1 = 0$, $c_2 = 0$

$\therefore$ S is linearly independent set

Exercise: Let S = {(2,2),(0,1)}. Is the set S linearly independent or dependent?

Example: Let S ={(1,1,–1),(2,3,–4),(4,3,–2)}. Show that S is non linearly independent (dependent) set.

Solution: Let c_1 and c_2 and c_3 be a constants

$c_1 (1,1,-1) + c_2 (2,3,-4) + c_3 (4,3,-2) = (0,0,0)$

$c_1 + 2c_2 + 4c_3 = 0$

$c_1 + 3c_2 + 3c_3 = 0$

$-c_1 - 4c_2 - 2c_3 = 0$

The coefficients matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ -1 & -4 & -2 \end{bmatrix}$, because $|A| = 0$

$\therefore$ The linear system or the set has no trivial solution.

Thus S is non linearly independent (dependent) set.

Remark: If the system of equations is homogeneous and the determinant of the coefficients matrix equal to zero, then this means that the system has a solution other than the zero solution, but if the determinant not equal to zero, this means that the system has a unique solution (the zero solution).

To make sure of this, we will find the values of c_1, c_2 and c_3 in the previous example

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ -1 & -4 & -2 \end{bmatrix} \xrightarrow[R_3=r_3+r_1]{R_2=r_2-r_1} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow[R_3=r_3+2r_2]{R_1=r_1-2r_2} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$c_1 + 6c_3 = 0 \Rightarrow c_1 = -6c_3$

$c_2 - c_3 = 0 \Rightarrow c_2 = c_3$

Let $c_3 = 1$, then $c_1 = -6$ and $c_2 = 1$.

$-6(1,1,-1) + 1(2,3,-4) + 1(4,3,-2) = (0,0,0)$

Thus S is non linearly independent (dependent) set.

Exercise: Show that S = {(0,1,1),(2,1,2),(1,2,1)} is linearly independent set by using the previous remark.

Examples:

(1) If $S=\{\overrightarrow{X_1},\overrightarrow{X_2}\}$ is a linearly independent set in the vector space V. Prove that the set $\{\overrightarrow{X_1}+\overrightarrow{X_2},\overrightarrow{X_1}-\overrightarrow{X_2}\}$ is also linearly independent.

Solution: Let c_1 and c_2 be a constants

$$c_1(\overrightarrow{X_1}+\overrightarrow{X_2})+c_2(\overrightarrow{X_1}-\overrightarrow{X_2})=\vec{O}$$

$$(c_1+c_2)\overrightarrow{X_1}+(c_1-c_2)\overrightarrow{X_2}=\vec{O}$$

Because $\overrightarrow{X_1}$ and $\overrightarrow{X_2}$ are linearly independent, so

$c_1 + c_2 = 0$

$c_1 - c_2 = 0$

-------------------- by addition

$2c_1 = 0 \Rightarrow c_1 = 0$, $c_2 = 0$

$\therefore$ The set $\{\overrightarrow{X_1}+\overrightarrow{X_2},\overrightarrow{X_1}-\overrightarrow{X_2}\}$ is also linearly independent.

(2) If $\overrightarrow{E_1}$ and $\overrightarrow{E_2}$ are two vectors in the vector space $\mathbb{R}^2$ such that $\overrightarrow{E_1}=(1,0)$ and $\overrightarrow{E_2}=(0,1)$. Prove that $S=\{\overrightarrow{E_1},\overrightarrow{E_2}\}$ is a linearly independent set. Also prove that the set $\{\overrightarrow{E_1},\overrightarrow{E_2},...,\overrightarrow{E_n}\}$ in $\mathbb{R}^n$ is linearly independent, where

$\overrightarrow{E_1}=(1,0,0,...,0),\overrightarrow{E_2}=(0,1,0,0,...,0),...,\overrightarrow{E_n}=(0,0,...,0,1)$

Solution: Let c_1 and c_2 be a constants

$$c_1\overrightarrow{E_1}+c_2\overrightarrow{E_2}=\vec{O}$$

$$c_1(1,0)+c_2(0,1)=(0,0)$$

$$(c_1,0)+(0,c_2)=(0,0)$$

$$(c_1,c_2)=(0,0)\Rightarrow c_1=0,\ c_2=0$$

$\therefore$ S is a linearly independent set

In the same way (or more generally) the set $\{\overrightarrow{E_1},\overrightarrow{E_2},...,\overrightarrow{E_n}\}$ is a linearly independent set.

(3) Let $P_1(t) = t^2 + t + 2$, $P_2(t) = 2t^2 + t$, $P_3(t) = 3t^2 + 2t + 2$. Is the set $S = \{P_1(t),P_2(t),P_3(t)\}$ linearly independent or not?

Solution: Let c_1 and c_2 and c_3 be a constants

$$c_1P_1(t)+c_2P_2(t)+c_3P_3(t)=0$$

$$c_1(t^2 + t + 2) + c_2(2t^2 + t) + c_3(3t^2 + 2t + 2) = 0$$

$(c_1 + 2c_2 + 3c_3)\, t^2 + (c_1 + c_2 + 2c_3)\, t + (2c_1 + 2c_3) = 0\, t^2 + 0\, t + 0$

$c_1 + 2c_2 + 3c_3 = 0$

$c_1 + c_2 + 2c_3 = 0$

$2c_1 \quad + 2c_3 = 0$

This homogeneous system has infinity of solutions **(check it)**

One of these solutions is $c_1 = 1$, $c_2 = 1$, $c_3 = -1$. So that $P_1(t) + P_2(t) - P_3(t) = \overrightarrow{O}$.

$\therefore$ The set S is non linearly independent (dependent).

Exercises:

(1) Show whether the following sets are linearly independent in V, where V is the vector space of all polynomials of second degree or less

(a) $S_1=\{x, 2x-1, 1\}$ **(b)** $S_2=\{2x^2, 3x^2\}$ **(c)** $S_3=\{1, 2x+3, x^2+2x+1, 3x^2-2x\}$

(2) Show whether the following sets are linearly independent in the vector space V, where V is the vector space of 2×2 matrices

(a) $S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

(b) $S_2 = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -4 \\ -7 & 3 \end{bmatrix} \right\}$

Theorem: S is a set of nonzero vectors in a vector space V. S is non linearly independent if and only if one of these vectors is a linear combination of all other vectors in S.

Proof: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_{i-1}}, \overrightarrow{X_i}, \overrightarrow{X_{i+1}}, ..., \overrightarrow{X_k}\}$

Because S is non linearly independent

$\therefore$ There exists a constants $c_1, c_2, \ldots, c_i, \ldots, c_k$ not all zero ($c_i \neq 0$)

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_{i-1}\overrightarrow{X_{i-1}} + c_i\overrightarrow{X_i} + c_{i+1}\overrightarrow{X_{i+1}} + ... + c_k\overrightarrow{X_k} = \overrightarrow{O}$$

$$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_{i-1}\overrightarrow{X_{i-1}} + ... + c_{i+1}\overrightarrow{X_{i+1}} + ... + c_k\overrightarrow{X_k} = -c_i\overrightarrow{X_i}$$

$$\left(\frac{-c_1}{c_i}\right)\overrightarrow{X_1} + \left(\frac{-c_2}{c_i}\right)\overrightarrow{X_2} + ... + \left(\frac{-c_{i-1}}{c_i}\right)\overrightarrow{X_{i-1}} + ... + \left(\frac{-c_{i+1}}{c_i}\right)\overrightarrow{X_{i+1}} + ... + \left(\frac{-c_k}{c_i}\right)\overrightarrow{X_k} = \overrightarrow{X_i}$$

$\therefore$ $\overrightarrow{X_i}$ is a linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_{i-1}}, \overrightarrow{X_{i+1}}, ..., \overrightarrow{X_k}$.

Conversely, $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_i}, \overrightarrow{X_{i+1}}, , \ldots, \overrightarrow{X_k}\}$

If $\overrightarrow{X_i}$ is a linear combination of the vectors $\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_{i+1}}, , \ldots, \overrightarrow{X_k}$

$\therefore$ There exists a constants $c_1, c_2, \ldots, c_{i-1}, \ldots, c_{i+1}, \ldots, c_k$ such that

$$\overrightarrow{X_i} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_{i-1}\overrightarrow{X_{i-1}} + \ldots + c_{i+1}\overrightarrow{X_{i+1}} + \ldots + c_k\overrightarrow{X_k}$$

$$\overrightarrow{O} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_{i-1}\overrightarrow{X_{i-1}} + (-1)\overrightarrow{X_i} + c_{i+1}\overrightarrow{X_{i+1}} + \ldots + c_k\overrightarrow{X_k}$$

$\therefore$ The set $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_{i-1}}, \overrightarrow{X_i}, \overrightarrow{X_{i+1}}, , \ldots, \overrightarrow{X_k}\}$ is non linearly independent

Examples:

(1) The set S = {(2,6),(3,9),(1,3)} is non linearly independent (dependent) since

1(3,9) = 1(2,6) + 1(1,3)

That is the vector (3,9) is a linear combination of the other vectors.

or solve using the definition

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + c_3\overrightarrow{X_3} = \overrightarrow{O} \quad \Rightarrow \quad 1(2,6) + 1(1,3) - 1(3,9) = (0,0)$

(2) Let $f(x) = x^2$, $g(x) = x$, $h(x) = 1$ and $j(x) = (x + 1)^2$, for all $x \in \mathbb{R}$. Show that the set $S = \{f, g, h, j\}$ is non linearly independent in the vector spaces for all real functions

Solution:

$j(x) = (x + 1)^2 = x^2 + 2x + 1$

$\quad = 1\cdot f(x) + 2\cdot g(x) + 1\cdot h(x)$

$\quad j = 1\cdot f + 2\cdot g + 1\cdot h$

$\therefore$ $S = \{f, g, h, j\}$ is non linearly independent.

Theorem: In $\mathbb{R}^n$ any set containing more than n elements is a non linearly independent set.

Proof: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_m}\}$ set in $\mathbb{R}^n$ such that m > n, if

$\overrightarrow{X_1} = (a_{11}, a_{12}, \ldots a_{1n})$, $\overrightarrow{X_2} = (a_{21}, a_{22}, \ldots a_{2n})$, …, $\overrightarrow{X_m} = (a_{m1}, a_{m2}, \ldots a_{mn})$

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_m\overrightarrow{X_m} = \overrightarrow{O}$,

$c_1(a_{11}, a_{12}, \ldots a_{1n}) + c_2(a_{21}, a_{22}, \ldots a_{2n}) + \ldots + c_m(a_{m1}, a_{m2}, \ldots a_{mn}) = (0, 0, \ldots, 0)$

$c_1a_{11} + c_2a_{21} + \ldots + c_ma_{m1} = 0 \qquad \ldots(1)$

$c_1a_{12} + c_2a_{22} + \ldots + c_ma_{m2} = 0 \qquad \ldots(2)$

$\vdots \qquad \vdots \qquad \vdots$

$c_1a_{1n} + c_2a_{2n} + \ldots + c_ma_{mn} = 0 \qquad \ldots(n)$

The set of homogeneous equations (homogeneous system) contains (n) of equations and (m) of unknowns i.e.
The number of unknowns (m) > The number of equations (n).

$\therefore$ The homogeneous system has a non-trivial solution.

$\therefore$ S is non linearly independent

Example: Is the set S = {(1,1,1), (1,2,1), (1,3,5), (0,1,–1)} in $\mathbb{R}^3$ linearly independent or dependent set?

Solution: Because 4 > 3 (m > n), so S is non linearly independent (linearly dependent).

Basis and Dimension

Definition of the basis: A set of the vectors $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}\}$ in a vector space V is said to be the basis of the vector space V if **(1)** S spans V
(2) S is linearly independent

Example: Let $S=\{\overrightarrow{E_1},\overrightarrow{E_2}\}$ where $\overrightarrow{E_1}$ = (1,0) and $\overrightarrow{E_2}$ = (0,1), then S is a basis for $\mathbb{R}^2$.

Solution:

(1) Let c_1 and c_2 be a constants

$$c_1\overrightarrow{E_1}+c_2\overrightarrow{E_2}=\overrightarrow{O}$$
$$c_1(1,0)+c_2(0,1)=(0,0)$$
$$(c_1,0)+(0,c_2)=(0,0)$$
$$(c_1,c_2)=(0,0)\Rightarrow c_1=0,\ c_2=0$$

$\therefore$ S is linearly independent set.

(2) Let $(x,y)\in\mathbb{R}^2$ and k_1 , k_2 constants

$$(x,y)=k_1\overrightarrow{E_1}+k_2\overrightarrow{E_2}$$
$$=k_1(1,0)+k_2(0,1)$$
$$(x,y)=(k_1,k_2)\Rightarrow k_1=x\ ,\ k_2=y$$

$\therefore$ $(x,y) = x\,(1,0) + y(0,1)$

$\therefore$ S spans $\mathbb{R}^2$

Thus S is a basis for $\mathbb{R}^2$.

Remark: S in this case called the normal basis for $\mathbb{R}^2$.

Exercise: The set $S=\{\overrightarrow{E_1},\overrightarrow{E_2},\overrightarrow{E_3}\}$ where $\overrightarrow{E_1}$ = (1,0,0) $\overrightarrow{E_2}$ = (0,1,0) and $\overrightarrow{E_3}$ = (0,0,1) is a basis for $\mathbb{R}^3$.

In general:

The set S = {(1,0,…,0),(0,1,…,0),…,(0,0,…1)} is a basis for $\mathbb{R}^n$.

Example: Let S_1 = {(1,0),(1,1)} and S_2 = {(–2,1),(2,3)}

(a) Show that S_1 is a basis for $\mathbb{R}^2$.

(b) Show that S_2 is a basis for $\mathbb{R}^2$.

Solution (a):

(1) Let c_1 and c_2 be a constants

$c_1(1,0)+c_2(1,1)=(0,0)$

$(c_1,0)+(c_2,c_2)=(0,0)$

$(c_1+c_2,c_2)=(0,0)\Rightarrow c_2=0$

$c_1+0=0\Rightarrow c_1=0$

$\therefore$ S_1 is linearly independent set.

(2) Let $(a,b)\in\mathbb{R}^2$ and k_1 , k_2 constants

$(a,b)=k_1(1,0)+k_2(1,1)$

$=(k_1,0)+(k_2,k_2)$

$(a,b)=(k_1+k_2,k_2)\Rightarrow k_2=b$

$k_1+k_2=a\Rightarrow k_1+b=a\Rightarrow k_1=a-b$

$\therefore$ $(a,b) = (a-b)(1,0) + b(1,1)$

$\therefore$ S_1 spans $\mathbb{R}^2$

Thus S_1 is a basis for $\mathbb{R}^2$.

Solution (b):

(1) Let c_1 and c_2 be a constants

$c_1(-2,1)+c_2(2,3)=(0,0)$

$(-2c_1,c_1)+(2c_2,3c_2)=(0,0)$

$(-2c_1+2c_2,c_1+3c_2)=(0,0)$

$-2c_1+2c_2=0\Rightarrow -2c_1=-2c_2\Rightarrow c_1=c_2$

$c_1+3c_2=0\Rightarrow c_1=-3c_2\Rightarrow -3c_2=c_2\Rightarrow 4c_2=0\Rightarrow c_2=0$

$c_2=0,c_1=c_2\Rightarrow c_1=0$

$\therefore$ S_2 is linearly independent set.

(2) Let $(a,b) \in \mathbb{R}^2$ and k_1 , k_2 constants

$(a,b) = k_1(-2,1) + k_2(2,3)$

$$= (-2k_1 + 2k_2, k_1 + 3k_2) \Rightarrow \left.\begin{array}{l} -2k_1 + 2k_2 = a \\ k_1 + 3k_2 = b \end{array}\right\} \Rightarrow \begin{array}{l} -k_1 + k_2 = \frac{1}{2}a \\ k_1 + 3k_2 = b \end{array}$$

$$-k_1 + k_2 = \frac{1}{2}a$$
$$k_1 + 3k_2 = b$$

---------------------- by addition

$$4k_2 = \frac{1}{2}a + b \quad \Rightarrow \quad k_2 = \frac{1}{4}(\frac{1}{2}a + b) \quad \Rightarrow \quad k_1 = \frac{1}{4}(-\frac{3}{2}a + b)$$

$\therefore$ (a,b) = $\frac{1}{4}(-\frac{3}{2}a + b)(-2,1)$ + $\frac{1}{4}(\frac{1}{2}a + b)(2,3)$

$\therefore$ S_2 spans $\mathbb{R}^2$

Thus S_2 is a basis for $\mathbb{R}^2$.

Example: Find the basis for the space of the solutions of system of homogeneous equations.

$x_1 + x_2 - x_3 = 0$ …(1)

$x_1 + 2x_2 + x_3 + x_4 = 0$ …(2)

$3x_1 + 5x_2 + x_3 + 3x_4 = 0$ …(3)

$2x_1 + x_2 - 4x_3 - x_4 = 0$ …(4)

Solution:

Using the Causs elimination method or the Causs Jordan elimination method, we get that $x_1 = 3x_3 + x_4$, $x_2 = -2x_3 - x_4$. So the solution space is

W = $\{3x_3 + x_4, -2x_3 - x_4, x_3, x_4;\ x_3, x_4 \in \mathbb{R}\}$

= $\{(3x_3, -2x_3, x_3, 0) + (x_4, -x_4, 0, x_4);\ x_3, x_4 \in \mathbb{R}\}$

= $\{x_3(3,-2,1,0) + x_4(1,-1,0,1);\ x_3, x_4 \in \mathbb{R}\}$

$\therefore$ Every vector in W is a linear combination of the vectors $(1,-1,0,1)$ and $(3,-2,1,0)$

$\therefore$ B = $\{(3,-2,1,0), (1,-1,0,1)\}$ spans W.

Also, B = $\{(3,-2,1,0), (1,-1,0,1)\}$ is linearly independent set

$\therefore$ The set B is a basis for W.

Exercises:

(1) Show that

(a) $B_1 = \{(1,1,0),(0,1,1),(0,0,1)\}$ basis for $\mathbb{R}^3$.

(b) $B_2 = \{(1,2,-1),(2,2,1),(1,1,3)\}$ basis for $\mathbb{R}^3$.

(c) $B_3 = [(1,0,1,0),(0,1,-1,2),(1,0,0,1),(0,2,2,1)\}$ basis for $\mathbb{R}^4$.

(2) Let the set $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, is B a basis for $M_{2\times 2}(\mathbb{R})$?

(3) Let the set $B = \{1,x,x^2\}$. Show that B is a basis for the vector space V where V is the vector space of all polynomials of second degree or less

(4) Let the set $B = \{1 - x, 1 + x + x^2, 1 - x - x^2, 1 + 2x + x^2\}$. Is B a basis for the vector space V where V is the vector space of all polynomials of second degree or less?

Theorem: Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n}\}$ be a basis for the vector space V, then every vector in V can be written in only one form as a linear combination of S vectors.

Proof:

Every vector $\vec{X}$ in V can be written as a linear combination of S vectors because S spans V. Let

$$\vec{X} = c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + ... + c_n\overrightarrow{X_n} \qquad \ldots(1)$$

$$\vec{X} = d_1\overrightarrow{X_1} + d_2\overrightarrow{X_2} + ... + d_n\overrightarrow{X_n} \qquad \ldots(2)$$

By subtracting equation (2) from equation (1) we get that

$$\vec{O} = (c_1 - d_1)\overrightarrow{X_1} + (c_2 - d_2)\overrightarrow{X_2} + ... + (c_n - d_n)\overrightarrow{X_n}$$

Because S is linearly independent we have

$c_1 - d_1 = 0 \Rightarrow c_1 = d_1$,

$c_2 - d_2 = 0 \Rightarrow c_2 = d_2, \ldots$

$c_n - d_n = 0 \Rightarrow c_n = d_n$

i.e. $c_i = d_i \quad (1 \le i \le n)$.

Theorem: (without proof)

If $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n}\}$ is a basis for the vector space V and $T = \{\overrightarrow{Y_1}, \overrightarrow{Y_2}, ..., \overrightarrow{Y_r}\}$ is linearly independent set of the vectors of V then $n \ge r$.

Corollary: If $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}\}$ and $T=\{\overrightarrow{Y_1},\overrightarrow{Y_2},...,\overrightarrow{Y_m}\}$ are two basis for the vector space V then m = n.

Proof:

Because S is a basis for V and T is linearly independent set (since T is a basis for the vector space V)

$\Rightarrow$ $n \geq m$ (by previous theorem) …(1)

Because T is a basis for V and S is linearly independent set (since S is a basis for the vector space V)

$\Rightarrow$ $m \geq n$ (by previous theorem) …(2)

From (1) and (2) we get that m = n.

Examples:

(1) The set S = {(1,0),(0,1)} is a basis for $\mathbb{R}^2$, so S contains only two vectors

$\Rightarrow$ any basis for $\mathbb{R}^2$ must contains two vectors.

(2) The set S = {(1,0,0,…,0),(0,1,0,…,0),…,(0,0,0,…,1)} is a basis for $\mathbb{R}^n$, so S contains n vectors

$\Rightarrow$ any basis for $\mathbb{R}^n$ must contains n vectors (by corollary)

Theorem: (without proof)

Let $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_m}\}$ be a set of nonzero vectors generating subspace W of vector space V, then a subset of S is a basis for W.

Example: Let W be a subspace of $\mathbb{R}^4$ generator by the vectors

$$\overrightarrow{X_1}=(1,2,-2,1)\ ,\ \overrightarrow{X_2}=(-3,0,4,3)\ ,\overrightarrow{X_3}=(2,1,1,-1)\ ,\ \overrightarrow{X_4}=(-3,3,-9,6)$$

Find the set B such that $B\subseteq\{\overrightarrow{X_1},\overrightarrow{X_2},\overrightarrow{X_3},\overrightarrow{X_4}\}=S$ and B basis for W.

Solution: We test whether S is linearly independent or dependent set

In the case S is linearly independent set, then S is the basis we need.

If

$$c_1\overrightarrow{X_1}+c_2\overrightarrow{X_2}+c_3\overrightarrow{X_3}+c_4\overrightarrow{X_4}=\overrightarrow{O} \qquad \ldots(1)$$

$$c_1(1,2,-2,1)+c_2(-3,0,4,3)+c_3(2,1,1,-1)+c_4(-3,3,-9,6)=(0,0,0,0)$$

By solving this system of equations (Using the Causs elimination method or the Causs Jordan elimination method), we get $c_1 = -c_2 - 9c_4$ ، $c_3 = 2c_2 + 3c_4$

$\therefore$ The system has infinite number of solutions.

$\therefore$ S is non linearly independent set.

Take $c_2 = 1$ and $c_4 = 0$, then $c_1 = -1$ and $c_3 = 2$

$-\overrightarrow{X_1} + \overrightarrow{X_2} + 2\overrightarrow{X_3} + 0\overrightarrow{X_4} = \overrightarrow{O}$ by compensation in equation (1)

$\overrightarrow{X_2} = \overrightarrow{X_1} - 2\overrightarrow{X_3} - 0\overrightarrow{X_4}$

This means that $\overrightarrow{X_2}$ is linear combination for the vectors $\overrightarrow{X_1}$, $\overrightarrow{X_3}$ and $\overrightarrow{X_4}$.

Thus $B_1 = \{\overrightarrow{X_1}, \overrightarrow{X_3}, \overrightarrow{X_4}\}$ spans W

We test whether B_1 is an independent or dependent set.
In the case that B_1 is independent set, then B_1 is the required basis.

If $d_1\overrightarrow{X_1} + d_2\overrightarrow{X_3} + d_3\overrightarrow{X_4} = \overrightarrow{O}$ …(2)

$d_1(1,2,-2,1) + d_2(2,1,1,-1) + d_3(-3,3,-9,6) = (0,0,0,0)$

By solving this system of equations we get that $d_1 = -3d_3$ ، $d_2 = 3d_3$

$\therefore$ The system has infinite number of solutions.

$\therefore$ B_1 is non linearly independent set.

Take $d_3 = 1$, then $d_1 = -3$ and $d_2 = 3$

$-3\overrightarrow{X_1} + 3\overrightarrow{X_3} + \overrightarrow{X_4} = \overrightarrow{O}$ by compensation in equation (2)

$\overrightarrow{X_4} = 3\overrightarrow{X_1} - 3\overrightarrow{X_3}$

This means that $\overrightarrow{X_4}$ is linear combination for the vectors $\overrightarrow{X_1}$ and $\overrightarrow{X_3}$.

Thus $B_2 = \{\overrightarrow{X_1}, \overrightarrow{X_3}\}$ spans W.

We test whether B_2 is an independent or dependent set.
In the case that B_2 is independent set, then B_2 is the required basis.
Note that B_2 is independent set **(Prove that)**

$\therefore$ $B_2 = B = \{\overrightarrow{X_1}, \overrightarrow{X_3}\}$ is a basis for W.

Dimension of Space

Definition of the dimension: Let V be a no zero vector space, V is called a space with a finite dimension if it has a base the number of its elements is a natural number.

The number of the vectors in the basis is called the dimension and denoted by dim (V) i.e. dim (V) = the number of the vectors in the basis of the vector space V.

If $V = \{\overrightarrow{O}\}$, then dim (V) = 0.

If $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, ..., \overrightarrow{X_n}\}$ basis for the vector space V, then dim (V) = n

Remark: The spaces that we will discuss are vector spaces with finite dimensions (that is, their distance = a natural number). There are vector spaces of infinite dimensions.

Examples:

(1) $\dim(\mathbb{R}^2)=2$, $\dim(\mathbb{R}^3)=3$, $\dim(\mathbb{R}^n)=n$.

(2) dim (P_2) = 3, where P_2 is a quadratic polynomial.

(3) dim (P_3) = 4, where P_3 is a polynomial of the third degree.

Generally: $\boxed{\dim(P_n) = n + 1}$, where P_n is a polynomial of degree (n).

(4) Find the dimension for the vector space $M_{2\times 3}(\mathbb{R})$.

Solution: $M_{2\times 3}(\mathbb{R})=\left\{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}; a,b,c,d,e,f \in \mathbb{R}\right\}$

$$S=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

S is a basis for $M_{2\times 3}(\mathbb{R})$ because

(a) $c_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+c_2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}+c_4\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}+c_5\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}+$

$c_6\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$

$\therefore$ S is linearly independent set.

(b) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}=k_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+k_2\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}+k_3\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}+k_4\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}+$

$k_5\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}+k_6\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}=\begin{bmatrix} k_1 & k_2 & k_3 \\ k_4 & k_5 & k_6 \end{bmatrix} \Rightarrow k_1 = a,\ k_2 = b,\ k_3 = c, k_4 = d, k_5 = e, k_6 = f$

$\therefore$ S spans $M_{2\times 3}(\mathbb{R})$.

Thus S is a basis for $M_{2\times 3}(\mathbb{R})$.

$\therefore\ \dim(M_{2\times 3}(\mathbb{R}))=6$ (the number of the vectors in S is 6)

Generally: $\boxed{\dim(M_{m\times n}(\mathbb{R})) = m \times n}$

(5) Let V be the vector space of all polynomials of degree 2 or less. Find the dimension for this vector space?

Solution:

Let $p \in V$ ، $p(x) = a_0 + a_1x + a_2x^2$, $\forall\ a_0, a_1, a_3 \in \mathbb{R}, x \in \mathbb{R}$ and $S = \{1, x, x^2\}$

$\therefore$ S is a basis for the vector space V because

(a) $c_1 \cdot 1 + c_2 \cdot x + c_3 \cdot x^2 = \vec{O} = 0 + 0 \cdot x + 0 \cdot x^2 \Rightarrow c_1 = c_2 = c_3 = 0$

$\therefore$ S is linearly independent set.

(b) $k_1 \cdot 1 + k_2 \cdot x + k_3 \cdot x^2 = a_0 + a_1x + a_2x^2 \Rightarrow k_1 = a_0$, $k_2 = a_1$, $k_3 = a_2$

$\therefore$ $S = \{1, x, x^2\}$ spans V

Thus S is a basis for V.

$\therefore$ dim (V) = 3 (the number of the vectors in S is 3)

(6) Find the dimension for the subspace W generated by the vectors {(1,0,–1,5), (3,2,1,0), (0,–1,0,1), (–1,–5,–3,13)} in $\mathbb{R}^4$?

Solution: To find the dimension for subspace W generated by the vectors {(1,0,–1,5), (3,2,1,0), (0,–1,0,1), (–1,–5,–3,13)} we must first find a basis for W.

Let S = {(1,0,–1,5), (3,2,1,0), (0,–1,0,1), (–1,–5,–3,13)}

Therefore, we must make sure that S is linearly independent or linearly dependent set

So, if

$c_1(1,0,-1,5) + c_2(3,2,1,0) + c_3(0,-1,0,1) + c_4(-1,-5,-3,13) = (0,0,0,0)$, we get

$$\begin{aligned}
c_1 + 3c_2 \qquad\quad - c_4 &= 0 \\
2c_2 - c_3 - 5c_4 &= 0 \\
-c_1 + c_2 \qquad\quad - 3c_4 &= 0 \\
5c_1 \qquad\quad + c_3 + 13c_4 &= 0
\end{aligned}$$

From this it follows that this system has a non-trivial solution (because the determinant of the coefficients matrix = zero) **(check it)**.

And by solving this system by the Causs-Jordan elimination method, we get that:

$c_1 = -2c_4$, $c_2 = c_4$, $c_3 = -3c_4$

Take $c_4 = 1 \Rightarrow c_1 = -2$, $c_2 = 1$, $c_3 = -3$

$\therefore$ S is a non linearly independent set (linearly dependent) because the constants are not zeros.

$-2(1,0,-1,5) + 1(3,2,1,0) - 3(0,-1,0,1) + 1(-1,-5,-3,13) = (0,0,0,0)$

We take the vector of one factor for convenience

$(-1,-5,-3,13) = 2(1,0,-1,5) - 1(3,2,1,0) + 3(0,-1,0,1)$
$= (2,0,-2,10) - (3,2,1,0) + (0,-3,0,3)$
Thus W generated by the set $B = \{(1,0,-1,5), (3,2,1,0), (0,-1,0,1)\}$
Now we must check whether B is linearly independent set or linearly dependent set.
So if:
$k_1(1,0,-1,5) + k_2(3,2,1,0) + k_3(0,-1,0,1) = (0,0,0,0)$

$$\begin{aligned} k_1 + 3k_2 \quad &= 0 \\ 2k_2 - k_3 &= 0 \\ -k_1 + k_2 \quad &= 0 \\ 5k_1 \quad + k_3 &= 0 \end{aligned}$$

By solving this system, we get $k_1 = k_2 = k_3 = 0$.
$\therefore$ B is a linearly independent set.
But the set B spans the subspace W, so B is a basis for W and from that we obtain dim (W) = 3 (the number of the vectors in B is 3)

Exercises:

(1) Find the dimension for the subspace of $\mathbb{R}^2$ span by the vectors (2,4), (4,2), (0,0)?

(2) Find the dimension for the subspace of $\mathbb{R}^3$ span by the vectors
(a) (1,0,0), (–1,2,1), (3,2,2) **(b)** (2,3,4) ,(1,1,–1)

(3) Find the dimension for the vector space span by $\{1 + x,\ 1 + x + x^2,\ 1 - x - x^2,\ 1 + 2x + x^2\}$ for polynomials of second degree or less.

Theorem: (without proof)
If S is a set of linearly independent vectors in the finite dimension vector space V, there exists a basis T for the vector space V contains S.

Example: Find the basis for $\mathbb{R}^3$ contain the vector $\overrightarrow{X_1} = (1,0,2)$?

Solution: Let $S = \{\overrightarrow{X_1}\}, S_1 = \{\overrightarrow{X_1}, \overrightarrow{E_1}, \overrightarrow{E_2}, \overrightarrow{E_3}\}$ where $\overrightarrow{E_1} = (1,0,0), \overrightarrow{E_2} = (0,1,0), \overrightarrow{E_3} = (0,0,1)$

S_1 spans $\mathbb{R}^3$. We find the set T such that $T \subseteq S_1$, $\overrightarrow{X_1} \in T$ and T is a basis for $\mathbb{R}^3$.

S_1 is non linearly independent set (it contains four vectors from $\mathbb{R}^3$ and $\dim((\mathbb{R}^3)) = 3$).

Moreover if $c_1\overrightarrow{X_1} + c_2\overrightarrow{E_1} + c_3\overrightarrow{E_2} + c_4\overrightarrow{E_3} = \overrightarrow{O}$ we get that $c_2 = -c_1$, $c_4 = -2c_1$, $c_3 = 0$

Take $c_1 = 1 \Rightarrow c_2 = -1$, $c_3 = 0$, $c_4 = -2$

$\overrightarrow{X_1} - \overrightarrow{E_1} + 0\overrightarrow{E_2} - 2\overrightarrow{E_3} = \overrightarrow{O} \quad \Rightarrow \quad \overrightarrow{X_1} - 0\overrightarrow{E_2} - 2\overrightarrow{E_3} = \overrightarrow{E_1}$

$\therefore$ $T = \{\overrightarrow{X_1}, \overrightarrow{E_2}, \overrightarrow{E_3}\}$ spans $\mathbb{R}^3$. But T is linearly independent set of vectors in $\mathbb{R}^3$. Therefore T is a basis for $\mathbb{R}^3$ and contains the set $S = \{\overrightarrow{X_1}\}$.

Exercises:

(1) Find a basis for $\mathbb{R}^3$ contains the two vectors $\overrightarrow{X_1} = (1,0,2)$ and $\overrightarrow{X_2} = (0,1,2)$.

(2) Find a basis for $\mathbb{R}^4$ contains the two vectors $\overrightarrow{X_1} = (1,0,1,0)$ and $\overrightarrow{X_2} = (-1,1,-1,0)$.

(3) Find a basis for $\mathbb{R}^4$ contains the two vectors $\overrightarrow{X_1} = (1,0,1,0)$ and $\overrightarrow{X_2} = (0,1,-1,0)$.

Rank of Matrix

Definition: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ matrix of degree m×n and

$\overrightarrow{X_1} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \overrightarrow{X_2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \overrightarrow{X_n} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$, the subspace W span by the set $\{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}\}$ is called matrix column space A or the space span by the column of the matrix A. The dimension of the matrix column space A is called the column rank for the matrix A.

Example: Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & 8 \\ 2 & 3 & 7 \\ -1 & 2 & 0 \end{bmatrix}$. Find the column rank for the matrix A?

Solution: The subspace of the columns of the matrix A is

$$W = \left\{ c_1 \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 2 \\ 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 8 \\ 7 \\ 0 \end{bmatrix}, c_1, c_2, c_3 \in \mathbb{R} \right\}.$$

Let $S=\left\{\begin{bmatrix}1\\3\\2\\-1\end{bmatrix},\begin{bmatrix}-2\\2\\3\\2\end{bmatrix},\begin{bmatrix}0\\8\\7\\0\end{bmatrix}\right\}$ the spans set for W.

If

$$c_1\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+c_2\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}+c_3\begin{bmatrix}0\\8\\7\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$. Then we get

$c_1 - 2c_2 = 0$
$3c_1 + 2c_2 + 8c_3 = 0$
$2c_1 + 3c_2 + 7c_3 = 0$
$-c_1 + 2c_2 = 0$
$\Rightarrow\ c_1 = 2c_2,\ c_3 = -c_2$

Let $c_2 = 1 \ \Rightarrow\ c_1 = 2$ and $c_3 = -1$
$\therefore$ The set S is non linearly independent.

$$2\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}-\begin{bmatrix}0\\8\\7\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\ \Rightarrow\ \begin{bmatrix}0\\8\\7\\0\end{bmatrix}=2\begin{bmatrix}1\\3\\2\\-1\end{bmatrix}+\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}$$

Let $S'=\left\{\begin{bmatrix}1\\3\\2\\-1\end{bmatrix},\begin{bmatrix}-2\\2\\3\\2\end{bmatrix}\right\}$ be the set spans W. It is a linearly independent set.

$\therefore$ S' is a basis for W. Thus dim (W) = 2.
$\therefore$ The dimension of the column space of the matrix A = 2 (the column rank of A).

Example: Let $A=\begin{bmatrix}1&1&1&0\\2&3&4&-1\\-1&2&-2&3\\4&1&5&2\end{bmatrix}$. Find the column rank for the matrix A?

Solution: The column subspace of the matrix A is

$$W=\left\{c_1\begin{bmatrix}1\\2\\-1\\4\end{bmatrix}+c_2\begin{bmatrix}1\\3\\2\\1\end{bmatrix}+c_3\begin{bmatrix}1\\4\\-2\\5\end{bmatrix}+c_4\begin{bmatrix}0\\-1\\3\\2\end{bmatrix},c_1,c_2,c_3,c_4\in\mathbb{R}\right\}$$ and let

$$S=\left\{\begin{bmatrix}1\\2\\-1\\4\end{bmatrix},\begin{bmatrix}1\\3\\2\\1\end{bmatrix},\begin{bmatrix}1\\4\\-2\\5\end{bmatrix},\begin{bmatrix}0\\-1\\3\\2\end{bmatrix}\right\}$$ be the set spans W, if we have

$$c_1\begin{bmatrix}1\\2\\-1\\4\end{bmatrix}+c_2\begin{bmatrix}1\\3\\2\\1\end{bmatrix}+c_3\begin{bmatrix}1\\4\\-2\\5\end{bmatrix}+c_4\begin{bmatrix}0\\-1\\3\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$c_1+c_2+c_3 \qquad = 0$$

$$2c_1+3c_2+4c_3-c_4 = 0$$

$$-c_1+2c_2-2c_3+3c_4=0$$

$$4c_1+c_2+5c_3+2c_4 = 0$$

$$\begin{vmatrix}1&1&1&0\\2&3&4&-1\\-1&2&-2&3\\4&1&5&2\end{vmatrix}=\begin{vmatrix}3&4&-1\\2&-2&3\\1&5&2\end{vmatrix}-\begin{vmatrix}2&4&-1\\-1&-2&3\\4&5&2\end{vmatrix}+\begin{vmatrix}2&3&-1\\-1&2&3\\4&1&2\end{vmatrix}+0\neq 0$$

$\therefore$ This homogeneous linear system has a trivial solution, i.e. $c_1 = c_2 = c_3 = c_4 = 0$

$\therefore$ S is a linearly independent set.

$\therefore$ S is a basis for W. Thus dim (W) = 4.

$\therefore$ The dimension of the column space of the matrix A = 4 (the column rank of A).

Definition: Let A be a matrix of degree m×n, if A is row equivalent to a matrix B (where B is a reduced echelon form matrix (r.e.f)) then the number of non-zero rows of the matrix B is called the row rank of A.

Examples:

(1) Let $A=\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-1&3\end{bmatrix}$. Find the row rank of A?

Solution: $\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-1&3\end{bmatrix}\xrightarrow[R_3=r_3+r_1]{R_2=r_2-2r_1}\begin{bmatrix}1&2&-1\\0&0&0\\0&1&2\end{bmatrix}\xrightarrow{R_1=r_1-2r_3}\begin{bmatrix}1&0&-5\\0&0&0\\0&1&2\end{bmatrix}$

$\xrightarrow{r_2\leftrightarrow r_3}\begin{bmatrix}1&0&-5\\0&1&2\\0&0&0\end{bmatrix}=B$

$\therefore$ The matrix B is a reduced echelon form matrix (r.e.f). Thus the row rank of A = 2.

(2) Let $A=\begin{bmatrix}1&-2&0\\3&2&8\\2&3&7\\-1&2&0\end{bmatrix}$. Find the row rank of A?

Solution:

$$\begin{bmatrix}1&-2&0\\3&2&8\\2&3&7\\-1&2&0\end{bmatrix}\xrightarrow[R_4=r_4+r_1]{R_2=r_2-3r_1,\ R_3=r_3-2r_1}\begin{bmatrix}1&-2&0\\0&8&8\\0&7&7\\0&0&0\end{bmatrix}\xrightarrow[R_3=\frac{1}{7}r_3]{R_2=\frac{1}{8}r_2}\begin{bmatrix}1&-2&0\\0&1&1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\xrightarrow[R_3=r_3-r_2]{R_1=r_1+2r_2}\begin{bmatrix}1&0&2\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}=B$$

$\therefore$ The matrix B is a reduced echelon form matrix (r.e.f). Thus the row rank of A = 2.

Remarks:

(1) The rank of the zero matrix of degree m×n = 0.

(2) The rank of the identity matrix of degree n×n = n. (because it is a reduced echelon form matrix (r.e.f) and all rows are linearly independent).

(3) If the degree of the matrix is m×n, then its rank is not greater than the smallest of the two numbers m and n, i.e. $r(A) \le \min\{m,n\}$.

Another definition of matrix rank: The rank of a matrix is the largest number of linearly independent rows (columns) in the matrix.

Theorem: (without proof)

Let A be a matrix of degree (m×n) the row rank and the column rank of a matrix A are equal.

Remark: The row rank of A = The column rank = The rank of a matrix A.

Example: Find the rank of a matrix A if $A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 3 \\ 3 & -5 & 1 \\ 1 & -4 & 5 \end{bmatrix} \sim B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution: Because the matrix A is equal to the matrix B which has reduced echelon form.

$\therefore$ The rank of a matrix A = 2 (the number of non zero rows) **(check that A ~ B)**.

Exercise: Find the rank of the matrix A where $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 4 & -1 \\ -1 & 2 & -2 & 3 \\ 4 & 1 & 5 & 2 \end{bmatrix}$? **(Ans. 4)**

Remark: To find the rank of the matrix do the following

(1) Transform the matrix A to reduced echelon form and let the resulting matrix is B.

(2) The rank of the matrix A = the number of the non zero rows in the matrix B.

Theorem: Let A and B be two matrices of degree (m×n) row equivalent then the two rows spaces are identical.

Proof: Because A and B are two row equivalent matrices, then B can be obtained from A rows after performing a finite number of elementary transformations on it.

That is, each row in B is a linear combination of A rows.

This means that the rows of B are a subset of the row space of A.

That is, the row space of B is contained in the row space of A ... (1)

In the same way we get that

The row space of A is contained in the row space of B ... (2)

From (1) and (2) we get that the row space of A is equal to the row space of B.

Theorem: (without proof)

Let A be a matrix of degree (m×n) then the non zero rows in a matrix B which is the matrix A after transform it to the reduced echelon form which is the basis for the row space of A.

Remark: We can use the above theorem to find the basis for the vector space V spans by the set of vectors $S=\{\overrightarrow{X_1},\overrightarrow{X_2},...,\overrightarrow{X_n}\}$ in $\mathbb{R}^n$, i.e. spans S= V ⊆ $\mathbb{R}^n$ as follows:

(1) Form the matrix A defined by the shape $A=\begin{bmatrix}\overrightarrow{X_1}\\ \overrightarrow{X_2}\\ \vdots \\ \overrightarrow{X_n}\end{bmatrix}$ whose rows represent the S vectors

(2) Transform the matrix A to the reduced echelon forms and let the resulting matrix is B.

(3) The non zero rows in the matrix B are the basis for the matrix A.

Example: Let S = {(1,–2,0,3,–4),(3,2,8,1,4),(2,3,7,2,3),(–1,2,0,4,–3)} and V be a subspace of $\mathbb{R}^5$. Find the basis for V?

Solution: Form the matrix A where the rows of it are the S vectors and V is the row space for it

$$A=\begin{bmatrix}1 & -2 & 0 & 3 & -4\\ 3 & 2 & 8 & 1 & 4\\ 2 & 3 & 7 & 2 & 3\\ -1 & 2 & 0 & 4 & -3\end{bmatrix}$$

And by using the elementary transformations on the rows, we convert the matrix A to the reduced echelon forms, and we get the matrix

$$B = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero rows in the matrix B represent the basis for the row space of the matrix A which is represent the basis for V. That is, $\{(0,0,0,1,-1),(0,1,1,0,1),(1,0,2,0,1)\}$ is the basis for V.

Remark: From the above theorem and example we note the following

(1) The result basis is not subset of the given vectors.

(2) The way to represent any vector as a linear combination of these basis elements is done in the same way that the vector is represented as a linear combination of the elements of the natural base.

That is: in the previous example when we want to represent the vector $\vec{X} = (5,4,14,6,3)$ we note that the base vectors contain the axis element 1 in the first, second, and fourth positions in the first, second and fourth vectors, respectively. Thus, we will use the base vectors with the first, second and fourth projections of the vector $\vec{X}$, as follows:

$$\vec{X} = (\underline{\underline{5}},\underline{\underline{4}},14,\underline{\underline{6}},3) = 5(1,0,2,0,1) + 4(0,1,1,0,1) + 6(0,0,0,1,-1)$$

$$= (5,0,10,0,5) + (0,4,4,0,4) + (0,0,0,6,-6)$$

Theorem: Let A be a matrix of degree $(n \times n)$, then

A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n.

Proof: $\Rightarrow$ Let A be a square invertible matrix

$\therefore$ A is a row equivalent to the identity matrix I_n (A square matrix of degree $n \times n$ has an inverse if it is a row equivalent to the identity matrix)

$\therefore$ The rank of the matrix A = n.

$\Leftarrow$ Let The rank of the matrix A = n

Since A is a row equivalent to the identity matrix I_n and $|I_n| \neq 0$

$\therefore$ A is invertible matrix.

Corollary (1): Let A be a matrix of degree (n×n), then

The rank of the matrix A = n $\Leftrightarrow$ $|A| \neq 0$

Proof: $\Rightarrow$ The rank of the matrix A = n

A is invertible matrix (previous theorem: let A be a matrix of degree (n×n), then A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n)

Thus $|A| \neq 0$.

$\Leftarrow$ Let $|A| \neq 0$, then A is invertible matrix

The rank of the matrix A = n (previous theorem: let A be a matrix of degree (n×n), then A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n)

Corollary (2): Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \ldots, \overrightarrow{X_n}\}$ be a set of n vectors in $\mathbb{R}^n$ and A columns (rows) are $\overrightarrow{X_1}$, $\overrightarrow{X_2}$, …, $\overrightarrow{X_n}$ then S is linearly independent set $\Leftrightarrow$ $|A| \neq 0$.

Proof: $\Rightarrow$ Suppose that S is linearly independent set

From definition of linearly independent

$c_1\overrightarrow{X_1} + c_2\overrightarrow{X_2} + \ldots + c_n\overrightarrow{X_n} = \overrightarrow{O}$ such that $c_1 = c_2 = \ldots = c_n = 0$

We can write this relation as

$$c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1a_{11} + c_2a_{12} + \ldots + c_na_{1n} \\ c_1a_{21} + c_2a_{22} + \ldots + c_na_{2n} \\ \vdots \\ c_1a_{n1} + c_2a_{n2} + \ldots + c_na_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow A\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This system is homogeneous this lead to $|A| \neq 0$ (because the column of the unknowns = 0) this is possible when $|A| \neq 0$.

$\Leftarrow$ Suppose that $|A| \neq 0$.

From Corollary (1), the rank of the matrix A = n $\Leftrightarrow$ $|A| \neq 0$.

From Theorem, A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n.

$\therefore$ S is linearly independent set when the columns (rows) of the matrix are the vectors $\overrightarrow{X_1}$, $\overrightarrow{X_2}$, …, $\overrightarrow{X_n}$ invertible matrix this means $|A| \neq 0$.

Corollary (3): The homogeneous system $A\vec{X}=\vec{O}$ former of n linear equation and n unknowns has non zero solution $\Leftrightarrow$ the rank of the matrix A < n.

That is : $A\vec{X}=\vec{O}$ has non zero solution $\Leftrightarrow$ the rank of the matrix A < n.

Proof: From Corollary (1), the rank of the matrix A < n $\Leftrightarrow$ $|A| = 0$.

This is: the rank of the matrix A < n $\Leftrightarrow$ A is a non invertible matrix.

$\therefore$ The homogeneous system $A\vec{X}=\vec{O}$ has non zero solution $\Leftrightarrow$ A is a non invertible matrix.

Example: Consider the homogeneous system

$$\begin{aligned} 2x_1 \quad\quad\;\; + x_3 &= 0 \\ 3x_1 + 3x_2 + x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

Solution: The coefficient matrix is $A=\begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 1 \\ 1 & -3 & 1 \end{bmatrix}$.

$$A=\begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \xrightarrow[R_2=r_2-3r_3]{R_1=r_1-r_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 12 & -2 \\ 1 & -3 & 1 \end{bmatrix} \xrightarrow[R_3=r_3-r_1]{R_2=\frac{1}{2}r_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 6 & -1 \\ 0 & -6 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3=r_3+r_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 6 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2=\frac{1}{6}r_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1=r_1-3r_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is in a reduced echelon forms. Therefore the rank of the matrix A < 3.

Theorem: Let A be a matrix of degree n×n, then

The homogeneous system $A\vec{X}=\vec{O}$ has non zero solution $\Leftrightarrow$ A is non invertible matrix.

Proof: $\Rightarrow$ Suppose that A is invertible matrix that is mean A^{-1} exists.

$A^{-1}(A\vec{X})=A^{-1}\vec{O}$ By multiplying both sides of the homogeneous system equation by A^{-1}

$(A^{-1}A)\vec{X} = \vec{O}$

$I_n.\vec{X} = \vec{O} \Rightarrow \vec{X} = \vec{O}$

This means that the unique solution for this homogeneous system is the zero solution $\vec{X} = \vec{O}$ which is a contradiction since the homogeneous system has non zero solution.

$\therefore$ A is non invertible matrix.

$\Leftarrow$ **(Home work)**

Example: Find the rank of the matrix $A=\begin{bmatrix}1&2&0\\0&1&3\\2&1&3\end{bmatrix}$. And from it conclude whether it is invertible matrix or not and if the homogeneous system $A\vec{X}=\vec{O}$ has non zero solution or not?

Solution: We transform the matrix A in the reduced echelon forms.

$$A=\begin{bmatrix}1&2&0\\0&1&3\\2&1&3\end{bmatrix}\xrightarrow{R_3=r_3-2r_1}\begin{bmatrix}1&2&0\\0&1&3\\0&-3&3\end{bmatrix}\xrightarrow[R_3=r_3+3r_2]{R_1=r_1-2r_2}\begin{bmatrix}1&0&-6\\0&1&3\\0&0&12\end{bmatrix}\xrightarrow{R_3=\frac{1}{12}r_3}\begin{bmatrix}1&0&-6\\0&1&3\\0&0&1\end{bmatrix}$$

$$\xrightarrow[R_2=r_2-3r_3]{R_1=r_1+6r_3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3$$

Thus the rank of the matrix A = 3.

$\therefore$ A is invertible matrix (By previous theorem: Let A be a matrix of degree (n×n), then A is invertible matrix $\Leftrightarrow$ the rank of the matrix A = n)

or A square matrix of degree n×n has an inverse if it is row equivalent to the identity matrix

So by the previous theorem (Let A be a matrix of degree n×n, then the homogeneous system $A\vec{X}=\vec{O}$ has non zero solution $\Leftrightarrow$ A is non invertible matrix).

The homogeneous system has no solution only the zero solution.

Exercise: Find the rank of the matrix $B=\begin{bmatrix}1&0&-6\\0&1&3\\0&0&0\end{bmatrix}$ And from it conclude whether it is invertible matrix or not and if the homogeneous system $A\vec{X}=\vec{O}$ has non zero solution or not?

Corollary: (without prove)

The linear system has solution $A\vec{X}=\vec{B}$ $\Leftrightarrow$ The rank of the matrix A = the rank of [A:B].

Example: Consider the linear system $\begin{bmatrix}2&1&3\\1&2&2\\0&1&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$.

Since the rank of the matrix A = the rank of [A:B], so the linear system has solution

Example: Consider the linear system $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -3 & 4 \\ 2 & -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

The rank of the matrix A = 2 and the rank of [A:B] = 3.

$\therefore$ the rank of the matrix A $\neq$ the rank of [A:B]

$\therefore$ The system has no solution.

Exercises:

(1) Let $S = \{\overrightarrow{X_1}, \overrightarrow{X_2}, \overrightarrow{X_3}, \overrightarrow{X_4}, \overrightarrow{X_5}\}$, where $\overrightarrow{X_1} = (1,2,3)$, $\overrightarrow{X_2} = (2,1,4)$, $\overrightarrow{X_3} = (-1,-1,2)$, $\overrightarrow{X_4} = (0,1,2)$ and $\overrightarrow{X_5} = (1,1,1)$ find the basis for the subspace?

(2) Find the row and column rank for the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 1 & -5 & -2 & 1 \\ 7 & 8 & -1 & 2 & 5 \end{bmatrix}$?

(3) Is the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 3 \\ 0 & 8 & 0 \end{bmatrix}$ invertible or not invertible matrix?

(4) Does the following system have a solution or not

$x_1 - 2x_2 - 3x_3 + 4x_4 = 1$

$4x_1 - x_2 - 5x_3 + 6x_4 = 2$

$2x_1 + 3x_2 + x_3 - 2x_4 = 2$

(5) Let $S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$, is it is linearly independent in $\mathbb{R}^3$?

Chapter Five

Linear Transformations

Linear Transformations and Matrices

Def Let V and W be V.spaces. A function $L: V \to W$ is called Linear transformation of V in to W if

ⓐ $L(u+v) = L(u) + L(v)$ for u and v in V

ⓑ $L(cu) = c\,L(u)$ for u in V and c is a real number

if $V = W$, the linear trans. $L: V \to V$ is also called linear operator.

Examples and Remarks

1) $L: V \to W$ is a linear trans $\Longleftrightarrow L(au + bv) = a\,L(u) + b\,L(v)$ For any vectors $u, v \in V$ and for any scalars a, b.

② Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Then L is a linear trans.

③ Let $L: P_1 \to P_2$ defined by

$$L(p(t)) = t\,p(t)$$

Then L is a linear trans.

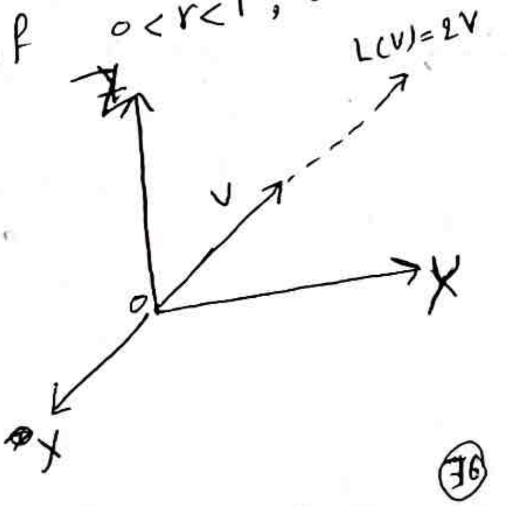
④ Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

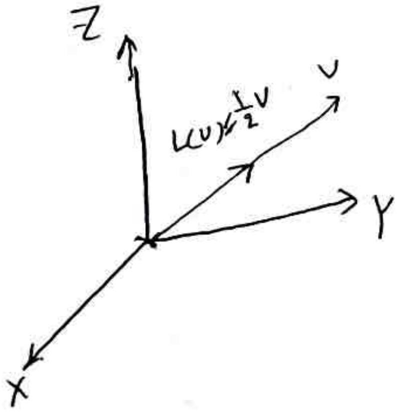
$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = r \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \text{ where } r \in \mathbb{R}.$$

Then L is a linear operator.

If $r > 1$, L is called a dilation

If $0 < r < 1$, L 〃 〃 a contraction





(5) Let $V = C[a,b]$ be the v.sp of all real valued functions that are integrable over the interval $[a,b]$

Let $L : V \longrightarrow \mathbb{R}$ defined by

$$L(f) = \int_a^b f(x)\,dx$$

Then L is a linear trans.

(6) Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by, A be a fixed 2×3 real matrix.

Let $L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A_{2\times 3} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

Then L is a lin. trans

<u>proof</u> Let $X, Y \in \mathbb{R}^3$, Let $a, b \in \mathbb{R}$

$$\begin{aligned} L(aX + bY) &= A(aX + bY) \\ &= A(aX) + A(bY) \\ &= a(AX) + b(AY) \\ &= a\,L(X) + b\,L(Y) \end{aligned}$$

(7) Let $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} 2a_1 + 1 \\ 2a_2 \\ a_3 \end{pmatrix}$$

show that L is not a lin-tran.

(8) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$L(a_1 \;\; a_2) = (a_1^2 \;\; 2a_2)$$

show that L is not a lin-trans.

<u>ملاحظه</u> المتجهات في $\mathbb{R}^n$ يمكن ان تكتب على شكل مصفوفات صفيه من النمط $1 \times n$ او مصفوفات عموديه من النمط $n \times 1$ او على شكل $(a_1, a_2, \ldots, a_n)$

3

Def A Linear transformation $L: V \to W$ is called 1-1 if it is 1-1 function

EX Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$L\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_1 - a_2 \end{pmatrix}$$

Show that L is 1-1

Def Let $L: V \to W$ be a lin-trans of a V.sp V in to a V.sp W. The Kernel of L (denoted by $\operatorname{Ker} L$) is the subset $\{v \in V : L(v) = 0_W\}$.

Note that $\operatorname{Ker} L$ is a subspace of V (check)

Hence $\operatorname{Ker} L \neq \varphi$.

EX Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ s.t $L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Find Ker L

Sol
$$\begin{aligned}
\operatorname{Ker} L &= \{ v \in \mathbb{R}^3 : L(v) = 0_{\mathbb{R}^2} \} \\
&= \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \quad '' \quad a_1 = a_2 = 0 \right\} \\
&= \left\{ \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix} : a_3 \in \mathbb{R} \right\}
\end{aligned}$$

Th Let $L: V \to W$ be a lin. trans. of a V.sp V into a V.sp W. Then

ⓐ $\operatorname{Ker} L$ is a subspace of V

ⓑ L is 1-1 $\iff \operatorname{Ker} L = \{0_V\}$

proof EXC.

EX Let $L : P_2 \to R$ be a lin. trans. defined by (4)

$$L(at^2+bt+c) = \int_0^1 (at^2+bt+c)\,dt$$

Find Ker L, find dim Ker L, IS L one to one.

Sol $\ker L = \{ v \in P_2 : L(v) = 0_R \}$

let $v = at^2 + bt + c \in \ker L$

$$L(v) = \int_0^1 (at^2+bt+c)\,dt$$

$$= \frac{at^3}{3} + \frac{bt^2}{2} + ct \Big]_0^1 = \frac{a}{3} + \frac{b}{2} + c$$

$$L(v)=0 \Rightarrow \frac{a}{3} + \frac{b}{2} + c = 0$$

$$\Rightarrow c = -\frac{a}{3} - \frac{b}{2}$$

$$\Rightarrow v = at^2 + bt + \left(-\frac{a}{3} - \frac{b}{2}\right)$$

To find dim Ker T ?

$$\ker T = \left\{ at^2 + bt + \left(-\frac{a}{3} - \frac{b}{2}\right) : a, b \in \mathbb{R} \right\}$$

$$= \left\{ a\left(t^2 - \frac{1}{3}\right) + b\left(t - \frac{1}{2}\right) : a, b \in \mathbb{R} \right\}$$

$S = \left\{ t^2 - \frac{1}{3}, t - \frac{1}{2} \right\}$ generates Ker T.

To show S is Lin-indep.

Assume $\alpha_1\left(t^2 - \frac{1}{3}\right) + \alpha_2\left(t - \frac{1}{2}\right) = 0$

$$\alpha_1 t^2 + \alpha_2 t - \frac{\alpha_1}{3} - \alpha_2 \frac{1}{2} = 0$$

$\Rightarrow \alpha_1 = \alpha_2$, so S is Lin-indep.

او نقول S هي Lin-indep لان كل من المتجهين في S ليس مضاعفا الى الاخر

$\therefore$ S is basis for Ker L, hence dim Ker = 2

Now L is not 1-1 since $\ker L \neq \{0\}$

Def if $L: V \longrightarrow W$ be a Lin trans of a V.sp V into a V.sp W, then the range of L (or image of V under L) denoted by range L, is defined by

$$\text{rang}\, L = \{w \in W;\ \exists v \in V \text{ s.t } L(v) = w\}.$$

Note that L is onto if $\text{rang}\, L = W$.

Theorem if $L: V \longrightarrow W$ be a linear trans. of V.sp V into a V.sp W, then range of L is a subspace of W.

proof Let $w_1, w_2 \in \text{rang}\, L$. Then $w_1 = L(v_1)$, $w_2 = L(v_2)$ for some $v_1, v_2 \in V$

Hence $w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2)$

$\therefore w_1 + w_2 \in \text{range}\, L$

Also $aw = a L(v)$ For any Real no. a

$= L(av)$

$\therefore aw \in \text{rang}\, L$.

EX Let $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Is L onto ? Find dim Range L.

Sol Let $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in \mathbb{R}^2$. It clear that for $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$ (where a_3 any number in $\mathbb{R}$), $L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

To find dim range L ?

Since L is onto, range $L = \mathbb{R}^2$ $\therefore$ dim range $= 2$

EX Let $L: P_2 \to \mathbb{R}$ defined by $L(at^2+bt+c) = \int_0^1 (at^2+bt+c)\,dt$

Is L - onto. Find dim range L.

Is $\exists$ v s.t $L(v) = r$?

Sol Let $r \in \mathbb{R}$. we can find v s.t $v = at^2 + bt + c$, so $L(v) = \frac{at^3}{3} + \frac{bt^2}{2} + c$

Let $a = 0$, $b = 0$, $c = r$. Hence $L(r) = r$

$\therefore L$ is onto, so range $L = \mathbb{R}$ and dim range $L = 1$

EX Let $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

a) is L onto ?
b) Find a basis for range L ?.
c) Find Ker L
d) is L 1-1 ?

Sol (a) Let $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, where $a, b, c \in \mathbb{R}$.

To find V s.t $L(V) = w$, so we seek a solution of the lin-sys

$$AV = w$$

ie
$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right)$$

By transform this mat to r.r.e.f, we get

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array}\right)$$

The solution exists only when $c-b-a = 0$

$\therefore L$ is not onto.

(b) range $L = \{L(V) : V \in \mathbb{R}^3\}$

$$= \left\{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3\right\}.$$

$$= \left\{\begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 + 2a_3 \\ 2a_1 + a_2 + 3a_3 \end{pmatrix} : \text{" " }\right\}$$

Now $\begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 + 2a_3 \\ 2a_1 + a_2 + 3a_3 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (7)

$\therefore S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ spans range L

Notice that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\therefore S' = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ spans range L

Also S' is Lin-indep, since any vector is not multiple of the other.

$\therefore S'$ is a basis of range L

$\therefore$ dim range $L = 2$.

© $\operatorname{Ker} L = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : L \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ \quad " \quad : \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 + a_3 \\ 2a_1 + a_2 + 3a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$\therefore \left. \begin{array}{l} a_1 + a_3 = 0 \\ a_1 + a_2 + 2a_3 = 0 \\ 2a_1 + a_2 + 3a_3 = 0 \end{array} \right\} \Rightarrow \begin{array}{l} a_1 = -a_3 \\ a_2 = -a_3 \end{array}$

$\therefore \operatorname{Ker} L = \left\{ \begin{pmatrix} -a \\ -a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$

$= \left\{ a \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$

So $\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis for $\operatorname{Ker} L$

$\therefore \dim \operatorname{Ker} L = 1$

Thus L is not 1-1

Theorem Let $L: V \to W$ be a lin. trans of n-dim. V.sp V into V.sp W. Then

$$\dim \operatorname{Ker} L + \dim \operatorname{range} L = \dim V$$

ملحوظة المبرهنة تعطى بدون برهان

Example Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be a lin-trans defined by

$$L\left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right] = \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{pmatrix}$$

verify the previous th.

Sol
$$\operatorname{Ker} L = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : L\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} : \begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

But
$$\left.\begin{aligned} a_1 + a_3 &= 0 \\ a_1 + a_2 &= 0 \\ a_2 - a_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a_1 &= -a_3 \\ a_2 &= a_3 \end{aligned}$$

$$\therefore \operatorname{Ker} L = \left\{ \begin{pmatrix} -a \\ a \\ a \end{pmatrix} : a \in \mathbb{R} \right\} = \left\{ a\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$

$\therefore \dim \operatorname{Ker} L = 1$

Now, any vector in Range L is of the form $\begin{pmatrix} a_1 + a_3 \\ a_1 + a_2 \\ a_2 - a_3 \end{pmatrix}$

$$\Rightarrow a_1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + a_3\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\therefore S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ generate Range L

But $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, so $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ basis for range L

$\therefore \dim \operatorname{range} L = 2$

$$\dim \mathbb{R}^3 = 3 = \dim \ker L + \dim \operatorname{rang} L = 1 + 2$$

Chapter Six

Eigen Values and Eigen Vectors

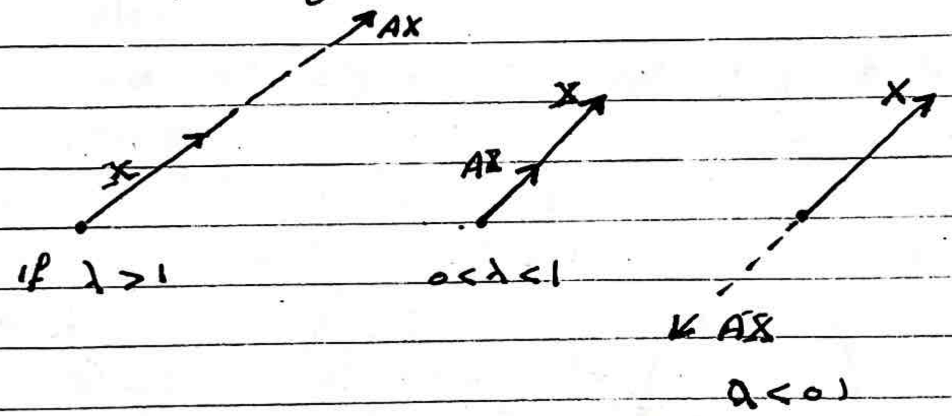
1 Eigenvalues and Eigen vectors

Definition Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there exists a nonzero vector X in $\mathbb{R}^n$ such that

$$AX = \lambda X$$

The vector x is called ~~ano~~ an eigenvector corresponding to λ.

geometricaly, the vector AX is the same direction as X depending on λ.



Computation of eiganvectors and Eigenvalues.

Let A be $n \times n$ matrix with eigenvalue λ and corresponding eigen vector X. Thus $AX = \lambda X$

Thus this equation Canbe written as

$$AX - \lambda X = 0$$

$$(A - \lambda I)X = 0$$

This matrix equation is a homogenous linear system

Note that this system has a nonzero solution if $|A - \lambda I| = 0$

Hence, solving the equation $|A-\lambda I_n|=0$ for λ (leads to all eigenvalues of A).

On expanding $|A-\lambda I|=0$, we get a poly. in λ. This polynomial is called the characteristic poly. of A. The equation $|A-\lambda I_n|=0$ is called the characteristic equation of A.

The eigenvalues are then substituted back in the eq. $(A-\lambda I_n)X=0$ to find the corresponding eigenvectors.

Examples

① Find all eigenvalues and eigenvectors of the matrix

$$A=\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}$$

Sol.

$$A-\lambda I_2=\begin{pmatrix}-4 & -6\\ 3 & 5\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}-4-\lambda & -6\\ 3 & 5-\lambda\end{pmatrix}$$

The ch. poly. $|A-\lambda I|=(-4-\lambda)(6-\lambda)-(-6)3$
$$=\lambda^2-\lambda-2$$

Now, to solve the ch. equation $\lambda^2-\lambda-2=0$

$\therefore\ (\lambda-2)(\lambda+1)=0$

$\Rightarrow \lambda=2$ or $\lambda=-1$ (the eigenvalues of A)

The corresponding eigenvectors are found by using these values of λ in the equation

$$(A-\lambda I_2)X = 0$$

So, if $\lambda = 2$, we solve the eq. $(A-2I_2)X=0$

Hence $\left[\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} -6 & -6 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad -6x_1 - 6x_2 = 0$$
$$3x_1 + 3x_2 = 0$$

Hence $x_1 = -x_2$, so the solution of this system eq. are $x_1 = -r$ $(x_2 = r)$, where r is a scalar

Thus the eigen vectors of A corresponding to $\lambda = 2$ are nonzero vectors of the form

$$r\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

when $\lambda = -1$, we solve the eq. $(A+1I_2)X = 0$

i.e. $\left[\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -3 & -6 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad -3x_1 - 6x_2 = 0$$
$$3x_1 + 6x_2 = 0$$

Thus $x_1 = -2x_2$, so the solutions are $x_1 = -2s$ and $x_2 = s$, where s is a sclar.

Hence the eigenvectors of A corresponding to $\lambda = -1$ are nonzero vectors of the form

$$s\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Note observe that the set of eigenvectors of $\lambda = 2$ together with zero vector, ie

$S = \{ r\begin{pmatrix} -1 \\ 1 \end{pmatrix} : r \in \mathbb{R} \}$ is a subspace of $\mathbb{R}^2$ with dimension 1.

Also the set of eigenvectors corresponding to $\lambda = -1$ with the zero vector; i.e

$S = \{ s\begin{pmatrix} -2 \\ 1 \end{pmatrix} : s \in \mathbb{R} \}$ is a subspace of $\mathbb{R}^2$ with $\dim(S) = 1$.

Theorem Let A be an $n \times n$ matrix and λ an eigenvalue of A. The set of all eigenvectors corresponding to λ, together with zero vector is a subspace of $\mathbb{R}$. (This space is called the eigenspace of λ.

Proof. Let X_1, X_2 be two vectors corresponding to λ, so $AX_1 = \lambda X_1$, $AX_2 = \lambda X_2$

Let c, d be any scalars.

T.P $cX_1 + dX_2$ is a vector in the eigen sp. of λ

$$\begin{aligned} A(cX_1 + dX_2) &= AcX_1 + AdX_2 \\ &= cAX_1 + dAX_2 \\ &= c(\lambda X_1) + d(\lambda X_2) \\ &= \lambda(cX_1) + \lambda(dX_2) \\ &= \lambda(cX_1 + dX_2). \end{aligned}$$

Thus $cX_1 + dX_2$ is a vector in the eigenspace of λ.
So eigenspace of λ is a subspace of $\mathbb{R}^n$.

<u>Example</u> Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

<u>Solution</u> $A - \lambda I_3 = \begin{pmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{pmatrix}$

The ch. poly.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} \xlongequal{-R_2+R_1} \begin{vmatrix} 1-\lambda & -1+\lambda & 0 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$\xlongequal{C_1+C_2} \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 9-\lambda & 2 \\ 2 & 4 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)[(9-\lambda)(2-\lambda) - 8] = (1-\lambda)(\lambda^2 - 11\lambda + 10)$$

بالفك حول R_1

5-

$= (1-\lambda)(\lambda-10)(\lambda-1) = -(\lambda-10)(\lambda-1)^2$

To solve the ch. eq. of A :

$-(\lambda-10)(\lambda-1)^2 = 0$

$\Longrightarrow \lambda = 10$ or $\lambda = 1$

$\therefore$ eigenvalues of A are 10 , 1 .

To find eigenvectors corresponding to $\lambda = 10$ we slove the eq: $(A - 10 I_3)X = 0$

$$\therefore \begin{pmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-5x_1 + 4x_2 + 2x_3 = 0$$
$$4x_1 - 5x_2 + 2x_3 = 0$$
$$2x_1 + 2x_2 - 8x_3 = 0$$

Hence $x_1 = 2r$, $x_2 = 2r$ and $x_3 = r$, where r is any scalar.

$\therefore$ Thus eigenvectors of $\lambda = 10$; $r\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $r \in \mathbb{R} - \{0\}$ & eigem space of $\lambda = 10$ is $\left\{ r\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} : r \in \mathbb{R} \right\}$ and It is One dimensional subspace of $\mathbb{R}^3$.

If $\lambda = 1$, then $(A - I)X = 0$, lead to .

$$\begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_2 = s \ , \ x_3 = 2t \ , \ x_1 = -s-t$ where s, t are scalar.

$\therefore$ Thus eigen space of $\lambda = 1$ is the space

$\left\{ \begin{pmatrix} -s-t \\ s \\ 2t \end{pmatrix} : s, t \in \mathbb{R} \right\}$ and its dim. is 2.

since $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} , \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right\}$ is a basis for this subsp.

Exercises

6'