4) Legendre polynomial:

(1)

The Legendre differential equation is given by:-

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{---} (1)$$

The solution of this equation gives the legendre polynomial $P_n(x)$ which is apolynomial in x of degree n, the rodriges formula is given by:-

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \quad \text{---} (2)$$

or

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \quad \text{---} (3)$$

where
$$N = \begin{cases} \frac{n}{2} & \text{for } n \text{ is even} \\ \frac{n-1}{2} & \text{for } n \text{ is odd} \end{cases}$$

Note: $P_n(1) = 1$, $P_n(-1) = (-1)^n$ — (4)

هذه حفظ

Ex:- Find $P_0(x), P_1(x), P_2(x)$

Sol/ From eq. (2) we get $\Rightarrow P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

$$\therefore P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0}(x^2-1)^0 = 1$$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx}(x^2-1) = \frac{1}{\cancel{2}} \cdot \cancel{2}x = x$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8} \frac{d}{dx}[2(x^2-1) \cdot 2x]$$

$$= \frac{1}{2} \frac{d}{dx}[(x^2-1) \cdot x] = \frac{1}{2}[2x^2 + (x^2-1)]$$

$$= \frac{1}{2}[x^2 - 1 + 2x^2]$$

$$= \frac{1}{2}[3x^2 - 1]$$

H.w Find $P_3(x), P_4(x), P_5(x)$ (2)

Generating function for Legendre polynomial

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n \quad \text{---} (5)$$

show that

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

Differntiating the G.f of eq.(5) with respect to (t) for both side, we obtain:-

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}} \cdot (-2x+2t) = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

$$\cancel{+}\frac{1}{\cancel{2}} \frac{\cancel{+2}(x-t)}{(1-2xt+t^2)^{\frac{1}{2}}} \cdot \frac{1}{(1-2x+t^2)} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

$$\frac{(x-t)}{(1-2xt+t^2)^{\frac{1}{2}}} = (1-2x+t^2) \sum_{n=0}^{\infty} n P_n(x) t^{n-1} \quad \text{---} (6)$$

put eq.(5) in (6), we obtain:-

$$(x-t) \sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2) \sum_{n=0}^{\infty} n P_n(x) t^{n-1}$$

$$\sum_{n=0}^{\infty} x P_n(x) t^n + \sum_{n=0}^{\infty} 2xn P_n(x) t^n = \underbrace{\sum_{n=0}^{\infty} n P_n(x) t^{n-1}}_{\text{replacing } (n \to n+1)} + \sum_{n=0}^{\infty} n P_n(x) t^{n+1}$$

$$+ \underbrace{\sum_{n=0}^{\infty} P_n(x) t^{n+1}}_{\text{replacing } (n \to n-1)}$$

$$\sum_{n=0}^{\infty} (1+2n) x P_n(x) t^n = \sum_{n=0}^{\infty} (n+1) P_{(n+1)}(x) t^n + \sum_{n=0}^{\infty} (n-1) P_{(n-1)}(x) t^n$$

$$+ \sum_{n=0}^{\infty} P_{(n-1)}(x) t^n \quad \text{---} (9)$$

$$\sum_{n=0}^{\infty}(1+2n)\,x\,P_n(x)\,t^n = \left[\sum_{n=0}^{\infty}(n+1)\,P_{(n+1)}(x) + \sum_{n=0}^{\infty}(n-1)\,P_{(n-1)}(x) + \sum_{n=0}^{\infty}P_{(n-1)}(x)\right]t^n$$

$$\sum_{n=0}^{\infty}(1+2n)\,x\,P_n(x)\,\cancel{t^n} = \sum_{n=0}^{\infty}\left[(n+1)\,P_{n+1}(x) + n\,P_{(n-1)}(x)\right]\cancel{t^n} \quad \text{——— (b)}$$

$$\therefore\ (1+2n)\,x\,P_n(x) = (n+1)\,P_{(n+1)}(x) + n\,P_{(n-1)}(x) \quad \text{——— (c)}$$

$$P_{n+1}(x) = \frac{(1+2n)}{(n+1)}\,x\,P_n(x) - \frac{n}{n+1}\,P_{(n-1)}(x)$$

"proved"

The properties of L.P :-

1) The L.P in algebria form satisfay the orthogonalty relation i.e:-

$$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = 0 \qquad (\text{if } m \neq n)$$

2) $$\int_{-1}^{+1} P_m(x)\,P_n(x)\,dx = \frac{2}{2n+1} \qquad (\text{if } m = n)$$

i.e $$\int_{-1}^{+1} |P_n(x)|^2\,dx = \frac{2}{2n+1}$$

H.W show that

$$P_2(\cos\theta) = \frac{1}{4}(1+3\cos 2\theta)$$

Hint :- $\cos^2\frac{1}{2}\alpha = \frac{1}{2}(1+\cos\alpha)$

Ex:- Prove the orthognality of Legendre polynomial

$$\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0 \qquad (m \neq n)$$

Sol. taken the Legendre differentoal equation.

$$(1-x^2)\, y'' - 2xy' + n(n+1)\, y = 0. \quad \text{———(1)}$$

The solution of eq.(1) is $P_n(x)$ -- i.e:-

$y = P_n(x)$ then and $y = P_m(x)$

$$\left.\begin{aligned} &[(1-x^2)\, P_m'' - 2x P_m' + m(m+1) P_m = 0] * P_n \\ &[(1-x^2)\, P_n'' - 2x P_n' + n(n+1) P_n = 0] * P_m \end{aligned}\right\} \text{———(2)}$$

∴ eq.(2) give

$$\left.\begin{aligned} &(1-x^2)\, P_n P_m'' - 2x P_n P_m' + m(m+1) P_m P_n = 0 \\ &(1-x^2)\, P_m P_n'' - 2x P_m P_n' + n(n+1) P_n P_m = 0 \end{aligned}\right\} \text{———(3)}$$

نطرح المعادلة (3)

Sub.

$$(1-x^2)[P_n P_m'' - P_m P_n''] - 2x[P_n P_m' - P_m P_n'] + P_n P_m[m(m+1) - n(n+1)] = 0$$

$$\therefore \underbrace{(1-x^2)[P_n P_m'' - P_m P_n''] - 2x[P_n P_m' - P_m P_n']}_{\text{هذا الحد هو مشتقة التالي}} = P_n P_m [n(n+1) - m(m+1)] \quad \text{———(4)}$$

L.H.S is:-

$$L.H.S = \frac{d}{dx}\left[(1-x^2)(P_n P_m' - P_m P_n')\right] \quad \text{———(5)}$$

Put eq.(5) in eq.(4) we obtain :-

$$\frac{d}{dx}\left[(1-x^2)(P_n P_m' - P_m P_n')\right] = P_n P_m\left[n(n+1) - m(m+1)\right]$$

$$d\left[(1-x^2)(P_n P_m' - P_m P_n')\right] = P_n P_m\, dx\left[n(n+1) - m(m+1)\right] \quad \text{—— (6)}$$

Integrating both sides of eq. (6),

$$\frac{\left[(1-x^2)(P_n P_m' - P_m P_n')\right]_{-1}^{+1}}{n(n+1) - m(m+1)} = \int_{-1}^{+1} P_n P_m\, dx$$

من التعويض عن قيم ١ و ١- نحصل على

$$\int_{-1}^{+1} P_n P_m\, dx = 0$$

The associated Legendre polynomial

The differential equation is

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \quad \text{—— (1)}$$

is called Associated Legendre polynomial.

If $m=0$, eq. (1) reduces to L. eq.

The solution of eq. (1) gives the associated Legendre polynomial $P_n^m(x)$ which is a polonomial in x of degree n.

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \quad \text{—— (2)}$$

Notes:-

* If $m > n \Rightarrow P_n^m(x) = 0$
* $P_n^{-m} = (-1)^m P_n^m$
* $-n \leq m \leq n$

Ex:- proved $P_2^3(x) = 0$

Sol.- $\because P_2(x) = \frac{1}{2}(3x^2-1)$

using Rodregus formula ; $P_n^m = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$

$\therefore P_2^3(x) = (1-x^2)^{3/2} \frac{d^3}{dx^3}\left[\frac{1}{2}(3x^2-1)\right]$

$= (1-x^2)^{3/2} \frac{d^2}{dx^2}\left[\frac{1}{2}6x\right] = (1-x^2)^{3/2} \frac{d}{dx}(3)$

$\therefore P_2^3(x) = (1-x^2)^{3/2} * 0 = 0$

---

* Note:-

The $P_n^m(x)$ have the orthogonality property:-

* $\int_{-1}^{+1} P_n^m(x)\, P_k^m(x)\, dx = 0$, if $(n \neq k)$

* $\int_{-1}^{+1} P_n^m(x)\, P_k^m(x)\, dx = \int_{-1}^{+1} |P_n^m(x)|^2\, dx = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!}$, if $(n=k)$

Ex:- Find the associated L.P. for $P_2^1(x)$

$\because P_n^m = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$

$\therefore P_2^1(x) = (1-x^2)^{1/2} \frac{d}{dx} P_2(x)$, $\because P_2(x) = \frac{1}{2}(3x^2-1)$

$= (1-x^2)^{1/2} \frac{d}{dx}\left[\frac{1}{2}(3x^2-1)\right]$

$= (1-x^2)^{1/2} * 3x$

H.w Find $P_3^2(x)$?

Ch.(3)
Integral Transforms

Integral Transforms

The integral transforms $F(s)$ of a function $f(x)$ with the kernel $k(s,x)$ is defined as:-

$$I[f(x)] = F(s) = \int_a^b f(x)\, k(s,x)\, dx$$

For example:-

1- Laplace transform with the kernel $k(s,x) = e^{-sx}$

$$L[f(x)] = F(s) = \int_0^\infty f(x) \cdot e^{-sx}\, dx \qquad (1)$$

2- Fourier Complex transform with the kernel $k(s,x) = e^{-isx}$

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx. \qquad (2)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(s)\, e^{-isx}\, ds \qquad \text{(Inversion formula)}$$

3- Fourier Sine transform with the kernel $k(s,x) = \sin sx$

$$F_s[f(x)] = F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx \qquad (3)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(s) \sin sx \, ds \qquad \text{(Inversion formula)}$$

4- Fourier Cosine transform with the kernel $k(s,x) = \cos sx$

$$F_c[f(x)] = F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx \qquad (4)$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx\, ds \qquad \text{(Inversion formula)}$$

Note that integral transforms are used in solving the partical differential equation with boundary conditions.

List of formulae of Fourier Integrals

1. Fourier integral for $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{+\infty} f(t) \cos u(t-x)\, du\, dt \quad \text{---} (5)$$

2. Fourier Sine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin ux\, du \int_0^{\infty} f(t) \sin ut\, dt \quad \text{---} (6)$$

3- Fourier Cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos ux\, du \int_0^{\infty} f(t) \cos ut\, dt \quad \text{---} (7)$$

proof of eq. (6) and (7)

It is known that $\cos u(t-x) = \cos(ut-ux)$

$$= \cos ut \cos ux + \sin ut \sin ux$$

using eq. (5) we get

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) [\cos ut \cos ux + \sin ut \sin ux]\, du\, dt$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{+\infty} f(t) \cos ut \cos ux\, du\, dt + \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{+\infty} f(t) \sin ut \sin ux\, du\, dt$$

Case 1: If $f(t) \cos ut$ is odd, then the first integral will equal to zero and we obtain.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin ux\, du \int_0^{\infty} f(t) \sin ut\, dt \quad \text{---} (8) \text{ (Fourier sine integral)}$$

Case 2: if $f(t) \sin ut$ is odd, then the second integral will equal to zero and we obtain.

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos ux\, du \int_0^{\infty} f(t) \cos ut\, dt \quad \text{(Fourier Cosine integral)} \quad \text{---} (9)$$

Note that we have used the following relations:-

1) For odd functions: $\int_{-a}^{a} f(x)\,dx = 0$

2) For even functions: $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

Fourier Complex Integral (Fourier Integral Theorem)

It has the form

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\,du \int_{-\infty}^{\infty} f(t)\,e^{iut}\,dt \qquad \text{---(10)}$$

proof,

It is known that $\int_{-\infty}^{\infty} \sin u(t-x)\,du = 0$ [Since, $\sin u(t-x)$ is odd]

Obviously we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \sin u(t-x)\,du = 0$$

or

$$\frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \sin u(t-x)\,du = 0 \qquad \text{---(11)}$$

Adding eq's (5) and (11), we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos u(t-x)\,du\,dt + \frac{i}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} \sin u(t-x)\,du$$

لأن صدر
التكامل أصبح من $-\infty$ الى ∞

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} [\cos u(t-x) + i\sin u(t-x)]\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_{-\infty}^{\infty} e^{iu(t-x)}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\,du \int_{-\infty}^{\infty} f(t)\,e^{iut}\,dt$$

(proved)

Solved problems

1. Express the function $f(x)=\begin{cases}1 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1\end{cases}$

as a Fourier integral. Hence evaluate

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda$$

Sol. The Fourier integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(t)\cos\lambda(t-x)\,dt\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\int_{-1}^{1}\cos\lambda(t-x)\,dt\,d\lambda=\frac{1}{\pi}\int_0^\infty\left[\frac{\sin\lambda(t-x)}{\lambda}\right]_{-1}^{1}d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{\sin\lambda(1-x)-\sin\lambda(-1-x)}{\lambda}\,d\lambda$$

$$=\frac{1}{\pi}\int_0^\infty\frac{\sin\lambda(1-x)+\sin\lambda(1+x)}{\lambda}\,d\lambda$$

Using the identity $\sin\alpha+\sin\beta=2\sin\frac{1}{2}(\alpha+\beta).\cos\frac{1}{2}(\alpha-\beta)$

we get

$$f(x)=\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda\Longrightarrow\int_0^\infty\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda=\frac{\pi}{2}f(x)$$

$$\therefore\int_0^\infty\frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda=\begin{cases}\frac{\pi}{2} & \text{for } |x|<1\\ 0 & \text{for } |x|>1\end{cases}$$

2. Find the Fourier sine integral for $f(x)\ e^{-\beta x}$, hence show that $\frac{\pi}{2} e^{-\beta x} = \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2+\lambda^2} d\lambda$.

Sol The Fourier sine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \, d\lambda \int_0^\infty f(t) \sin \lambda t \, dt$$

$$= \frac{2}{\pi}\int_0^\infty \sin \lambda x \, d\lambda \int_0^\infty e^{-\beta t} \sin \lambda t \, dt$$

From table of integral we have the following identities

$$\left.\begin{aligned} \int_0^\infty e^{-ax} \sin bx \, dx &= \frac{b}{a^2+b^2} \quad \text{if } a>0 \quad &\text{--- (1)}^* \\ \int_0^\infty e^{-ax} \cos bx \, dx &= \frac{a}{a^2+b^2} \quad \text{if } a>0 \quad &\text{--- (2)}^* \end{aligned}\right\}$$

Use the identity of eq. (1) and let $f(x) = e^{-\beta x}$ in the L.H.S

$$e^{-\beta x} = \frac{2}{\pi}\int_0^\infty \sin \lambda x \, d\lambda . \frac{\lambda}{\beta^2+\lambda^2}$$

$$\frac{\pi}{2} e^{-\beta x} = \int_0^\infty \frac{\lambda \sin \lambda x}{\beta^2+\lambda^2} d\lambda$$

3- Using Fourier cosine integral representation of an appropriate function, show that $\int_0^\infty \frac{\cos wx}{k^2+w^2} dw = \frac{\pi e^{-kx}}{2k}$, when $f(x) = e^{-kx}$

sol The Fourier cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos ux \, du \int_0^\infty f(t) \cos ut \, dt$$

Replacing u by w and putting $f(x) = e^{-kx}$ and $f(t) = e^{-kt}$, we get

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \cos wx \, dw \int_0^\infty e^{-kt} \cos wt \, dt$$

using eq. (2) we, obtain $\Longrightarrow$

$$e^{-kx} = \frac{2}{\pi}\int_0^\infty \cos wx \, dw \cdot \frac{k}{k^2+w^2}$$

$$\therefore \int_0^\infty \frac{\cos wx}{k^2+w^2}\, dw = \frac{\pi e^{-kx}}{2k}$$

Fourier Transforms

From eq.(10), we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\, du \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \qquad (u=s) \quad \text{——(1)}$$

Now putting

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \quad \text{——(2)}$$

we get from eq.(1)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds \quad \text{——(3)}$$

In eq. (2), $F(s)$ is called the Fourier transform of $f(x)$

In eq. (3), $f(x)$ is called the inverse Fourier transform of $F(s)$.

For reason of symmetry, we multiply both $f(x)$ and $F(s)$ by $\sqrt{\frac{1}{2\pi}}$ istead of having the factor $\frac{1}{2\pi}$ in only one function.

Thus we obtain the definition of Fourier transforms as:-

$$\boxed{\begin{aligned} F(s) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt \\ f(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\, e^{-ist}\, ds \end{aligned}}$$

Fourier Sine and Cosine Transforms

From eq. (6), we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin sx\, ds \int_0^\infty f(t)\sin st\, dt \qquad (u=s)$$

$$\boxed{\begin{aligned} F(s) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\, dt \\ f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\sin sx\, ds \end{aligned}}$$

where:-

1) $F(s)$ is called Fourier sine transform of $f(x)$
2) $f(x)$ is called the inverse Fourier sine transform of $F(s)$

From eq. (7), we have

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos sx\, ds \int_0^\infty f(t)\cos st\, dt \qquad (u=s)$$

$$\boxed{\begin{aligned} F(s) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\, dt \\ f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\cos sx\, ds \end{aligned}}$$

where

1) $F(s)$ is called Fourier cosine transform of $f(x)$
2) $f(x)$ is called inverse Fourier cosine transform of $F(s)$.

Solved Problems

1. Find the Fourier transform of $f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$

Sol: the Fourier transform of a function $f(x)$ is given by

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} 1\cdot e^{isx}\, dx = \frac{1}{\sqrt{2\pi}}\frac{e^{isx}}{is}\Big|_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{is}\left[e^{isa} - e^{-isa}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{2}{s}\,\frac{1}{2i}\left[e^{isa} - e^{-isa}\right] = \sqrt{\frac{2}{\pi}}\,\frac{1}{s}\sin sa.$$