Changing avariable may be sometimes make the evaluation of double Integral easier:-

$$X = X(u,v) \quad , \quad y = y(u,v)$$

$$\iint_A f(x,y)\,dx\,dy = \iint_{A'} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)}$$

A = area in x-y plane

A' = area in u-v plane

J = Jacobian

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

* For triple Integral.

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J|\,du\,dv\,dw$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Ex:- Find the triple integral by asuitable change of avariable $I = \iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$

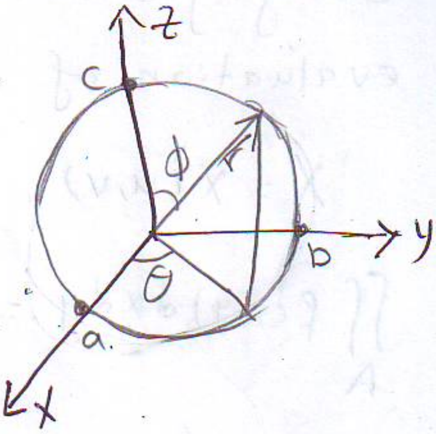
V = region bounded by ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Sol:

الاحداثيات الكروية

$$\left.\begin{aligned} x &= ar\sin\phi\cos\theta \\ y &= br\sin\phi\sin\theta \\ z &= cr\cos\phi \end{aligned}\right\}$$ حفظ



The relation between (x,y,z) and (r,θ,ϕ) show in the figure, The range of r,θ,ϕ is:-

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

The Jacobian is:-

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$J = \begin{vmatrix} a\sin\phi\cos\theta & ar\cos\phi\cos\theta & -ar\sin\phi\sin\theta \\ b\sin\phi\sin\theta & br\cos\phi\sin\theta & br\sin\phi\cos\theta \\ c\cos\phi & -cr\sin\phi & 0 \end{vmatrix}$$

$\therefore J = a\sin\phi\cos\theta\,[bcr^2\sin^2\phi\cos\theta] - acr\cos\phi\cos\theta \cdot$

$[bcr\sin\phi\cos\theta\cos\phi] + -ar\sin\phi\sin\theta[(-b\sin\phi\sin\theta \cdot$

$-cr\sin\phi) - (brc\cos\phi^2\sin\theta)]$

$J = abcr^2\sin^3\phi\cos^2\theta - abcr^2\cos^2\phi\cos^2\theta\sin\phi -$

$ar\sin\phi\sin\theta(bcr\sin^2\phi\sin\theta - brc\cos^2\phi\sin\theta)$

$-bcr\sin\theta(\sin^2\phi + \cos^2\phi)$

$$J = abcr^2 \sin^3\phi \cos^2\theta - abc\,r^2 \cos^2\phi \cos^2\theta \sin\phi$$
$$+ abc\,r^2 \sin\phi \sin^2\theta$$

$$= abc\,r^2 \sin\phi (\sin^2\phi \cos^2\theta + \cos^2\phi \cos^2\theta + \sin^2\theta)$$

$$= abc\,r^2 \sin\phi \, \underbrace{[\cos^2\theta \underbrace{[\sin^2\phi + \cos^2\phi]}_{1} + \sin^2\theta}_{1}$$

$$\therefore \boxed{J = abc\,r^2 \sin\phi}$$ هذا القانون حفظ

in spherical

$$\therefore I = \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} [r^2 (a^2 \sin^2\phi \cos^2\theta + b^2 \sin^2\phi \sin^2\theta + c^2 \cos^2\phi)$$
$$abc\,r^2 \sin\phi \, dr \, d\theta \, d\phi$$

$$= abc \frac{r^5}{5}\Big|_0^1 \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} (a^2 \sin^3\phi \cos^2\theta + b^2 \sin^3\phi \sin^2\theta$$
$$+ c^2 \cos^2\phi \sin\phi) \, d\theta \, d\phi$$

$$= \frac{abc}{5} \int_0^{\pi} [a^2 \sin^3\phi \left(\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} + b^2 \sin^3\phi$$
$$\left[\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} + c^2 \cos^2\phi \sin\phi [\theta]_0^{2\pi} d\phi$$

$$= \frac{abc}{5} \int_0^{\pi} (\sin^3\phi (a^2\pi + b^2\pi) + (c^2 \cos^2\phi \sin\phi)(2\pi) d\phi$$

$$= \frac{\pi\, abc}{5} \left[(a^2+b^2) \left(-\frac{1}{3} (\cos\phi \sin^2\phi + 2\cos\phi \right)_0^{\pi} \right.$$
$$\left. + 2c^2 \left[- \cos^3\phi \right]_0^{\pi} \right]$$

$$= \frac{4\pi\, abc}{15} (a^2+b^2+c^2)$$

Note قوانين متطابقات مهمة

$$① \int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$2) \int \sin^n x \, dx = \frac{-\cos x \sin^{n-1} x}{n} - \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

Example :- Find the unit normal $\vec{n}$ to the surface of asphere of radius (a), Centred at the origion

$\because X = a \sin\phi \cos\theta$ $\qquad a = b = c \equiv a$

$y = a \sin\phi \sin\theta$

$z = a \cos\phi$

$$\vec{r} = \hat{i} x + \hat{j} y + \hat{k} z$$

$$r_\phi = \frac{\partial r}{\partial \phi} = \hat{i} \frac{\partial x}{\partial \phi} + \hat{j} \frac{\partial y}{\partial \phi} + \hat{k} \frac{\partial z}{\partial \phi}$$

$$= \hat{i} a \cos\phi \cos\theta + \hat{j} a \cos\phi \sin\theta - \hat{k} a \sin\phi$$

$$r_\theta = \frac{\partial r}{\partial \theta} = \hat{i} \frac{\partial x}{\partial \theta} + \hat{j} \frac{\partial y}{\partial \theta} + \hat{k} \frac{\partial z}{\partial \theta}$$

$$= \hat{i} (-a \sin\phi \sin\theta) + \hat{j} (a \sin\phi \cos\theta) - 0$$

$$\vec{n} = \frac{\vec{r}_\phi \times \vec{r}_\theta}{|\vec{r}_\phi \times r_\theta|}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos\phi \cos\theta & a \cos\phi \sin\theta & -a \sin\phi \\ -a \sin\phi \sin\theta & a \sin\phi \cos\theta & 0 \end{vmatrix}$$

$$= i(a^2 \sin^2\phi \cos\theta) + \hat{j}(a^2 \sin^2\phi \sin\theta) + \hat{k}[a^2 \sin\phi \cos\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1}]$$

$$= \hat{i}(a^2 \sin^2\phi \cos\theta) + \hat{j}(a^2 \sin^2\phi \sin\theta) + \hat{k}(a^2 \sin\phi \cos\theta)$$

9

$$|r\theta| = \frac{\partial r}{\partial\theta} = \sqrt{a^2 \sin^2\phi \underbrace{\sin^2\theta + a^2 \sin^2\phi \cos^2\theta}_{=1} + 0}$$

$$= \sqrt{a^2 \sin^2\phi}$$

$$= a^{1} \sin\phi$$

$$|r\phi| = \frac{\partial r}{\partial\phi} = \sqrt{a^2 \cos^2\theta \cos^2\phi + a^2 \cos^2\phi \sin^2\theta + a^2 \sin^2\phi}$$

$$= \sqrt{a^2 \cos^2\phi \underbrace{(\cos^2\theta + \sin^2\theta)}_{\Rightarrow 1} + a^2 \sin^2\phi}$$

$$= \sqrt{a^2(\cos^2\phi + \sin^2\phi)} = \sqrt{a^2} = a$$

$$\therefore |\vec{r}\theta \times \vec{r}\phi| = a^2 \sin\phi$$

$$\therefore \vec{n} = \frac{\hat{i}\, a^2 \sin^2\phi \cos\theta + \hat{j}\, a^2 \sin^2\phi \sin\theta + \hat{k}\, a^2 \sin\phi \cos\phi}{a^2 \sin\phi}$$

$$= \frac{\cancel{a \sin\phi}\,(i\, a \sin\phi \cos\theta + j\, a \sin\phi \sin\theta + \hat{k}\, a \cos\phi}{\cancel{a^2}\ \cancel{\sin\phi}}$$

$$\hat{n} = \frac{\vec{r}}{a}$$

also, $\phi = x^2 + y^2 + z^2 = a^2$

$$\nabla\phi = \text{grad}\,\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla\phi = 2\vec{r}$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{r}}{|2r|} = \frac{\vec{r}}{a}$$

Surface and Volume Integrals:-

$\phi(x,y,z)$: scalar function, the surface integral of over S given by:-

$$I = \iint_S \phi(x,y,z)\, ds \quad \text{——} (1)$$

تحويل الفضاء (x,y,z) الى مستوي (u,v)

$$= \iint_{S'} \phi(x(u,v), y(u,v), z(u,v))\, |r_u \times r_v|\, du\, dv$$

S = in xyz - space

S' = in uv plane

if ϕ, scalar function

For the vector function $\bar{V}$, vector field

$$\iint_S \vec{V}\cdot\vec{n}\, ds = \iint_S \bar{V}(u,v)\cdot(r_u \times r_v)\, du\, dv$$

Example, Find $\iint_S \vec{V}\cdot n\, ds$ where $\vec{V} = 4x\hat{i} + y\hat{j} + \hat{k}z$ and S is the plane $(2x+y+2z=6)$ in position at $z = 3 - x - \frac{y}{2}$ —— (1)'

$2x + y = 6$ —— (2)

Sol. $\iint_S \vec{V}\cdot\vec{n}\, ds = \iint_S \overrightarrow{V}(u,v)\cdot(r_u \times r_v)\, du\, dv$

Let
$$\left.\begin{aligned} u &= x \\ v &= y \\ \therefore z &= 3 - u - \frac{v}{2} \end{aligned}\right\} \text{——} (3)$$

eq. (2) becom $2u + v = 6$ —— (4)

So, the vector position $\vec{r}$ in any point on S is:-

$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$\vec{r} = \hat{i}u + \hat{j}v + \hat{k}(3 - u - \frac{v}{2})$ —— (5)

$\vec{r}_u = \frac{\partial r}{\partial u} = \hat{i} - \hat{k}$

$\vec{r}_v = \frac{\partial r}{\partial v} = \hat{j} - \frac{1}{2}\hat{k}$

$\vec{r}_u \times \vec{r}_v = \hat{i} + \frac{1}{2}\hat{j} + \hat{k}$ —— (6)

$$\iint_S \vec{V}\cdot\vec{n}\, ds = \iint_S [4u\hat{i} + v\hat{j} + (3 - u - \frac{v}{2})\hat{k}]\cdot(\hat{i} + \frac{1}{2}\hat{j} + \hat{k})\, dv\, du$$

$$= \int_{u=0}^{3}\int_{v=0}^{6-2u} (4u + \cancel{\frac{1}{2}v} + 3 - u - \cancel{\frac{v}{2}})\, dv\, du$$

$$= \int_0^3\int_0^{6-2u} (3u + 3)\, dv\, du$$

$$= 3\int_0^3 (uv + v)_0^{6-2u}\, du$$

$$= 3\int_0^3 [u((6-2u) - 0) + (6 - 2u - 0)\, du$$

$$= 3\int_0^3 (6u - 2u^2 + 6 - 2u)\, du$$

$$= 3\int_0^3 (4u - 2u^2 + 6)\, du$$

$$= \cancel{12}\frac{u^2}{\cancel{2}} - \cancel{6}\frac{u^3}{\cancel{3}} + 18u\Big|_0^3$$

$$= 6\times 9 - 2\times 3^3 + 18\times 3 = 54$$

Volume Integral :-

$$\iiint_V \phi(x,y,z)\,dx\,dy\,dz = \iiint_V \phi(x,y,z)\,dV$$

$$\vec{r} = \hat{i}\,x(u,v,w) + \hat{j}\,y(x,v,w) + \hat{k}\,z(u,v,w)$$

$$dV = |(\vec{r_u} \times \vec{r_v}) \cdot \vec{r_w}|\ du\,dv\,dw$$

$$\iiint_V \phi(x,y,z)\,dx\,dy\,dz = \iiint_V \phi(x(u,v), y(u,v), z(u,v))\,|(\vec{r_u} \times \vec{r_v}) \cdot \vec{r_w}|\ du\,dv\,dw$$

ϕ = scalar function

$$\iiint_V \vec{V}\,d\vec{V} = \iiint_V (iV_x + jV_y + \hat{k}V_z)\,|(\vec{r_u} \times \vec{r_v}) \cdot \vec{r_w}|\ du\,dv\,dw$$

V = Vector

① Divergent theorm نظرية التباعد

التفريق

$$\iint_S \vec{V} \cdot n\,ds = \iiint_V \operatorname{div} \vec{V}\,dV = \iiint_V \vec{\nabla} \cdot \vec{V}\,dV$$

$\vec{n}$ = is the unit out word normal to S هي نظرية تحويل التكامل السطحي الى حجم

$$\iint \underbrace{V \cdot n}_{\Downarrow}\ \underset{\Downarrow}{ds} \equiv \iiint_V \underbrace{\operatorname{div} V}_{\Downarrow}\ dV$$

تحدد حسب شكل المقطع

$ds \Rightarrow dx\,dy = \int du\,dv$, $\int d\theta\,d\phi$

$dV = dx\,dy\,dz \Rightarrow$ or $\int dr\,d\theta\,d\phi = |r_r \times r_\phi| \cdot r_\theta = r^2 \sin\phi\,dr\,d\theta\,d\phi$

$$\operatorname{div}\vec{V} = i\frac{\partial}{\partial x}V_x + j\frac{\partial}{\partial y}V_y + k\frac{\partial}{\partial z}V_z$$

Ex:- Evaluate $\int F.n\, ds$ where $F = 4xz\,i - y^2 j + yz\,k$ and S is the surface of the cube bounded by $x=0, x=1, y=0, y=1$, $z=0$ and $z=1$ by using divergence theorem.

Sol

$$\int_S \vec{F}.n\, ds = \int_V \text{div} F\, dv \quad \left(\int_V \nabla . F\, dv\right)$$

$$\text{div}\,\vec{F} = \vec{\nabla}.\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y$$

$$\therefore \int_V \text{div} F\, dV = \iiint (4z-y)\, dx\, dy\, dz = \int_0^1\int_0^1 (2z^2 - yz)\Big|_0^1 dx\, dy$$

$$= \int_0^1\int_0^1 (2-y)\, dx\, dy = \int_0^1 \left(2y - \frac{y^2}{2}\right)\Big|_0^1 dx$$

$$= \int_0^1 \frac{3}{2}\, dx = \frac{3}{2}.$$

Ex:- prove that $\int_S F.n\, ds = -4\pi \int \rho\, dV$, where $F = \nabla\phi$ and $\nabla^2\phi = -4\pi\rho$.

Sol

$$\int_S F.n\, ds = \int_V \text{div} F\, dv = \int_V \nabla . F\, dV = \int \nabla.(\nabla\phi)\, dV$$

$$= \int_V \nabla^2\phi\, dV = \int -4\pi\rho\, dV = -4\pi \int_V \rho\, dV$$

H.W ① prove that $\int_V \frac{dV}{r^2} = \int \frac{r.n}{r^2}\, ds$.

② prove that $\int_V \nabla \times B\, dv = \int_S (n \times B)\, ds$

Green's theorem

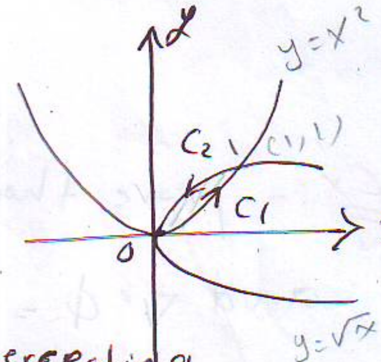
If M and N are continuous functions of x and y having continuous derivative in a region S bounded by a closed curve C, then

$$\int_C (M\,dx + N\,dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Ex :- Verify Green theorem in the plane for

$$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

where C is the boundary of the region defined by $y = \sqrt{x}$ and $y = x^2$



Sol

$y = \sqrt{x}$

$\therefore y^2 = x$

and $y = x^2$ are two parabolas intersecting at $(0,0)$ and $(1,1)$

Now, along the curve C_1, $x^2 = y$ --i.e

$dy = 2x\,dx$, and x varies from 0 to 1.

$\therefore$ Along C_1,

$$\int_0^1 [(3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx]$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx = -1 \quad \text{—— (1)}$$

Along C_2, $x = y^2$ --i.e, $dx = 2y\,dy$, y varies from 1 to 0

$\therefore$ Along C_2

$$\int_1^0 [(3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy] = \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= \frac{5}{2} \quad \text{—— (2)}$$

The line integral along C

$$= -1 + \frac{5}{2} = \frac{3}{2}.$$

Again $\quad M = 3x^2 - 8y^2, \therefore \frac{\partial M}{\partial y} = -16y$

and $\quad N = 4y - 6xy \therefore \frac{\partial N}{\partial x} = -6y$

$$\therefore \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = +10y$$

$$\therefore \iint_S \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y \, dx\,dy$$

$$= \int_0^1 \left[5y^2\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)\, dx = \frac{3}{2}$$

H.W Evaluate $\int_C (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$

where C is the boundary of the surface in xy plane enclosed by x-axis and semi-circle $x^2+y^2=1$

Equation of Continuity of Flowing Fluid (Conservation of Mass)

$$m = \iiint_V \rho \, dV \quad \text{—— (1)}$$

the rate of influx of mass

$$Q = \frac{dm}{dt} = \iiint_V \frac{\partial \rho}{\partial t} \, dV \quad \text{—— (2)}$$

$$Q = -\iint_S \rho \, \mathbf{v} \cdot \mathbf{n} \, ds \quad \text{—— (3)}$$

$\vec{V}$ = Velocity of the fluid, ($-$ve = outword normal) using Guass theorm get:-

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV = -\iiint_V \operatorname{div}(\rho \vec{V}) \, d\vec{V}$$

$$\therefore \iiint_V \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V)\right) dV = 0 \quad \text{——} (5)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho V) = 0 \quad \text{——} (6)$$

$$\nabla \cdot (\rho \vec{V}) = \rho \left(\frac{\partial \vec{V}_x}{\partial x} + \frac{\partial \vec{V}_y}{\partial y} + \frac{\partial \vec{V}_z}{\partial z}\right) + \vec{V}_x \frac{\partial \rho}{\partial x} + \vec{V}_y \frac{\partial \rho}{\partial y} + \vec{V}_z \frac{\partial \rho}{\partial z}$$

$$= \rho \nabla \vec{V} + \vec{V} \nabla \cdot \rho \quad \text{——} (7)$$

$$\therefore \frac{\partial \rho}{\partial t} + V \nabla \rho + \rho \nabla \cdot V = 0 \quad \text{——} (8) \text{ equation}$$

ρ = constant for incompresible
السائل غير قابل للانضغاط fluid

$$\frac{\partial \rho}{\partial t} = 0$$

so eq. (8) become

$$\nabla \cdot V = \operatorname{div} V = 0$$