chapter 1

Vector and Tensor analysis:-

Vector: Define as aquantity has both magnetude and direction

$$\bar{V} = \hat{i}V_x + \hat{j}V_y + \hat{k}V_z \quad \text{---(1)} \qquad \vec{V}(V_x, V_y, V_z)$$

$\bar{V}$ -vector, Define as atensor of order one

Scalar: Has only magnitude, Represent atensor of zero order.

* Second order tensor in 3-D space is represent by 9-numbers (components)

$$\bar{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{---(2)}$$

2nd order tensor

Vector:-

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = |A||B| \cos\theta$$, θ angle between $\vec{A}$ and $\vec{B}$

$$\vec{C} = \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} = |A||B| \sin\theta$$

$$\hat{n} = \frac{\vec{c}}{|c|} = \frac{\vec{c}}{c}$$, $\hat{n}$: unit vector orthogonal to $\vec{A}$ an $\vec{B}$

$$C = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\bar{A}\cdot\bar{B}\times\bar{C} = \bar{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\bar{A}\cdot\bar{B}\times\bar{C} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$\frac{d}{dt}(\bar{A}+\bar{B}) = \frac{d\bar{A}}{dt} + \frac{d\bar{B}}{dt}$$

$$\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \bar{A}\cdot\frac{d\bar{B}}{dt} + \frac{d\vec{A}}{dt}\cdot\bar{B}$$

$$\frac{d}{dt}(\vec{A}\times\bar{B}) = \bar{A}\times\frac{d\bar{B}}{dt} + \frac{d\vec{A}}{dt}\times\vec{B}$$

$$\frac{d}{dt}(\phi A) = \phi\frac{d\bar{A}}{dt} + \bar{A}\frac{d\phi}{dt}$$

ϕ : scalar

V : Vector

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

اكرا

① $\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$ = grand ϕ = vector

② $\vec{\nabla}\cdot V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ = div $\vec{V}$ = Scalar

③ $\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$ = Curl $\bar{V}$ or (rot V)

= vector

* $\nabla\times(A\times B) = (\bar{B}\cdot\bar{\nabla})\bar{A} - \bar{B}(\bar{\nabla}\cdot\bar{A}) - (\bar{A}\cdot\bar{\nabla})\bar{B} + \underline{\bar{A}(\vec{\nabla}\cdot\vec{B})}_{(1)}$

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ Laplacian operator

2) $\vec{\nabla}\cdot(\vec{w}\times\vec{r})=0$

Sol//

$$\vec{\nabla}\cdot(\vec{w}\times\vec{r}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_x & w_y & w_z \\ x & y & z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}(w_y \cdot z - y w_z) + \frac{\partial}{\partial y}(x w_z - z w_x) + \frac{\partial}{\partial z}(y w_x - x w_y)$$

$$= 0$$

3) $\vec{\nabla}\times\vec{w}\times\vec{r} = 2w$

from eq.(1)

$$\vec{\nabla}\times\vec{w}\times\vec{r} = (\vec{r}\cdot\vec{\nabla})\vec{w} - \vec{r}(\vec{\nabla}\cdot\vec{w}) - (\vec{w}\cdot\vec{\nabla})\vec{r} + \vec{w}(\vec{\nabla}\cdot\vec{r}) \quad ---(2)$$

$$(\vec{w}\cdot\vec{\nabla})r = \left[w_x\frac{\partial}{\partial x} + w_y\frac{\partial}{\partial y} + w_z\frac{\partial}{\partial z}\right]\vec{r}$$

$$= \hat{i}w_x + \hat{j}w_y + \hat{k}w_z = w$$

$$\vec{\nabla}\cdot\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

$\hat{i}\cdot\hat{i}=1$
$\hat{j}\cdot\hat{j}=1$
$\hat{k}\cdot\hat{k}=1$

$$(\vec{r}\cdot\vec{\nabla})\vec{w} = (\hat{i}x + \hat{j}y + \hat{k}z)\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\vec{w}$$

$$= x\frac{\partial w_x}{\partial x} + y\frac{\partial w_y}{\partial y} + z\frac{\partial w_z}{\partial z}$$

نفسه

$$\vec{r}(\vec{\nabla}\cdot\vec{w}) = x\frac{\partial w_x}{\partial x} + y\frac{\partial w_y}{\partial y} + z\frac{\partial w_z}{\partial z}$$

∴ اكد الحدود دوال في ماعدا طرحهما = zero

$$\therefore \vec{\nabla}\times\vec{w}\times\vec{r} = 0 - w + 3w = 2w$$

H.W ① Find angle between two vector

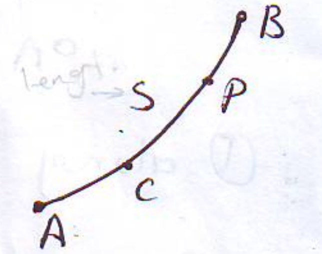
$\bar{A} = 3\hat{i} + 6\hat{j} + 9\hat{k}$, $\vec{B} = -2\hat{i} + 3\hat{j} + \hat{k}$

② Find a vector perpendicular to both $\bar{A} = 2\hat{i} + \hat{j} - \hat{k}$ & $\bar{B} = \hat{i} + 3\hat{j} - 2\hat{k}$

Integral of a scalar function :-

$\phi(x,y,z)$, scalar function along c from point $\bar{A}$ to $\vec{B}$

$$r(s) = i\,x(s) + j\,y(s) + \hat{k}\,z(s) \quad \text{---}(1)$$



Integral along a Curve C.

$$I = \int_a^b \phi(x,y,z)\, ds$$

$$I = \int_a^b \phi(x(s), y(s), z(s))\, ds$$

$$= \int_{AB} \phi(s)\, ds = \int_c \phi_s\, ds \quad \text{---}(2)$$

The representation of the curve c by the arc length S in not always simple for integration. It might be more convenient to use a new Parameter (t) instead of S, the line element ds in then given by :-

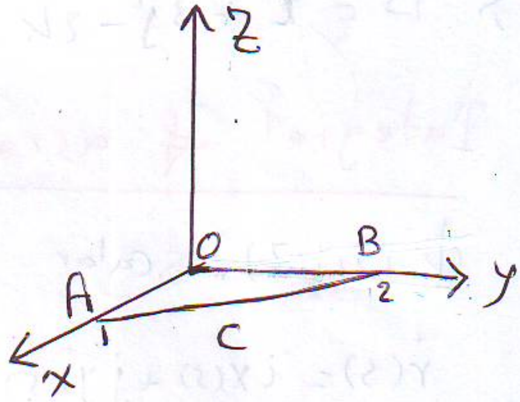
$$ds = \sqrt{dr \cdot dr} = \sqrt{\frac{dr}{dt} \cdot \frac{dr}{dt}}\, dt \quad \text{---}(3)$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \quad \text{---}(4)$$

Ex. : Evaluate the line Integral

$$I = \int_C \phi(x,y,z)\, ds \quad \text{——— (1)}, \text{ where } \phi = x^2 + y^2$$

C is the triangle OAB at point A(1,0,0), Point B(0,2,0)



Sol.

$$I = \int_{OA} \phi\, ds + \int_{AB} \phi\, ds + \int_{BO} \phi\, ds \quad \text{——— (2)}$$

① along OA , $y = z = 0$, so

$$\int_{OA} \phi\, ds = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3} \quad \text{——— (3)}$$

② along AB

$y = -2(x-1)$, $z = 0$

$\because$ slope $= \frac{\Delta y}{\Delta x}$

$= \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{2-0}{0-1} = -2$

$\therefore$ slope $= \frac{y-0}{x-1}$

$-2 = \frac{y}{x-1}$

$\therefore y = -2(x-1)$

Can be rewrite in new parametric from

$$\left.\begin{aligned} x &= 1-t \\ y &= 2t \\ z &= 0 \end{aligned}\right\} \quad \text{——— (4)}$$

A, B correspond to $t=0$ and $t=1$ respectively.

s are AB then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \quad \text{——— (5)}$$

$$\frac{dx}{dt} = -1 , \quad \frac{dy}{dt} = 2 , \quad \frac{dz}{dt} = 0$$

$$\therefore \frac{ds}{dt} = \sqrt{(-1)^2 + (2)^2 + (0)^2} = \sqrt{1+4} = \sqrt{5} \quad \text{—— (6)}$$

$$\int_{AB} \phi \, ds = \int_{t_1=0}^{t_2=1} \left[(1-t)^2 + 4t^2\right] \frac{ds}{dt} \cdot dt$$

$$= \sqrt{5} \int_0^1 (1 - 2t + t^2 + 4t^2)\, dt$$

$$= \sqrt{5} \int_0^1 (1 - 2t + 5t^2)\, dt$$

$$= \sqrt{5} \left[t - \frac{\cancel{2}t^2}{\cancel{2}} + \frac{5t^3}{3} \right]_0^1$$

$$= \sqrt{5}\left(1 - 1 + \frac{5}{3}\right) = \frac{5\sqrt{5}}{3} \quad \text{—— (7)}$$

(3) along BO

$$\int_{BO} \phi \, ds = \int_2^0 y^2 \, ds = \left.\frac{y^3}{3}\right|_2^0 = -\frac{8}{3}$$

$$\therefore I = \frac{1}{3} + \frac{5\sqrt{5}}{3} - \frac{8}{3} = \frac{5\sqrt{5} - 7}{3}$$

Line Integral of avector Function :-

Let $\bar{V}(x(s), y(s), z(s))$ defined at all points of asmooth curve C, and the $\vec{n}$ be the unit vector tangent to C.

$$\vec{n} = \frac{\left|\frac{dr}{dt}\right|}{\left|\frac{dr}{dt}\right|} \quad , \text{ use } (t) \text{ instead of } (s)$$

So the scalar line integral of $\bar{V}$ along C is:-

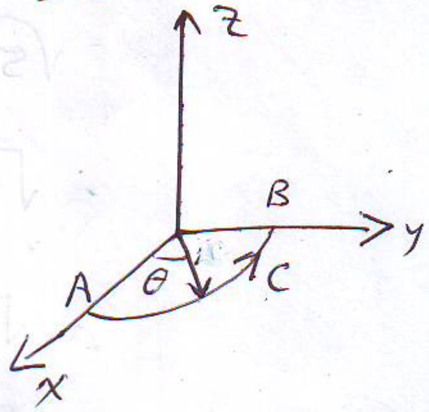
$$I = \int_C \bar{V} \cdot \vec{n} \, ds = \int_C \bar{V} \cdot dr \qquad dr = \vec{n}\, ds$$

Ex:-

Evaluate $\int_A^B \bar{V} \cdot dr$, where

$\vec{V} = 2x\,\hat{i} + (2x+2y+2z)\hat{k}$, and long the arc of Circle $x^2+y^2=1$, $z=0$, $A(1,0,0)$, $B(0,1,0)$, where $\vec{r}$ is vector point on path c given by:-

$\vec{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$



sol.

$$\int_A^B \vec{V} \cdot \vec{dr} \quad , \quad x = \cos\theta, \; y = \sin\theta$$

$$= \int_0^{\pi/2} (2\cos\theta\,\hat{i} + (2\cos\theta + 2\sin\theta)\hat{k}) \cdot (-i\sin\theta + j\cos\theta)\,d\theta$$

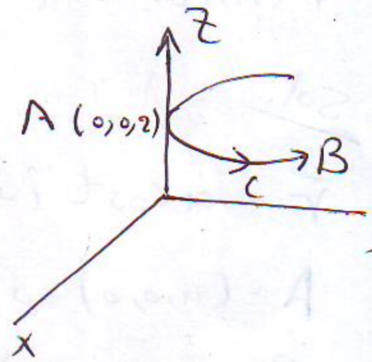
$$= \int_0^{\pi/2} -2\sin\theta\cos\theta\,d\theta = \int_0^{\pi/2} \sin 2\theta\,d\theta$$

$$= -\frac{1}{2}\cos 2\theta \Big|_0^{\pi/2}$$

$$= -\frac{1}{2}\left[\cos\frac{2\pi}{2} - \cos 0\right]$$

$$= -\frac{1}{2}[-1-1] = -1$$

Ex :- Calculate the work done by the force $F = (x, -2, 2y)$ is displacing a particle along the parabola $y = 2x^2$; $z = 2$ From the point $(0,0,2)$ to $(1,2,2)$



solve

$\therefore F = x\hat{i} - 2\hat{j} + 2y\hat{k}$

by chose $x = t$ get $y = 2t^2$, $z = 2$

$\bar{r}$ = vector of position at any point in parabola given by :-

$\bar{r} = \hat{i}t + 2t^2\hat{j} + 2\hat{k}$ —— (1)

$dr = 1\hat{i} + 4t\hat{j}$

$$I = \int_A^B \vec{F} \cdot dr = \int_0^1 (t\hat{i} - 2\hat{j} + 4t^2\hat{k}) \cdot (\hat{i} + 4t\hat{j})\, dt$$

$$= \int_0^1 (t - 8t)\, dt = \frac{t^2}{2}(1-8)\Big|_0^1$$

$$= \frac{t^2}{2}(-7) = \boxed{\frac{-7}{2}}$$

Vector line Integral :-

$$I = \int_C \vec{V} \cdot \vec{ds}$$

$$I = \int_C \vec{V} \times \vec{ds}$$

ex:- show that $\vec{\nabla}\cdot\vec{V} \neq \vec{V}\cdot\vec{\nabla}$

$$\vec{V} = \hat{i}V_x + \hat{j}V_y + \hat{k}V_z \quad , \quad \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Sol. $\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$ — Scalar —(1)

$\hat{i}\cdot\hat{i} = 1$
$\hat{j}\cdot\hat{j} = 1$
$\hat{k}\cdot\hat{k} = 1$

$\vec{V}\cdot\vec{\nabla} = V_x\frac{\partial}{\partial x} + V_y\frac{\partial}{\partial y} + V_z\frac{\partial}{\partial z}$ Operator —(2)

$\therefore \vec{\nabla}\cdot\vec{V} \neq \vec{V}\cdot\vec{\nabla}$

ex:- $r = \hat{i}x + \hat{j}y + \hat{k}z$, $W = \hat{i}W_x + \hat{j}W_y + \hat{k}W_z$

show:-

1) $\nabla\phi_{(r)} = \frac{1}{r}\frac{d\phi}{dr}\vec{r}$ $\qquad \hat{r} = \frac{\vec{r}}{|r|}$, $r = |r| = \sqrt{x^2+y^2+z^2}$

Sol. $\nabla\phi_{(r)} = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= \hat{i}\frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial x} + \hat{j}\frac{\partial\phi}{\partial r}\cdot\frac{\partial r}{\partial y} + \hat{k}\frac{\partial\phi}{\partial r}\cdot\frac{d r}{d z}$ —(1)

$= \frac{d\phi}{dr}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$

$\because r = (x^2+y^2+z^2)^{\frac{1}{2}}$

$\therefore \frac{dr}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{1}{2}}\cdot\cancel{2}x = \frac{x}{r}$

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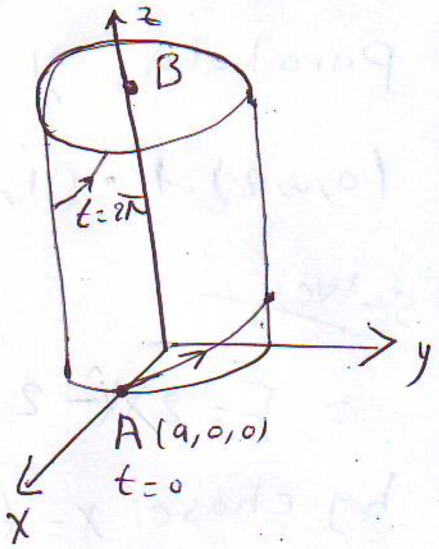
$\therefore \nabla\phi_{(r)} = \frac{d\phi}{dr}\cdot\frac{1}{r}\left[\hat{i}x + \hat{j}y + \hat{k}z\right]$

$= \frac{1}{r}\frac{d\phi}{dr}\vec{r}$

Ex:- Evaluate ① $\int \vec{r}\, ds$, ② $\int \vec{r} \times d\vec{r}$ from point $A(a,0,0)$ to point $B(a,0,2\pi b)$, on the circular helix



$r(x,y,z) = (a\cos t, a\sin t, bt)$

Sol.

$\vec{r} = a\cos t\, \hat{i} + a\sin t\, \hat{j} + bt\, \hat{k}$

$A = (a,0,0) \Rightarrow t = 0$

$B = (a,0,2\pi) \Rightarrow t = 2\pi$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\ dt$$

$$= \sqrt{a^2+b^2}\ dt$$

$$\int_c \vec{r}\, ds = \int_0^{2\pi} (a\cos t\, \hat{i} + a\sin t\, \hat{j} + bt\, \hat{k}) \sqrt{a^2+b^2}\ dt$$

$$= \sqrt{a^2+b^2}\left[\hat{i}\, a\sin t - \hat{j}\, a\cos t + b\frac{t^2}{2}\hat{k}\right]_0^{2\pi}$$

$$= \sqrt{a^2+b^2}\,[2\pi^2 b]\,\hat{k}$$ <u>vector</u>

$$\int \vec{r} \times d\vec{r} = \int_0^{2\pi} (\hat{i}\, a\cos t + \hat{j}\, a\sin t + k\, bt) \times (-\hat{i}\, a\sin t + \hat{j}\, a\cos t + bk)\, dt$$

$$\because \begin{vmatrix} i & j & \hat{k} \\ a\cos t & a\sin t & bt \\ -a\sin t & a\cos t & b \end{vmatrix}$$

$$\therefore \int \vec{r} \times d\vec{r} = \int_0^{2\pi} \hat{i}(a\sin t \cdot b - bt \cdot a\cos t) + \hat{j}(-bt \cdot a\sin t - b\, a\cos t) + \hat{k}((a\cos t \cdot a\cos t) - (a\sin t \cdot -a\sin t))\, dt$$

$$= \int_0^{2\pi} \hat{i}\,(ab(\sin t \underset{a}{-} bt\cos t)) - \hat{j}\,(ab(\cos t + t\sin t)) + \hat{k}a^2)\,dt$$

$\int u\,dv$

$$= \hat{i}ab\,[-t\sin t - 2\cos t) - \hat{j}ab(-t\cos t + 2\sin t) + \hat{k}a^2 t\Big]_0^{2\pi}$$

$$= 2\pi ab\,\hat{j} + 2\pi a^2\hat{k}$$

Summery خلاصة

① Integral of ascalar function

$$I = \int_C \phi\, ds \quad , \text{where} \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

ϕ is scalar function

② Integral of avector function

$$I = \int \vec{V}\cdot\vec{n}\, ds = \int \vec{V}\cdot\vec{dr} \qquad \vec{n}\,ds = \vec{dr}$$

$\vec{V}$ = vector function $\quad \hat{i},\hat{j},\hat{k}$ تعطى بدلالة

$\vec{r} = \hat{i}x + \hat{j}y$

$\vec{dr} = \hat{i}dx + \hat{j}dy \quad$ او يعطي r حسب السؤال

③ Vector line Integrals

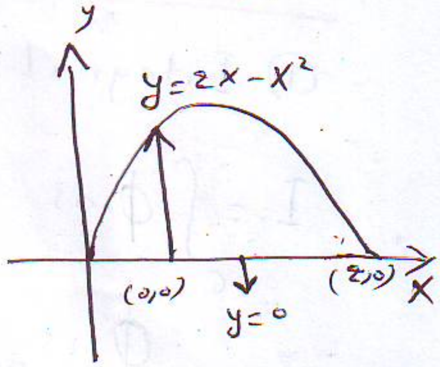
$$I = \int \vec{V}\, ds \quad \underline{\underline{or}} \quad \int \vec{r}\, ds \; , \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$$

$$I = \int \vec{V}\times\vec{dr} \quad \text{or} \quad \int \vec{r}\times dr$$

Repeat Integrals: التكاملات المتكررة

$$\int_a^b I(x)\,dx = \int_a^b \int_{u(x)}^{v(x)} f(x,y)\,dy\,dx$$

Ex:- A Thin plate bounded by prabola $y = 2x - x^2$ and $y = 0$ find its mass if the density $\rho(x,y) = \dfrac{1}{1+x}$



Sol. The parabola intersects $y=0$ at $x=0$, $x=2$

The mass M of the plate of unit thickness is:-

$$M = \int_{x=0}^{x=2} \int_{y=0}^{y=2x-x^2} \rho(x,y)\,dy\,dx$$

$$= \int_0^2 \int_0^{2x-x^2} \frac{1}{1+x}\,dy\,dx \Rightarrow \int_0^2 \frac{[y]_0^{2x-x^2}}{1+x}\,dx$$

$$= \int_0^2 \frac{(2x-x^2)}{1+x}\,dx = \int_0^2 \frac{x(2-x)}{1+x}\,dx$$

تم تحويلها عن طريق القسمة الطويلة

$$\begin{array}{r} -x \\ 1+x \,\big)\, 2x - x^2 \\ \pm x \pm x^2 \\ \hline 3x \end{array}$$

$$= \int_0^2 \left(-x + \frac{3x}{1+x}\right) dx$$

$$= \left[\frac{-x^2}{2} + 3x - 3\ln(1+x)\right]_0^2 = 4 - 3\ln 3$$

H.W Find the area bounded by the curve $y = x^2$, $y = 0$, $x = 0$ and $x = 1$