Chapter 2

Special function

1- Gamma Function :- It is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad , \text{where } x > 0 \quad \text{——(1)}$$

where $x > 0$, for convergence of the integral

from eq.(1) we obtain

$$\Gamma(x+1) = \int_0^{\infty} t^{x} e^{-t} dt \quad \text{———(2)}$$

integration ② by part

$$I = \int_a^b v\,du = vu\Big]_a^b - \int_a^b u\,dv \quad \text{——(3)}$$

$$\left.\begin{aligned} v &= t^{x} & dv &= x t^{x-1} dt \\ du &= e^{-t} dt & u &= -e^{-t} \end{aligned}\right\} \quad \text{——(4)}$$

put (4) into (3) we obtain

$$I = -t^{x} e^{-t}\Big]_0^{\infty} + x\int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{—— (5)}$$

from (2) and (5)

$$\Gamma(x+1) = I = x\,\Gamma(x)$$

$$\therefore \boxed{\Gamma(x+1) = x\,\Gamma(x)} \quad \text{——— (6)}$$

مهم

eq.(6) is called the recursion Formula for Gamma function.

If n is apositive integral and $n>1$

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1)$$
$$= n(n-1)(n-2)\,\Gamma(n-2)$$
$$= n(n-1)(n-2) \text{——} \Gamma(1)$$
$$= n!\,\Gamma(1) \text{——} (7)$$

show that $\Gamma(1)=1$

using eq.(1) we obtain

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\,dt = \int_0^\infty e^{-t}\,dt$$

$$\left\{ e^{-0}=1,\ e^{-\infty}=0 \right.$$

$$= -e^{-t}\Big]_0^\infty = -(0-1) = 1$$

equation (7) gives an important relation i.e:-

$$\Gamma(n+1) = n!$$

* The value of $\Gamma(\frac{1}{2})$ is often requried and may be obtain directily for definition ①

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\,dt$$

put $t=u^2 \implies dt = 2u\,du$

$$\therefore \Gamma(\tfrac{1}{2}) = \int_0^\infty \cancel{u^{-1}}\, e^{-u^2}\, 2\cancel{u}\,du$$

$$= 2\int_0^\infty e^{-u^2}\,du = \sqrt{\pi}$$

$$\therefore \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$ prove it

H.W

Ex:- Find $\Gamma(\frac{3}{2})$, $\Gamma(\frac{7}{2})$

Sol.

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\tfrac{7}{2}) = \frac{5}{2}\cdot\frac{3}{2}\frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{15}{8}\sqrt{\pi}$$

From eq.(6), we have

$$\Gamma(x) = \frac{\Gamma(x+1)}{x} \quad \text{--- (8)}$$

since eq.(8) is used for negative value of x for example **Find $\Gamma(-\frac{3}{2})$**

Sol

$$\Gamma(-\tfrac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{\Gamma(\frac{1}{2})}{(-\frac{3}{2})(-\frac{1}{2})} = \frac{4\sqrt{\pi}}{3}$$

نستمر نستخدم الصيغة الى ان نحصل على قيمة موجبة .

Ex:- evalute the integral $I = \int_0^\infty x^{5.2} e^{-x^2} dx$

نقارن التكامل مع المعادلة الاصلية رقم (1)

$$\because \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \text{— (1)}$$

Let $x^2 = t$, $2x\,dx = dt \Longrightarrow dx = \frac{dt}{2x}$

$$\therefore I = \int_0^\infty x^{5.2} e^{-t} \frac{dt}{2x}$$

$$= \frac{1}{2}\int_0^\infty x^{4.2} e^{-t} dt$$

$$= \frac{1}{2}\int_0^\infty t^{2.1} e^{-t} dt \quad \text{— (2)}$$

$\because x^2 = t$

$x^{4.2} = t^{2.1}$

نقارن معادلة (2) بالمعادلة (1)

$$x-1 = 2.1 \Rightarrow x = 3.1$$

$$\therefore I = \frac{1}{2}\int_0^1 t^{3.1-1} e^{-t}\, dt$$

$$= \frac{1}{2}\Gamma(3.1) = \frac{1}{2} \times 2.1\, \Gamma(2.1)$$

$$= \frac{1}{2} * 2.1 \times 1.1\, \Gamma(1.1)$$

$$= \frac{1}{2} * 2.1 * 1.1 * 0.1\, \Gamma(0.1)$$

H.w show that $\int_0^\infty \frac{e^{-mt}}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{m}}$

H.w solve $I = \int_0^\infty x^6 e^{-3x}\, dx$

2) Beta function:- It is defined as:-

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad \text{--- (1)}$$

where $m>0$, $n>1$ for convergence of the integral.

the Beta function $B(m,n)$ is symmetric in m & n

Since by putting $u=(1-x)$

Let $u = 1-x \Rightarrow x = 1-u$

$du = -dx$

$$B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$$

$$= -\int_1^0 (1-u)^{n-1} u^{m-1}\, dx$$

$$= \int_0^1 u^{m-1}(1-u)^{n-1}\, du = B(m,n)$$

$B(m,n) = B(n,m)$ هذا برهان

alternative for the B function is obtaind by putting $x = \sin^2\theta$

if $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$

Then eq 1 becomes

$$B(m,n) = \int_0^{\pi/2} \sin\theta^{2m-2} \cos\theta^{2n-2}\, 2\sin\theta\cos\theta\, d\theta$$

حدود التكامل
$x=0 \Rightarrow \theta = 0$
$x=1 \Rightarrow \theta = \frac{\pi}{2}$

$$B(m,n) = 2\int_0^{\pi/2} \sin\theta^{2m-1} \cos\theta^{2n-1}\, d\theta \quad \text{--- (2)}$$

* the relation between Beta function and Γ function

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad \text{--- (3)}$$

Ex:- Evaluate the integral $I = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Let $x^4 = u \Rightarrow x = u^{1/4}$

$4x^3\,dx = du \Rightarrow dx = \frac{du}{4x^3}$

$$\therefore I = \int_0^1 \frac{du}{4u^{3/4}(1-u)^{\frac{1}{2}}} = \frac{1}{4}\int_0^1 u^{-3/4}(1-u)^{-\frac{1}{2}}\, du$$

$m-1 = -3/4 \Rightarrow m = \frac{1}{4}$

$n-1 = -\frac{1}{2} \Rightarrow n = \frac{1}{2}$

from eq. 3

$$I = \frac{1}{4}B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{4}\,\frac{\Gamma(1/4)\,\Gamma(\frac{1}{2})}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{4}\,\frac{\Gamma(1/4)}{\Gamma(3/4)}$$

Ex:- If $I = \int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin(p\pi)}$

prove that $\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$ where $0 < p < 1$

هذا قانون مهم جداً سيستخدم في حل المسائل

Let $y = \frac{x}{1+x} \Rightarrow x = y(1+x)$

$x = y + yx \Rightarrow y = x - yx$

$y = x(1-y)$

$\therefore x = \frac{y}{1-y}$

$x = y(1-y)^{-1}$

$dx = y\cdot\left[-(1-y)^{-2}(-dy)\right] + dy\,(1-y)^{-1}$

$\therefore dx = \frac{y}{(1-y)^2}\cdot dy + \frac{1}{(1-y)} dy$

$dx = \left[\frac{y}{(1-y)^2} + \frac{1}{(1-y)}\right] dy$

The limits of integral will changes as

$x = \frac{y}{1-y}$

when $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = 1 \Rightarrow \int_0^{\infty} \Rightarrow \int_0^1$

$$\therefore I = \int_0^1 \frac{\left(\frac{y}{1-y}\right)^{p-1}}{1 + \frac{y}{1-y}} \left[\frac{y}{(1-y)^2} + \frac{1}{1-y}\right] dy$$

$$I = \int_0^1 \frac{\left(\frac{y}{1-y}\right)^{p-1}}{\frac{\cancel{1-y+y}}{\cancel{1-y}}} \left[\cancel{\frac{1}{1-y}}\left(\frac{y}{1-y} + 1\right)\right] dy$$

$$\therefore I = \int_0^1 \frac{y^{p-1}}{(1-y)^{p-1}} \frac{1}{(1-y)} dy$$

$$\therefore I = \int_0^1 \frac{y^{p-1}}{(1-y)^p}\, dy = \int_0^1 y^{p-1}(1-y)^{-p}\, dy$$

$$m-\cancel{1} = p-\cancel{1} \qquad , \quad n-1=-p$$
$$\therefore m = p \qquad \therefore n = 1-p$$

$$\therefore I = B(p, 1-p)$$

$$I = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(p+1-p)} = \Gamma(p)\Gamma(1-p)$$

$$\therefore \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}$$

H.W

1 Evaluate $I = \int_0^4 u^{3/2}(4-u)^{5/2}\, du$

2 Evaluate $I = \int_0^3 \frac{dx}{\sqrt{3x-x^2}}$