3) Bessel's Function

The Bessel's Differential eq. (Bessel's eq.) is:-

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad \text{---(1)}$$

when n is real constant. The solutions of this equations are called the Bessel's functions of order n. In general the solutions of eq.(1) is written as:-

$$y(x)=A\,J_n(x)+B\,J_{-n}(x) \quad \text{---(2)}$$

as we can see from eq.(2) there are two solution to eq.(1) which defines the Bessel's function of the first kind and order n. where the first solution is:-

$$\left.\begin{aligned} J_n(x) &= \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)} \\ \text{or} \quad &= \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!} \end{aligned}\right\} \quad \text{---(3)}$$

with n is apositive integral.

The seconed solution is

$$\left.\begin{aligned} J_{-n}(x) &= \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)} \\ \text{or} \quad &= \sum_{r=0}^{\infty}(-1)^r\frac{\left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!} \end{aligned}\right\} \quad \text{---(4)}$$

where n is anegative integral.

example :- show that

$$J_{-n}(x) = (-1)^n J_n(x)$$

we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

or

$$= \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!}$$

It is very clear that the first term of this series is zero, since the Γ function or factorial function of negative integral is infinity ∞. It follows that the series of this equation effectively begains with term when $r=n$, therefore the above equation can be written as

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{-n+2r}}{r!\,(-n+r)!}$$

Let the varible of summation be changed from (r) to (p) by putting

$$r = n+p \Rightarrow p = r-n \Rightarrow p = 0$$

$$\therefore J_{-n}(x) = \sum_{p=0}^{\infty} (-1)^{n+p} \frac{\left(\frac{x}{2}\right)^{-n+2n+2p}}{(n+p)!\,(-n+n+p)!}$$

$$= (-1)^n \sum_{p=0}^{\infty} (-1)^p \frac{\left(\frac{x}{2}\right)^{n+2p}}{p!\,(n+p)!}$$

$$= (-1)^n J_n(x)$$ ← هذا قانون يحفظ

Generating function:-

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$ ← هذا قانون حفظ

The proof:-

taking the left (L.H.S) of above eq.

$$e^{\frac{x}{2}(t-\frac{1}{t})} = e^{\frac{x}{2}t} \cdot e^{-\frac{x}{2t}} \quad \text{——(1)}$$

$$\because \quad e^{p} = \sum_{r=0}^{\infty} \frac{p^r}{r!} \quad \text{——(2)}$$ هذا قانون حفظ خاص e

using eq.(2) in (1) we obtain

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{r=0}^{\infty} \frac{(\frac{x}{2}t)^r}{r!} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2t})^k}{k!}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^r t^r}{r!} \cdot \frac{(-\frac{x}{2})^k\, t^{-k}}{k!}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^r t^r}{r!} \cdot \frac{(-1)^k (\frac{x}{2})^k\, t^{-k}}{k!}$$

$$= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{(\frac{x}{2})^{r+k}}{r!\,k!}\, t^{r-k} \quad \text{——(3)}$$

let $n = r-k \Rightarrow \therefore r = n+k$

so that, n varies from $(-\infty)$ to $(+\infty)$ and $r = n+k$

Now eq.(3) becomes

Ex: show that $J_0'(x) = -J_1(x)$

Sol.

$$\because J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

$$J_0(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{2r}}{r!\,(r)!}$$

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$$\therefore J_0'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{2\cancel{r}\left(\frac{x}{2}\right)^{2r-1} \cdot \frac{1}{2}}{r!\,\cancel{r}(r-1)!} \quad \text{——— (1)}$$

$$\because J_{-n}(x) = (-1)^n J_n(x)$$

$$\therefore J_n(x) = \frac{J_{-n}(x)}{(-1)^n}$$

$$\therefore J_1(x) = \frac{J_{-1}(x)}{(-1)^1} = -\sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{2r-1}}{r!\,(r-1)!} \quad \text{——— (2)}$$

From (1) and (2),

$$J_0'(x) = -J_1(x)$$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,k!} t^{n+k-k}$$

$$= \sum_{n=-\infty}^{\infty} \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{x}{2}\right)^{n+2k}}{k!\,(n+k)!}} t^n$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \quad \text{proved.}$$

Ex proved that:- ① $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ —(a)

(H.W) هذا → ② $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ —(b)

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The proof:-

$$\because J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$$

$$\therefore J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}+2r}}{r!\,\Gamma\left(\frac{3}{2}+r\right)}$$

$$= \left[\frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}} - \frac{\left(\frac{x}{2}\right)^{\frac{5}{2}}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} + \frac{\left(\frac{x}{2}\right)^{\frac{9}{2}}}{2!\,\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}} + \cdots\right]$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}} \left[1 - \frac{\left(\frac{x}{2}\right)^2}{\frac{3}{2}} + \frac{\left(\frac{x}{2}\right)^4}{2\,\frac{5}{2}\,\frac{3}{2}} + \cdots\right]$$

$$= \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right]$$

$$\because \sin x = \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$\sin x = x\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \text{———}\right]$$

$$\frac{\sin x}{x} = \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \text{———}\right]$$

$$\therefore J_{\frac{1}{2}}(x) = \frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{\frac{1}{2}\sqrt{\pi}} * \frac{\sin x}{x}$$

$$\therefore J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x \quad \text{proved.}$$

Recurrence Formulae

Theorem (1):-

$$\frac{d}{dx}\left\{x^n J_n(x)\right\} = x^n J_{n-1}(x)$$

The proof:- $\because$ from eq. (1) $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,\Gamma(n+r+1)}$

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{x^n \cdot x^{n+2r}}{r!\,\Gamma(n+r+1)\,2^{n+2r}}$$

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$$= \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2r}}{r!\,\Gamma(n+r+1)\,2^{n+2r}}$$

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)\,x^{n-1+2r}}{r!\,(n+r)\,\Gamma(n+r)\,2^{(n-1)+2r}}$$

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$$= x^n \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{(n-1)+2r}}{r!\,\Gamma((n-1)+r+1)}$$

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$$\therefore \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{(n-1)}(x)$$

proved.

Theorem(2):- $\frac{d}{dx}\left[X^{-n} J_n(x)\right] = -X^{-n} J_{n+1}(x)$ (7)

$$X^{-n} J_n(X) = \sum_{r=0}^{\infty} (-1)^r \frac{X^{-n} \cdot X^{n+2r}}{r!\, \Gamma(n+r+1)\, 2^{n+2r}}$$

$$= \sum_{r=0}^{\infty} (-1)^r \frac{X^{2r}}{r!\, \Gamma(n+r+1)\, 2^{n+2r}}$$

$$\frac{d}{dx}\left[X^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r \frac{\cancel{r}\, X^{2r-1}}{\cancel{r}\,(r-1)!\, \Gamma(n+r+1)\, 2^{n+2r-1}}$$

let $r-1=p \Rightarrow r=p+1$

(نضرب ونقسم X^n)

$$\therefore \frac{d}{dx}\left[X^{-n} J_n(x)\right] = \sum_{p=0}^{\infty} (-1)^{p+1} \frac{X^{-n}\, X^{n+2p+1}}{p!\, \Gamma(n+p+1+1)\, 2^{n+2p+2-1}}$$

$$= (-1)^1 X^{-n} \sum_{p=0}^{\infty} (-1)^p \frac{X^{(n+1)+2p}}{p!\, \Gamma((n+1)+p+1)\, 2^{(n+1)+2p}}$$

$$= -X^{-n} \sum_{p=0}^{\infty} (-1)^p \frac{\left(\frac{X}{2}\right)^{(n+1)+2p}}{p!\, \Gamma((n+1)+p+1)}$$

$$= -X^{-n} J_{n+1}(X)$$

Differentiating theorems 1 and 2 give

Frome theorem 1 $\Rightarrow$

$$X^n J_n'(x) + n X^{n-1} J_n(x) = X^n J_{n-1}(x) \quad \text{--- (1)}$$

Frome theorem 2 $\Rightarrow$

$$X^{-n} J_n'(x) - n X^{-n-1} J_n(x) = -X^{-n} J_{n+1}(X) \quad \text{--- (2)}$$

Multiplying eq.(1) by X^{-n} and eq.(2) X^{n} we obtain

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad ——(3)$$

$$J_n'(x) - \frac{n}{x} J_n(x) = -J_{n+1}(x)$$

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$$J_n'(x) + J_{n+1}(x) = \frac{n}{x} J_n(x) \quad ——(4)$$

From ③ $\Rightarrow J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x) \quad ——(5)$

From ④ $\Rightarrow J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \quad ——(6)$

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Adding eq.s (5) and (6) we obtain

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\therefore J_n'(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) \quad ——(7)$$

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which is the third formula for $J_n'(x)$

Subtracting eq.(5) and (6) we have.

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x)$$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Ex :- Using the Generating function to prove

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos(2\theta) + 2J_4(x)\cos 4\theta + \cdots$$

or

$$\cos(x\sin\theta) = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta)$$

solve :-

$$\because\ e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n \quad , \text{ let } \quad t = e^{i\theta}$$

$$\therefore\ e^{\frac{x}{2}(e^{i\theta}-e^{-i\theta})} = \sum_{-\infty}^{\infty} J_n(x)\,e^{in\theta}$$

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$$e^{\frac{i}{i}x\frac{1}{2}(e^{i\theta}-e^{-i\theta})} = \sum_{-\infty}^{\infty} J_n(x)\,e^{in\theta}$$

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\,e^{in\theta} \qquad \because \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta})$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(x)\left[\cos(n\theta) + i\sin(n\theta)\right]$$

Expanding the R.H.S we obtain:-

$$\begin{aligned}
\text{R.H.S} = J_0(x)\,[\cos 0 + i\sin 0] &+ J_1(x)\,[\cos\theta + i\sin\theta] \\
&+ J_{-1}(x)\,[\cos(-\theta) + i\sin(-\theta)] \\
&+ J_2(x)\,[\cos\theta(2\theta) + i\sin(2\theta)] \\
&+ J_{-2}(x)\,[\cos(-2\theta + i\sin(-\theta)] \\
&+ \cdots
\end{aligned}$$

Note:- $J_{-n}(x) = (-1)^n J_n(x)$

$$\begin{aligned}
\therefore \text{R.H.S} &= J_0(x) + J_1(x)[\cos\theta + i\sin\theta] + (-1)^1 J_1(x)[\cos\theta - i\sin\theta] \\
&\quad + J_2(x)[\cos(2\theta) + i\sin(2\theta)] + (-1)^2 J_2(x)[\cos 2\theta + i\sin 2\theta \\
&\quad + \cdots \\
&= J_0(x) + 2iJ_1(x)\sin\theta + 2J_2(x)\cos(2\theta) + 2iJ_3(x)\sin(3\theta) \\
&\quad + 2J_4(x)\cos(4\theta) + \cdots
\end{aligned}$$

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$$\therefore \cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos(2\theta) + 2J_4(x)\cos(4\theta) + \cdots$$

—— — proved

Note:- similarly, equating imaginary part of eq. (a) we get

$$\sin(x\sin\theta) = 2\sum_{n=1}^{\infty} J_{2n-1}(x)\sin(2n-1)\theta$$