Numerical Analysis

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Chapter Two

Solutions of Nonlinear Equations

In the next series of lectures, we shall discuss the problem of identifying the roots of equations in one Variable.

The basic formulation of the problem in the simplest case is this given a function $f(x)$ of one variable, find all x such that $f(x) = 0$.

وفي سلسلة المحاضرات القادمة سنناقش مشكلة تحديد جذور المعادلات في متغير واحد.

الصيغة الأساسية للمسالة في أبسط الحالات هي إعطاء دالة $f(x)$ لمتغير واحد، أوجد كل x بحيث يكون $f(x) = 0$.

We will review two methods for determining first approximations of possible positions for the equation $f(x) = 0$, where f is a continuous real function.

سنستعرض أسلوبين لتعيين تقريبات أولية لمواضع الجذور الحقيقية للمعادلة $f(x) = 0$، حيث f دالة حقيقية مستمرة.

1- (Method Graphic) تعين مواقع الجذور بالرسم البياني

If we draw the graph of the function $f(x) = y$, the points of intersection of the function curve with the x axis represent the roots of the equation. If the equation is cut, then the function graph can be considered the axis at the points $x_1,\ x_2, x_3, \dots\ x_n$. Each of these values represents a root. These values are first approximations to the roots.

إذا رسمنا مخطط الدالة $f(x) = y$ فإن نقاط تقاطع منحني الدالة مع محور x تمثل جذور المعادلة، فإذا قطع للمعادلة وبالتالي يمكن اعتبار مخطط الدالة المحور في النقاط $x_1,\ x_2, x_3, \dots\ x_n$. فإن كلاً من هذه القيم تمثل جذراً هذه القيم كتقريبات أولية للجذور .

Sometimes it is convenient to write the equation in the form $f_1(x) = f_2(x)$, where f_1, f_2 are functions that are easy to draw. If the two curves intersect at the point (x^*, y^*), then x^* is considered the root of the equation.

في بعض الأحيان يكون من الملائم كتابة المعادلة بالصيغة $f_1(x) = f_2(x)$ حيث f_1, f_2 دالتين يسهل رسمهما فإذا تقاطع المنحنيان في النقطة (x^*, y^*) فإن (x^*, y^*) تعتبر جذرا للمعادلة .

Example 1:

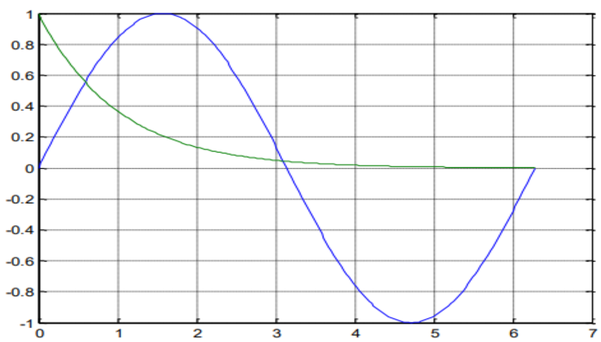
Determine the locations of the roots of the equation $e^x \sin(x) - 1 = 0$.

مثال: عين مواقع جذور المعادلة $e^x \sin(x) - 1 = 0$

Solution:

يمكن كتابة المعادلة السابقة بالصيغة المكافئة $\sin(x) = e^{-x}$

يمكن رسم الدالتين بنظام الماتلاب لتحديد مواقع الجذور



من الشكل الناتج سوف نلاحظ وجود تقاطع بين الدالتين الأول عند $x = 0.6$ والثاني عند $x = 3.1$ ويوجد عدد لا نهائي من التقاطعات الأخرى عندما تكون قريبة من $n\pi$.

2- Locating the position of roots (programming method):

تعين مواقع الجذور بطريقة البرمجة

To locate the position of roots of the function (equation) $f(x) = 0$ by using programming method, we use $f(x)$ be continuous function on the interval [a,b]. We divide the interval $[a, b]$ into n subintervals

لتحديد مواقع الجذور للدالة (المعادلة) $f(x) = 0$ باستخدام طريقة البرمجة، نستخدم $f(x)$ دالة مستمرة على الفترة $[a, b]$, نقسم الفترة [a,b] إلى n فترات الجزئية,

$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ where $x_i = a + ih, i = 0, 1, \ldots, n; h = \frac{b-a}{n}$.

If $f(x_i) \times f(x_{i+1}) < 0$ for any $0 \le i \le n$, then there exists c, $a < c < b$ for which $f(c) = 0$.

Example 2:

Find the approximate location of the function

$f(x) = x^4 - 7x^3 + 3x^2 + 26x - 10 = 0$ on the interval $[-8,8]$ with $n = 4$ and $n = 8$.

Solution:

If $n = 4$, $h = \frac{b-a}{n} = \frac{8-(-8)}{4} = \frac{16}{4} = 4.$

x	-8	-4	0	4	8
$f(x)$	+	+	-	-	+

There is a root between (-4,0) and (4,8).

If n=8, h=$\frac{b-a}{n} = \frac{8-(-8)}{8} = \frac{16}{8} = 2.$

x	-8	-6	-4	-2	0	2	4	6	8
$f(x)$	+	+	+	+	-	+	-	+	+

There is a root between (-2,0), (0,2), (2,4) and (4,6).

Example 3:

Find the approximate location of the function

$f(x) = x^3 + 4x^2 - 10 = 0$ on the interval [1,2] with $n = 5$.

Solution:

$n = 5, h = \frac{b-a}{n} = \frac{2-(1)}{5} = \frac{1}{5} = \frac{2}{10} = 0.2.$

x	1	1.2	1.4	1.6	1.8	2
$f(x)$	-	-	+	+	+	+

There is a root between (1.2,1.4).

Example 4:

Find the approximate location of the function

$f(x) = x^2 - 1 = 0$ on the interval $[-1,1]$ with $n = 10$.

Solution:

$n = 10, h = \frac{b-a}{n} = \frac{1-(-1)}{10} = \frac{2}{10} = 0.2.$

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$f(x)$	+	-	-	-	-	-	-	-	-	-	+

There is a root between (-1, -0.8) and (0.8,1).

Numerical methods: ***الطرق العددية***

Numerical Methods: are methods for solving problems numerically on a computer or calculator or by hand. الطرق العددية: هي طرق حل المسائل عددياً على الحاسوب أو الآلة الحاسبة أو باليد.

(i) Bisection Method: ***طريقة تنصيف الفترات***

Suppose a continuous function f defined on the interval [a,b] is given with $f(a)$ and $f(b)$ of different signal (i.e. $f(a)\times f(b) < 0$). Then by Intermediate value theorem ((If $f \in C[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ for which $f(c) = k$)) there exists a point $c \in (a,b)$ such that $f(c) = 0$. If we choose the midpoint $c = \frac{b+a}{2}$, then three possibilities arise:

لنفترض أن دالة مستمرة f معرفة على الفترة المغلقة [a,b] تعطي $f(a)$ و$f(b)$ بإشارات مختلفة أي $(f(a) \times f(b) < 0)$. فانه بواسطة نظرية القيمة المتوسطة .نحصل على إذا كان $f \in C[a,b]$ و k هو أي عدد يقع بين f (a) وf (b)، فهناك $c \in [a,b]$ بحيث $(f(c) = k)$ بحيث يكون $f(c) = 0$. إذا اخترنا نقطة المنتصف $c = (b+a)/2$، فستظهر ثلاثة احتمالات:

$$\text{If } f(a)\times f(c) \begin{cases} < 0 & \text{there is a root between a, c} \Rightarrow d=\frac{a+c}{2} \\ > 0 & \text{there is a root between b, c} \Rightarrow d=\frac{b+c}{2} \\ = 0 & \text{c is exact root ((Stop)).} \end{cases}$$

The two conditions stop are التوقف بشرطين $|x_{i+1} - x_i| \leq \in \ \vee \ |f(x_{i+1})| \leq \in$ for any i

Example 5:

Find an approximate root of $f(x) = x^2 - \frac{1}{2}$ in the interval (0,1) by using Bisection method if error $\in = 0.01$.

Solution:

$f(0) = 0^2 - \frac{1}{2} = -0.5$, $f(1) = 1^2 - \frac{1}{2} = 0.5$.

x	0	1	0.5	0.75	0.625	0.6875	0.7188	0.7032
$y = f(x)$	-0.5	0.5	-0.25	1.75	-0.1094	-0.0273	0.0166	-0.0056

$x_1 = \frac{a+b}{2} = \frac{0+1}{2} = 0.5, \quad f(0.5) = (0.5)^2 - \frac{1}{2} = -0.25.$

$|x_1 - b| = |0.5 - 1| = 0.5 > \in \ \vee \ |f(x_1)| = |-0.25| = 0.25 > \in$

$$x_2 = \frac{x_1 + b}{2} = \frac{0.5 + 1}{2} = 0.75 \ , \ f(0.75) = (0.75)^2 - \frac{1}{2} = 1.75$$

$|x_2 - x_1| = |0.5 - 0.75| = 0.625 > \in \vee |f(0.75)| = 1.75 > \in$

$$x_3 = \frac{x_2 + x_1}{2} = \frac{0.75 + 0.5}{2} = 0.625 \ , f(0.625) = (0.625)^2 - \frac{1}{2} = -0.1094$$

$|x_3 - x_1| = |0.0.625 - 0.75| = 0.7188 > \in \vee \ |f(0.625)| = |-0.1094| = 0.1094 > \in$

$$x_4 = \frac{x_3 + x_2}{2} = \frac{0.75 + 0.625}{2} = 0.6875 \ , f(0.6875) = (0.6875)^2 - \frac{1}{2} = -0.0273$$

$|x_4 - x_3| = |0.0.6875 - 0.75| = 0.7188 > \in \vee \ |f(0.6875)| = |-0.0273| = 0.0273 > \in$

$$x_5 = \frac{x_4 + x_3}{2} = \frac{0.6875 + 0.7188}{2} = 0.7023 \ , f(0.7023) = (0.7023)^2 - \frac{1}{2} = -0.0056$$

$|x_5 - x_4| = |0.7023 - 0.7188| = 0.0165 > \in \vee \ |f(0.7023)| = |-0.0056| = 0.0056 < \in$

Then $x_5 = 0.7023$ is the root.

Example 6:

Find an approximate root of $f(x) = xe^x - 2 = 0$ in the interval $[0.5, 0.9]$ by using Bisection method if error $\in = 0.007$.

Solution:

$f(0.5) = 0.5\, e^{(0.5)} - 2 = -1.1756 \ , f(0.9) = 0.9\, e^{(0.9)} - 2 = 0.2136$

x	0.5	0.9	0.7	0.8	0.85	0.875	0.8625	0.8562
$y = f(x)$	-1.1756	0.2136	–0.5904	–0.2196	–0.0113	0.0990	0.0433	0.0157

$x_1 = \frac{a+b}{2} = \frac{0.5+0.9}{2} = 0.7, \quad f(0.7) = 0.7e^{(0.7)} - 2 = -0.5904$

$|x_1 - b| = |0.7 - 0.9| = 0.2 > \in \ \vee |f(x_1)| = |f(-0.5904)| = |-0.5904|$
$= 0.5904 > \in$

$$x_2 = \frac{x_1 + b}{2} = \frac{0.7 + 0.9}{2} = 0.8 \ , \ f(0.8) = 0.8\, e^{(0.8)} - 2 = -0.2196$$

$|x_2 - x_1| = |0.8 - 0.7| = 0.1 > \in \vee |f(x_2)| = |-0.2196| = 0.2196 > \in$

$$x_3 = \frac{x_2 + b}{2} = \frac{0.8 + 0.9}{2} = 0.85 \ , f(0.85) = 0.85\, e^{(0.85)} - 2 = -0.0113$$

$|x_3 - x_2| = |0.85 - 0.8| = 0.05 > \in \vee \ |f(x_3)| = |-0.0113| = 0.0113 > \in$

$x_4 = \frac{x_3 + b}{2} = \frac{0.85 + 0.9}{2} = 0.875 \ , f(0.85) = 0.875\, e^{(0.875)} - 2 = 0.0990$

$|x_4 - x_3| = |0.875 - 0.85| = 0.025 > \in \vee \ |f(x_4)| = |0.0990| = 0.0990 > \in$

$x_5 = \frac{x_3 + x_4}{2} = \frac{0.85 + 0.875}{2} = 0.8625 \ , \ f(0.8625) = 0.8625 \ e^{(0.8625)} - 2 = 0.0433$

$|x_5 - x_4| = |0.8625 - 0.875| = 0.0125 > \in \vee \ \ |f(x_4)| = |0.0433| = 0.0433 > \in$

$x_6 = \frac{x_5 + x_3}{2} = \frac{0.8625 + 0.85}{2} = 0.8562 \ ,$

$f(0.8562) = (0.8562)75\, e^{(0.8562)} - 2 = 0.0157$

$|x_6 - x_5| = |0.8562 \ - \ 0.8625| = 0.0063 < \in$

Then $x_6 = 0.8562$ is the root.

Homework

1- Find an approximate root of $f(x) = x - 2^{-x} = 0$ in the interval (0,1) by using Bisection method with error $\in = 0.04$.

2- Find an approximate root of $f(x) = x^3 + 4x^2 - 10 = 0$ in the interval [1,2] by using Bisection method with error $\in = 0.04$.

(ii) The False-Position method : *طريقة الموضع الكاذب*

Suppose a continuous function f defined on the interval $[x_1, x_2]$ is given with $f(x_1)$ and $f(x_2)$ of opposite sign (i.e. $f(x_1). f(x_2) < 0$). To derive a formula for false-position method, approximate the graph of f by a straight line on $[x_1, x_2]$ connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ which intersect x-axis at $(x_3, 0)$ where x_3 is more approximate to the exact root than x_1 and x_2.To obtain a formula for x_3 we use the slop equality:

لنفترض أن دالة مستمرة ومعرفة على الفترة $[x_1, x_2]$ تعطي $f(x_1)$ و $f(x_2)$ بإشارات مختلفة $(f(x_1). f(x_2) < 0)$.لاشتقاق صيغة لطريقة الموضع الكاذب، قم بتقريب الرسم البياني لـ f بخط مستقيم $[x_1, x_2]$ وتوصيل $(x_2, f(x_2))$و $(x_1, f(x_1))$ التي يتقاطع مع المحور السيني عند $(x_3, 0)$حيث يكون x_3 أكثر تقريبًا للجذر الدقيق من x_1 و x_2. للحصول على صيغة x_3 نستخدم المتطابقة ادناه:

$$\frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \ \ \Rightarrow (x - x_2)(y_2 - y_1) = (y - y_2)(x_2 - x_1)$$

we put $y = 0$

$$(x - x_2)(y_2 - y_1) = -y_2(x_2 - x_1)$$

$$\Rightarrow \left((x-x_2)(y_2-y_1) = -y_2(x_2-x_1)\right) \div (y_2-y_1)$$

$$\Rightarrow (x-x_2) = -y_2\frac{(x_2-x_1)}{(y_2-y_1)} \quad \Rightarrow x_3 = x_2 - \frac{y_2(x_2-x_1)}{(y_2-y_1)}$$

To find another approximation: لإيجاد قيم أخرى

If $f(x_1) \times f(x_2)$ $\begin{cases} < 0 & \text{there is a root between } x_1, x_3 \\ > 0 & \text{there is a root between } x_2, x_3 \\ = 0 & \text{c is exact root ((Stop)).} \end{cases}$

$$\Rightarrow x_4 = x_3 - \frac{y_3(x_3-x_2)}{(y_3-y_2)}$$

$\vdots \qquad\qquad \vdots$

$\Rightarrow x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1}-x_i)}{(y_{i+1}-y_i)}$. And Stop condition is $|x_{i+2} - x_{i+1}| \le \in$ for any i.

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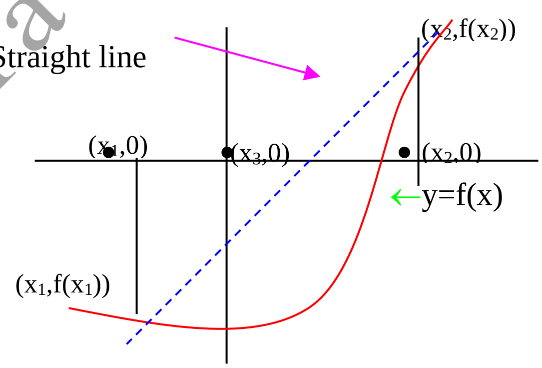


Example 7:

Find an approximate root of $f(x) = x.lnx - 1 = 0$ in interval $[1,2]$ by using False-Position method with error $\in = 0.06$.

Solution:

Let $x_1 = 1, x_2 = 2$

$f(1) = 1.ln1 - 1 = -1$ and $f(2) = 2.ln2 - 1 = 0.3863 \Rightarrow (f(x_1).f(x_2) < 0)$

x	1	2	1.7213	1.7615
$y = f(x)$	-1	0.3863	-0.0652	-0.0027

$$x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1}-x_i)}{(y_{i+1}-y_i)}$$

$$x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)} \quad \Rightarrow \quad x_3 = 2 - \frac{(0.3863)(2-1)}{\left(0.3863 - (-1)\right)} = 1.7213$$

$f(1.7213) = 1.7213.\ln(1.7213) - 1 = -0.0652$

$|x_3 - x_2| = |1.7213 - 2| = 0.2787 > \in = 0.06.$

$$x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)} \quad \Rightarrow x_4 = 1.7213 - \frac{(-0.0652)(1.7213-2)}{(-0.0652 - 0.3863)} = 1.7615$$

$f(1.7615) = 1.7615.\ln(1.7615) - 1 = -0.0027$

$|x_4 - x_3| = |1.7615 - 1.7213| = 0.0402 < \in$

Then $x_4 = 1.7615$ is the root.

Example 8:

Find an approximate root of $f(x) = x + e^{-2x} - 1 = 0$ in interval $[0.5,1]$ by using False-Position method with error $\in = 7 \times 10^{-3}$.

Solution:

Let $x_1 = 0.5, x_2 = 1$

$f(0.5) = 0.5 + e^{-2.(0.5)} - 1 = -0.1321$

$f(1) = 1 + e^{-2.(1)} - 1 = 0.1353 \qquad \Rightarrow (f(x_1).f(x_2) < 0)$

x	0.5	1	0.7470	0.7910	0.7961
$y = f(x)$	-0.1321	0.1353	-0.0285	-0.0034	-0.0004

$$x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1} - x_i)}{(y_{i+1} - y_i)}$$

$$x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)} \quad \Rightarrow \quad x_3 = 1 - \frac{(0.1353)(1-0.5)}{\left(0.1353 - (-0.1321)\right)} = 0.7470$$

$f(0.7470) = 0.7470 + e^{-2(0.7470)} - 1 = -0.0285$

$$|x_3 - x_2| = |0.7470 - 1| = 0.253 > \in = 0.007 x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)} \quad \Rightarrow x_4$$

$$= 0.7470 - \frac{(-0.0285)(0.7470-1)}{(-0.0285 - 0.1353)} = 0.7910$$

$f(0.7910) = 0.7910 + e^{-2(0.7910)} - 1 = -0.0034$

$$|x_4 - x_3| = |0.7910 - 0.7470| = 0.044 > \in$$

$$x_5 = x_4 - \frac{y_4(x_4 - x_3)}{(y_4 - y_3)} \quad \Rightarrow x_5 = 0.7910 - \frac{(-0.0034)(0.7910-1)}{(-0.0034-0.1353)} = 0.7961$$

$$f(0.7961) = 0.7961 + e^{-2(0.7961)} - 1 = -0.0004$$
$$|x_5 - x_4| = |0.7961 - 0.7910| = 0.005 < \in$$

Then $x_5 = 0.7961$ is the root.

Homework

1- Find an approximate root of $f(x) = x^2 - 2x - 1 = 0$ in interval $[-1,0]$ by using False-Position method with error $\in = 10^{-3}$.

2- Find an approximate root of $f(x) = e^x - 3x = 0$ in interval $[1,2]$ by using False-Position method with error $\in = 10^{-2}$.

(iii) Secant Method: طريقة القاطع

Suppose a continuous function f defined on the interval $[a,b]$ is given, the graph of the function f is approximated by a secant line, we get

لنفترض أن دالة مستمرة f معرفة على الفترة المغلقة $[a,b]$معطاة، ويتم تقريب الرسم البياني للدالة f بواسطة خط قاطع، نحصل على

$$\frac{f(b)-f(a)}{b-a} = \frac{f(b)-y}{b-x} \Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{f(b)-0}{b-x_1}$$

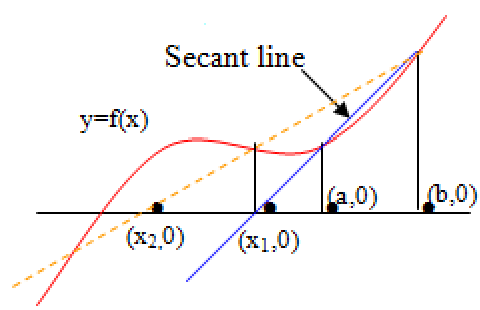
$$\Rightarrow x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

Similarly, $x_2 = \dfrac{bf(x_1)-x_1 f(b)}{f(x_1)-f(b)}$

In general, $x_{i+2} = \dfrac{x_i f(x_{i+1})-x_{i+1} f(x_i)}{f(x_{i+1})-f(x_i)}$;

i=0, 1, … x_{i+2} ,

and Stop condition is $|x_{i+2} - x_{i+1}| < \in$ for any i.



Example 9:

Find an approximate root of $f(x) = x^3 - 2x^2 - 5$ in interval $[2,3]$ by using Secant method with error $\in = 0.002$.

Solution:

Let $x_1 = 2,\ x_2 = 3,\ \ f(2) = (2)^3 - 2(2)^2 - 5 = -5$

$f(3) = (3)^3 - 2(3)^2 - 5 = 4, \qquad \Rightarrow (f(x_1).f(x_2) < 0)$

x	2	3	2.5556	2.6691	2.6923	2.6906
$y = f(x)$	-5	4	-1.3713	-0.23315	0.0185	-0.0005

$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}$,

$x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \dfrac{2(4) - (3)(-5)}{\left((4) - (-5)\right)} = \dfrac{23}{9} = 2.5556$

$f(2.5556) = (2.5556)^3 - 2(2.5556)^2 - 5 = -1.3713$

$= -0.0285$

$|x_3 - x_2| = |2.5556 - 3| = 0.4444 > \in = 0.007$.

$x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \Rightarrow x_4 = \dfrac{\left((3)(-1.3713) - (2.5556)(4)\right)}{(-1.3713 - 4)} = 2.6691.$

$f(2.6691) = (2.6691)^3 - 2(2.6691)^2 - 5 = -0.23315$

$|x_4 - x_3| = |2.6691 - 2.5556| = 0.1135 > \in.$

$x_5 = \dfrac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} \Rightarrow x_5 = \dfrac{\left((2.5556)(-0.23315) - (2.6691)(-1.3713)\right)}{(-0.23315 - (-1.3713))} = 2.6923.$

$f(2.6923) = (2.6923)^3 - 2(2.6923)^2 - 5 = 0.0185$

$|x_5 - x_4| = |2.6923 - 2.6691| = 0.0232 > \in.$

$x_6 = \dfrac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} \Rightarrow x_6 = \dfrac{\left((2.6691)(0.0185) - (2.6923)(-0.23315)\right)}{(0.0185 - (-0.23315))} = 2.6906.$

$f(2.6906) = (2.6906)^3 - 2(2.6906)^2 - 5 = -0.0005$

$|x_6 - x_5| = |2.6906 - 2.6923| = |-0.0017| = 0.0017 < \in$

Then $x_6 = 2.6906$ is the root.

Example 10:

Find an approximate root of $f(x) = x + e^{-2x} - 1 = 0$ in interval $[0.5,1]$ by using Secant method with error $\in = 0.007$.

Solution:

Let $x_1 = 0.5,\ x_2 = 1,\ \ f(0.5) = 0.5 + e^{-2(0.5)} - 1 = -0.1321$

$f(1) = 1 + e^{-2(1)} - 1 = 0.1353 \quad \Rightarrow (f(x_1) f(x_2) < 0)$

x	0.5	1	0.7470	0.7910	0.7970
$y = f(x)$	-0.1321	0.1353	-0.0285	-0.0034	-0.0001

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \frac{0.5(0.1353) - (1)(-0.1321)}{\left((0.1353) - (-0.1321)\right)} = 0.7470$$

$f(0.7470) = 0.7470 + e^{-2(0.7470)} - 1 = -0.0285$

$|x_3 - x_2| = |0.7470 - 1| = 0.253 > \in = 0.007 \;\; x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$

$$\Rightarrow x_4 = \frac{\left((1)(-0.0285) - (0.7470)(0.1353)\right)}{(-0.0285 - 0.1353)} = 0.7910.$$

$f(0.7910) = 0.7910 + e^{-2(0.7910)} - 1 = -0.0034$

$|x_4 - x_3| = |0.7910 - 0.7470| = 0.044 > \in$

$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} \Rightarrow x_5 = \frac{\left((0.7470)(-0.0034) - (0.7910)(-0.0285)\right)}{(-0.0034 - (-0.0285))} = 0.7970$

$f(0.7970) = 0.7970 + e^{-2(0.7970)} - 1 = -0.0001$

$|x_5 - x_4| = |0.7910 - 0.7990| = 0.006 < \in$

Then $x_5 = 0.7961$ is the root.

Homework By using Secant Method,

1- Find an approximate root of $f(x) = x^2 - 2x - 1 = 0$ between $[-1,0]$ by using Secant method with error $\in = 10^{-3}$.

2- Find an approximate root of $f(x) = x^4 - x - 10$ in interval $[1,2]$ by using Secant method with error $\in = 0.0001$.

(IV) Newton-Raphson method: طريقة نيوتن - رافسون

Suppose function f is continuous and differentiable on the interval $[a,b]$, Let x_0 be an approximation to the exact root x_1 such that $f'(x_1) \neq 0$ and h is error between x_1, x_0 such that $x_1 = x_0 + h$. Using the Taylor series to function f about point x_0 we get:

لنفترض أن الدالة f مستمرة وقابلة للاشتقاق على الفترة $[a,b]$، فليكن x_0هو جذر تقريبيًا للجذر المضبوط x_1 بحيث ان$f'(x_1) \neq 0$ و h يكون الخطأ بين x_0 و x_1 بحيث ان $x_1 = x_0 + h$. باستخدام متسلسلة تايلور للدالة f حول النقطة x_0 نحصل على:

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) + \cdots. \qquad \ldots(1)$$

put $x = x_1$ in formula (1) we get:-

$$f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{(x_1 - x_0)^2}{2!}f''(x_0) + \frac{(x_1 - x_0)^3}{3!}f'''(x_0)$$

$$f(x_1) = f(x_0 + h) = f(x_0) + (h)f'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \cdots. \qquad \ldots(2)$$

h is Small then h^2 , h^3, ... are very small can be deleted.

hصغير فان h^2 , h^3, ... صغير جدًا ويمكن حذفه.

$$\Rightarrow f(x_1) = f(x_0 + h) = f(x_0) + (h)f'(x_0)$$

$\because$ x_1 is exact root then $f(x_1) = 0 \Rightarrow 0 = f(x_0) + hf'(x_0)$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}, f'(x_0) \neq 0$$

We offset the value of h in formula $x_1 = x_0 + h$, we get:-

نقوم بتعويض قيمة h في الصيغة $x_1 = x_0 + h$ ، فنحصل على:-

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \ f'(x_0) \neq 0, \ x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, f'(x_1) \neq 0$$

Thus, the general formula of the **Newton-Raphson method** is

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, f'(x_i) \neq 0 \ , i = 1,2,3,\ldots}$$

The two conditions stop are $|x_{i+1} - x_i| \leq \in \vee \ |f(x_{i+1})| \leq \in$ for any i.

Note:

Sometime **Newton-Raphson method** does not converge to the required root:

1. If there is no real root of the function $f(x)$.
2. If the initial value so far from the exact root λ.
3. If $f'(x_i) = 0$.

ملاحظة:

في بعض الأحيان لا تتقارب طريقة نيوتن-رافسون مع الجذر المطلوب:

1. إذا لم يكن هناك جذر حقيقي للدالة f.
2. إذا كانت القيمة الأولية بعيدة عن الجذر المطلوب λ.
3. إذا كان $f'(x_i) = 0$.

Example 11:

Using **Newton-Raphson method**, find the solution to the equation $f(x) = x^3 + 5x - 3 = 0$ in interval $[0,1]$ with error $\in = 0.0002$.

Solution:

$f(0) = (0)^3 + 5(0) - 3 = -3$, $f(1) = (1)^3 + 5(1) - 3 = 3$

$f(x) = x^3 + 5x - 3 = 0$, $f'(x) = 3x^2 + 5 = 0$, $\boxed{x_0 = 0.5}$

x	0.5	0.5652	0.5641
$y = f(x)$	-0.375	0.0066	0.00002
$f'(x)$	5.75	5.9584	5.9546

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $f'(x_i) \neq 0$, $i = 1,2,3,\ldots$

$f(0.5) = (0.5)^3 + 5(0.5) - 3 = -0.375$, $f'(0.5) = 3(0.5)^2 + 5 = 5.75$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, f'(x_0) \neq 0 \Rightarrow x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$\Rightarrow x_1 = 0.5 - \frac{(-0.375)}{5.75} \Rightarrow x_1 = 0.5652$$

$f(0.5652) = (0.5652)^3 + 5(0.5652) - 3 = 0.0066$,

$f'(0.5652) = 3(0.5652)^2 + 5 = 5.9584$

$|x_1 - x_0| = |0.5652 - 0.5| = 0.0652 > \in \vee \; |f(x_1)| = |0.0066| > \in$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, f'(x_1) \neq 0 \Rightarrow x_2 = 0.5652 - \frac{(0.0066)}{(5.9584)} \Rightarrow x_2 = 0.5641$$

$f(0.5641) = (0.5641)^3 + 5(0.5641) - 3 = 0.00002$,

$f'(0.5641) = 3(0.5641)^2 + 5 = 5.9546$.

$|x_2 - x_1| = |0.5641 - 0.5652| = |-0.0011| = 0.0011 > \in \vee \; |f(x_2)| = |0.00002| <$

Then $x_2 = 0.5641$ is the Root required الجذر المطلوب .

Example 12:

Using **Newton-Raphson method**, find an approximate value of the solution to the equation $f(x) = x^2 - 5$ when $x = 2, \in = 0.0007$.

Solution:

$f(x) = x^2 - 5 \Rightarrow f'(x) = 2x$.

x	2	2.25	2.236111	2.236068
$y = f(x)$	-1	0.0625	0.000192	0.0000001

$f'(x)$	4	4.5	4.472222	4.472136

$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, $f'(x_i) \neq 0$, $i = 1,2,3,\ldots$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} , f'(x_0) \neq 0 \Rightarrow x_1 = 2 - \frac{f(-1)}{f'(4)} = 2.25$$

$f(2.25) = (2.25)^2 - 5 = 0.0625$, $f'(2.25) = 2(2.25) = 4.5$.

$|x_1 - x_0| = |2.25 - 2| = 0.25 > \in \vee \ |f(x_1)| = |0.0625| > \in$.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} , f'(x_1) \neq 0 \Rightarrow x_2 = 2.25 - \frac{(0.0625)}{(4.5)} \Rightarrow x_2 = 2.236111$$

$f(2.236111) = (2.236111)^2 - 5 = 0.000192$,

$f'(2.236111) = 2(2.236111) = 4.472222$.

$|x_2 - x_1| = |2.236111 - 2.25| = 0.013889 > \in \vee \ |f(x_2)| = |0.000192| < \in$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} , f'(x_2) \neq 0 \Rightarrow x_3 = 2.236111 - \frac{(2.236111)}{(4.472222)} \Rightarrow x_2 = 2.236068$$

$f(2.236068) = (2.236068)^2 - 5 = 0.0000001$,

$f'(2.236068) = 2(2.236068) = 4.472136$.

$|x_3 - x_2| = |2.236068 - 2.236111| = 0.000043 < \in \vee \ |f(x_2)| = |0.0000001| < \in$

Then $x_3 = 2.236068$ is the Root required الجذر المطلوب .

Homework Using Newton-Raphson method,

1- If $f(x) = x^3 - x + 1 = 0$ and $x_0 = 1$, find x_1 and x_2.

2- Find an approximate value of the solution to the equation $f(x) = x^2 - 8$ with error $\in = 3\times10^{-3}$.

Homework Find an approximate root of $f(x) = x^3 - x - 1 = 0$ in [1,2] with 10^{-5} first by Newton-Raphson method and then be the secant method.