Numerical Analysis

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Chapter Four

Interpolation and Extrapolation

Introduction

If we have a finite set of unknown function values $f(x_0), f(x_2), f(x_2), \cdots, f(x_m)$ and want to guess value of the function at point x^* ,if point x^*falls within the range of points$\{x_i\},\ i=0,1,2,\ldots,n$ then the process is called **The interpolation** and reverse called **the extrapolation**

There are several mathematical methods used in this field

الاندراج والاستكمال

مقدمة

إذا كان لدينا مجموعة محدودة من قيم الدالة غير المعروفة $f(x_0), f(x_2), f(x_2), \cdots, f(x_m)$ ونريد تخمين قيمة الدالة عند النقطة x^*، إذا كانت النقطةx^* تقع ضمن نطاق النقاط فتسمى العملية الاستيفاء والعكس يسمى الاستقراء

هناك العديد من الطرق الرياضية المستخدمة في هذا المجال

1- Lagrange polynomial

Let f be a real and continuous function in interval$[a,b]$ and its values are known in the points $x_0, x_1, x_2, \cdots, x_m$ which are numbered $m+1$. To guess value of function in one or several new points, we close the function to polynomial from class m Consistent with the function in the given points and thus find a new polynomial and make up the points needed to find their images .

لتكن f دالة حقيقية ومستمرة في الفترة $[a,b]$ وقيمها معروفة في النقاط $x_0, x_1, x_2, \cdots, x_m$ المرقمة $m+1$. لتخمين قيمة دالة f في نقطة واحدة أو عدة نقاط جديدة، نقوم بإغلاق الدالة إلى كثيرة حدود من

فئة متوافقة مع الدالة في النقاطm المعطاة وبالتالي العثور على كثيرة حدود جديدة وتكوين النقاط اللازمة للعثور على صورها .

The general formula of polynomial $P_m(x)$ (Lagrange polynomial)

الصيغة العامة لكثيرة الحدود (متعدد الحدود لاكرانج للاندراج)

$$P_m(x) = \sum_{j=0}^{m} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{m} \frac{(x - x_i)}{(x_j - x_i)}$$

$$f(x) \approx P_m(x)$$

$$= f(x_0)\frac{(x-x_1)(x-x_2)\cdots(x-x_m)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_m)}$$

$$+ f(x_1)\frac{(x-x_0)(x-x_2)\cdots(x-x_m)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_m)}$$

$$+\cdots+ f(x_m)\frac{(x-x_0)(x-x_1)\cdots(x-x_{m-1})}{(x_m-x_1)(x_m-x_2)\cdots(x_m-x_{m-1})}$$

And to find $f(x^*)$ without knowing form of the function we use the following law $P_m(x^*) = \sum_{j=0}^{m} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{m} \frac{(x^*-x_i)}{(x_j-x_i)}$

$$P_m(x^*) = f(x_0)\frac{(x^*-x_1)(x^*-x_2)\cdots(x^*-x_m)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_m)} + f(x_1)\frac{(x^*-x_0)(x^*-x_2)\cdots(x^*-x_m)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_m)} + \cdots + f(x_m)\frac{(x^*-x_0)(x^*-x_1)\cdots(x^*-x_{m-1})}{(x_m-x_1)(x_m-x_2)\cdots(x_m-x_{m-1})}$$

Note:-

1-Lagrange polynomial give a exact result at any point from the table points because the function is consistent with the polynomial in the table points.

2- We use Lagrange polynomial in both cases Interpolation and Extrapolation.

ملاحظة:-

1- متعددة حدود لاكرانج يعطي نتيجة دقيقة عند أي نقطة من نقاط الجدول لأن الدالة متوافقة مع متعددة الحدود في نقاط الجدول.

2- نستخدم متعددة حدود لاكرانج في كلتا الحالتين الاندراج والاستكمال.

Example 1:

Find the form of function $f(x)$ and then find $f(1.5), f(3.7), f(0.65)$ of the following table and using Lagrange polynomial

x	1	1.2	2
y	0	0.1823	0.6931

Solution:

$$P_2(x) = \sum_{j=0}^{2} f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{2} \frac{(x - x_i)}{(x_j - x_i)}$$

$$P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = 0 + 0.1823\frac{(x-1)(x-2)}{(1.2-1)(1.2-2)} + 0.6931\frac{(x-1)(x-1.2)}{(2-1)(2-1.2)}$$

$$P_2(x) = -0.273x^2 + 1.5121x - 1.2391 \Rightarrow$$

$$f(1.5) \approx P_2(1.5) = -0.273(1.5)^2 + 1.5121(1.5) - 1.2391 = 0.4148.$$

$$f(3.7) \approx P_2(3.7) = -0.273(3.7)^2 + 1.5121(3.7) - 1.2391 = -0.6183.$$

$$f(0.65) \approx P_2(0.65) = -0.273(0.65)^2 + 1.5121(0.65) - 1.2391 = -0.3715.$$

Homework

Find $f(1.7), f(5), f(2.5)$?

To find the function

$$f(x) \approx P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$+ f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= f(1.1)\frac{(x-1.7)(x-3)}{(1.1-1.7)(1.1-3)} + f(1.7)\frac{(x-1.1)(x-3)}{(1.7-1.1)(1.7-3)} + f(3)\frac{(x-1.1)(x-1.7)}{(3-1.1)(3-1.7)}$$

$$= (0)\frac{(x-1.2)(x-2)}{(1-1.2)(1-2)} + (0.1823)\frac{(x-1)(x-2)}{(1.2-1)(1.2-2)} + (0.6931)\frac{(x-1)(x-1.2)}{(2-1)(2-1.2)}$$

$$= (0) + \frac{(0.1823)}{(1.2-1)(1.2-2)}(x-1)(x-2) + \frac{(0.6931)}{(2-1)(2-1.2)}(x-1)(x-1.2)$$

$$= \frac{(0.1823)}{0.16}(x^2 - 3\text{x} + 2) + \frac{(0.6931)}{0.8}(x^2 - 2.2\text{x} + 1.2)$$

$$f\ (x) = -0.273\ x^2 + 1.5121\ \text{x} - 1.2391.$$

Example 2:

Find $f\ (2.3)$ of the following table and using Lagrange polynomial without knowing form of the function

x	1.1	1.7	3
y	0.6	15.2	20.3

Solution:

$$P_2(x^*) = \sum_{j=0}^{2} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{2} \frac{(x^* - x_i)}{(x_j - x_i)}$$

$f\ (2.3) \approx P_2(2.3)$

$$= f(x_0)\frac{(2.3-x_1)(2.3-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(2.3-x_0)(2.3-x_2)}{(x_1-x_0)(x_1-x_2)}$$
$$+ f(x_2)\frac{(2.3-x_0)(2.3-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(2.3) = 10.6\frac{(2.3-1.7)(2.3-3)}{(1.1-1.7)(1.1-3)} + 15.2\frac{(2.3-1.1)(2.3-3)}{(1.7-1.1)(1.7-3)}$$
$$+ 20.3\frac{(2.3-1.1)(2.3-1.7)}{(3-1.1)(3-1.7)}$$

$$f\ (2.3) \approx P_2(2.3) = -3.9053 + 16.3692 + 5.0174 = 18.3813\ \ .$$

Homework

Find $f(1.7), f(5), f(2.5)$?

To find the function

$$f(x) \approx P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= f(1.1)\frac{(x-1.7)(x-3)}{(1.1-1.7)(1.1-3)} + f(1.7)\frac{(x-1.1)(x-3)}{(1.7-1.1)(1.7-3)} + f(3)\frac{(x-1.1)(x-1.7)}{(3-1.1)(3-1.7)}$$

$$= (10.6)\frac{(x-1.7)(x-3)}{(1.1-1.7)(1.1-3)} + (15.2)\frac{(x-1.1)(x-3)}{(1.7-1.1)(1.7-3)} + (20.3)\frac{(x-1.1)(x-1.7)}{(3-1.1)(3-1.7)}$$

$$= \frac{(10.6)}{(1.1-1.7)(1.1-3)}(x-1.7)(x-3) + \frac{(15.2)}{(1.7-1.1)(1.7-3)}(x-1.1)(x-3) + \frac{(20.3)}{(3-1.1)(3-1.7)}(x-1.1)(x-1.7).$$

$$= \frac{(10.6)}{(1.1-1.7)(1.1-3)}(x^2-4.7\text{x}+5.1) + \frac{(15.2)}{(1.7-1.1)(1.7-3)}(x^2-4.1\text{x}+3.3) + \frac{(20.3)}{(3-1.1)(3-1.7)}(x^2-2.8\text{x}+1.87).$$

$$= (9.2982)(x^2-4.7\text{x}+5.1) - (19.4872)(x^2-4.1\text{x}+3.3) + (8.2186)(x^2-2.8\text{x}+1.87).$$

$$= (9.2982-19.4872+8.2186)x^2 + ((9.2982)(-4.7) + (-19.4872)(-4.1) + (8.2186)(-2.8))\text{x} + ((9.2982)(5.1) + (-19.4872)(3.3) + (8.2186)(1.87)).$$

$$f(x) = -1.9704\, x^2 + 13.1839\,\text{x} - 1.518158.$$

Example 3:

Find the function of the following table by using Lagrange polynomial and find $f(0.35), f(1.15), f(-0.15)$,

x	0	0.25	0.5	0.75	1
y	1	1.0625	1.25	1.5625	2

Solution:

To find the function $P_4(x^*) = \sum_{j=0}^{4} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{4} \frac{(x^*-x_i)}{(x_j-x_i)}$

$$f(x) \approx P_4(x) = f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$$
$$+ f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$
$$+ f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$
$$+ f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)}$$
$$+ f(x_4)\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$
$$= (1)\frac{(x-0.25)(x-0.5)(x-0.75)(x-1)}{(0-0.25)(0-0.5)(0-0.75)(0-1)}$$
$$+ (1.0625)\frac{(x-0)(x-0.5)(x-0.75)(x-1)}{(0.25-0)(0.25-0.5)(0.25-0.75)(0.25-1)}$$
$$+ (1.25)\frac{(x-0)(x-0.25)(x-0.75)(x-1)}{(0.5-0)(0.5-0.25)(0.5-0.75)(0.5-1)}$$
$$+ (1.5625)\frac{(x-0)(x-0.25)(x-0.5)(x-1)}{(0.75-0)(0.75-0.25)(0.75-0.5)(0.75-1)}$$
$$+ (2)\frac{(x-0)(x-0.25)(x-0.5)(x-0.75)}{(1-0)(1-0.25)(1-0.5)(1-0.75)}$$

$$= \left(\frac{1}{0.09375}\right)(x-0.25)(x-0.5)(x-0.75)(x-1)$$
$$+ \left(\frac{-1.0625}{0.0234375}\right)(x-0)(x-0.5)(x-0.75)(x-1)$$
$$+ \left(\frac{1.25}{0.015625}\right)(x-0)(x-0.25)(x-0.75)(x-1)$$
$$+ \left(\frac{-1.5625}{0.0234375}\right)(x-0)(x-0.25)(x-0.5)(x-1)$$
$$+ \left(\frac{2}{0.09375}\right)(x-0)(x-0.25)(x-0.5)(x-0.75)$$

$= (10.666667)(x^4 - 2.5x^3 + 3.3125x^2 - 1.625x + 0.9375) + (-45.3333333)(x^4 - 2.25x^3 + 1.625x^2 - 0.375x) + (80)(x^4 - 2x^3 + 1.1875x^2 - 0.1875x) + (-66.6666667)(x^4 - 1.75x^3 + 0.875x^2 - 0.125x) + (21.333333)(x^4 - 1.5x^3 + 0.6875x^2 - 0.09375x).$

$= (0)x^4 - (0)x^3 + (1)x^2 - (0)x + 1 = x^2 + 1.$

$f(x) = x^2 + 1.$

To find $f(0.35), f(1.15), f(-0.15),$

$f(0.35) \approx P_2(0.35) = (0.35)^2 + 1 = 1.1225.$

$f(1.15) \approx P_2(1.15) = (1.15)^2 + 1 = 2.3225$

$f(-0.15) \approx P_2(-0.15) = (-0.15)^2 + 1 = 1.0225$

Homework

Find $f(1.2), f(-0.5)$?

Finite differences with points equidistant from each other

الفروقات المنتهية ذات النقاط المتساوية الأبعاد فيما بينها

Let f be a real function whose values are known in $m+1$ points of equal dimensions $x_0, x_1, x_2, \cdots, x_m$ and that have known images $f(x_0), f(x_1), f(x_2), \cdots, f(x_m)$ or $y_0, y_1, y_2, \cdots, y_m$ definite the first difference at the point x_0 .

لتكن f دالة حقيقية تكون قيمها معروفة في $m+1$ في نقاط متساوية الأبعاد $x_0, x_1, x_2, \cdots, x_m$ ولها صور معروفة $y_0, y_1, y_2, \cdots, y_m$ أو تحديد الفرق الأول في هذه النقطة x_0 .

$\Delta f(x_0) = f(x_1) - f(x_0) \Rightarrow \Delta y_0 = y_1 - y_0$

$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, \cdots, \Delta y_{m-1} = y_m - y_{m-1}$

defined the second difference as follows:-

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - y_1 - y_1 + y_0$$
$$\Rightarrow \Delta^2 y_0 = y_2 - 2y_1 + y_0 \Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1)$$
$$= y_3 - y_2 - y_2 + y_1 \Rightarrow \Delta^2 y_1 = y_3 - 2y_2 + y_1$$
$$\vdots$$
$$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i \quad , i = 1,2,\cdots,n-2.$$

In general $\Delta^k y_i = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y_{i+k-j}$

Table of differences

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Divided Finite differences to three types:- **تنقسم الفروق المنتهية إلى ثلاث أنواع:-**

1- Forward differences **الفروقات التقدمية**
2- Backward differences **الفروقات التراجعية**
3- Central differences **الفروقات المركزية**

1- Forward differences **الفروقات التقدمية**

To find an image point located at the beginning of the table we use the Forward difference. للعثور على نقطة الصورة الموجودة في بداية الجدول نستخدم الفروقات التقدمية. نلاحظ ان

$$\Delta y_0 = y_1 - y_0 \Rightarrow y_1 = y_0 + \Delta y_0 \Rightarrow y_1 = (1+\Delta) y_0$$

$$y_2 = y_1 + \Delta y_1 \Rightarrow y_2 = (1+\Delta) y_1 \Rightarrow y_2 = (1+\Delta)^2 y_0$$

$\vdots$

$$y_m = (1+\Delta)^m y_0 \quad .$$

We get $y_m = \sum_{k=0}^{m} \binom{m}{k} \Delta^k y_0$

$y_m = f(x_0 + mh)$

$$= y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \cdots$$

$m = \frac{x_m - x_0}{h}$.

Example 4:

Find $f(0.3), f(0.8), f(1.2)$ of the following table use the forward difference.

x	0	1	2	3	4
y	1	2	9	28	65

Solution:

Table of the forward difference.differences

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$m = \frac{x_m - x_0}{h} \implies m = \frac{0.3-0}{1} = 0.3,$

$f(x) = f(x_0 + mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(0.3) = 1 + (0.3)(1) + \frac{(0.3)(0.3-1)}{2}(1) + \frac{(0.3)(0.3-1)(0.3-2)}{6}(6)$.

$f(0.3) = 1 + 0.3 - 0.63 + 0.357 = 1.027.$

Table of the forward difference

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$m = \frac{x_m - x_0}{h} \implies m = \frac{0.8-1}{1} = -0.2,$

$f(0.8) = 2 + (-0.2)(7) + \frac{(-0.2)(-0.2-1)}{2}(12) + \frac{(-0.2)(-0.2-1)(-0.2-2)}{6}(6)$.

$f(0.8) = 2 - 1.4 + 1.44 - 0.528 = 1.512.$

$m = \frac{x_m - x_0}{h} \implies m = \frac{1.2-1}{1} = 0.2,$

$f(1.2) = 2 + (0.2)(7) + \frac{(0.2)(0.2-1)}{2}(12) + \frac{(0.2)(0.2-1)(0.2-2)}{6}(6)$.

$f(1.2) = 2 + 1.4 - 0.96 + 0.288 = 2.728.$

To find the function

$m = \dfrac{x_m - x_0}{h} \implies m = \dfrac{x - 0}{1} = x$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(x) = 1 + (x)(1) + \frac{(x)(x-1)}{2}(1) + \frac{(x)(x-1)(x-2)}{6}(6) = 1 + x + 3x^2 - 3x + x^3 - 3x^2 + 2x = x^3 + 1$. $f(x) = x^3 + 1$

Example 5:

Find $f^{-1}(1.64), f^{-1}(0.18)$ of the following table use the forward differences .

x	1	2	3	4	5	6
y	0	0.7	1.1	1.4	1.6	1.7

Solution:

Table of the forward difference.differences

x	y	Δ	Δ^2	Δ^3	Δ^4
1	0				
		0.7			
2	0.7		-0.3		
		0.4		0.2	
3	1.1		-0.1		-
		0.3		0	0.2
4	1.4		-0.1		يهمل
		0.2		0	
5	1.6		-0.1		
		0.1			
6	1.7				

$m = \frac{x-x_0}{h} \Rightarrow m = \frac{x-1}{1} = x-1,$

$f(x) = f(x_0+mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(x) = 0 + (x-1)(0.7) + \frac{(x-1)(x-2)}{2}(-0.3) + \frac{(x-1)(x-2)(x-3)}{6}(0.2) + \frac{(x-1)(x-2)(x-3)(x-4)}{24}\left(-0.2^{\text{يهمل}}\right)$

$f(x) = 0 + 0.7x + 0.7 - 0.15x^2 + 0.45x - 0.3 + 0.0333x^3 - 0.2x^2 + 0.3667x - 0.1999.$

$f(x) = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001.$

$f(x) = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001 \Rightarrow f(x) = 1.64$
$\Rightarrow f^{-1}(1.64) = ?$
$\Rightarrow 1.64 = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001.$
$\Rightarrow 0.0333x^3 - 0.35x^2 + 1.5167x - 1.4399 = 0$
$\Rightarrow$ بالدستور . $\therefore f^{-1}(1.64) = 5.296.$

$f(x) = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001 \Rightarrow f(x) = 0.18$
$\Rightarrow f^{-1}(0.18) = ?$
$\Rightarrow 0.18 = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001.$
$\Rightarrow 0.0333x^3 - 0.35x^2 + 1.5167x + 0.0201 = 0$
$\Rightarrow$ بالدستور $\therefore f^{-1}(0.18) = 1.2.$

Homework

Find $f(0.7), f(2.3), f(4.2), f^{-1}(10), f^{-1}(30)$ of the following table use the forward difference and find To find the function .

x	0	5	10	15	20	25
y	7	11	14	18	24	32

2-Backward differences   الفروقات التراجعية

To find an image point located at the end of a table we use the Backward differences as follows

للعثور على نقطة صورة تقع في نهاية الجدول، نستخدم الفروقات التراجعية على النحو التالي

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \cdots, \nabla y_m = y_m - y_{m-1}$

$\nabla^2 y_m = \nabla(\nabla y_m) = \nabla y_m - \nabla y_{m-1} = (y_m - y_{m-1}) - (y_{m-1} - y_{m-2})$

$\Rightarrow \nabla^2 y_m = y_m - 2y_{m-1} + y_{m-2}.$

$\therefore \nabla^k y_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y_{i-j}$.

$\nabla^2 y_2 = \sum_{j=0}^{2} \binom{2}{j} (-1)^j y_{2-j} = \binom{2}{0} (-1)^0 y_2 + \binom{2}{1} (-1) y_1 + \binom{2}{2} (-1)^2 y_0 = y_2 - 2y_1 + y_0$.

From the definition we notice:-

$\Delta y_0 = \nabla y_{-1}, \Delta^2 y_0 = \nabla^2 y_{-1}, \Delta^3 y_0 = \nabla^3 y_{-1}$

$\nabla y_i = y_i - y_{i-1} \Rightarrow y_{i-1} = y_i - \nabla y_i = (1 - \nabla) y_i$

$y_{i-1} = (1 - \nabla) y_i \Rightarrow \ y_i = (1 - \nabla)^{-1} y_{i-1}$

And $\Delta y_{i-1} = y_i - y_{i-1} \Rightarrow y_i = y_{i-1} + \Delta y_{i-1} = (1 + \Delta) y_{i-1} \Rightarrow y_i = (1 + \Delta) y_{i-1}$

So $\boxed{(1-\nabla)^{-1} = (1+\Delta)}$

In general $\Delta^k y_i = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y_{i+k-j}$

Table of differences

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_0				
x_1	y_1		$\nabla^2 y_0$			
		∇y_1		$\nabla^3 y_0$		
x_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$	
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$
x_3	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$	
		∇y_3		$\nabla^3 y_2$		
x_4	y_4		$\nabla^2 y_3$			
		∇y_4				
x_5	y_5					

$$y_m = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \cdots$$

Example 6:

Find $f(3.8), f(3.2), f(2.8)$ of the following table use the Backward difference.

x	0	1	2	3	4
y	1	2	9	28	65

Solution:

Table of the Backward difference.

x	y	∇	∇^2	∇^3	∇^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{3.8-4}{1} = -0.2,$$

$$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots.$$

$$f(3.8) = 65 + (-0.2)(37) + \frac{(-0.2)(-0.2+1)}{2}(18) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{6}(6) \;.$$

$$f(3.8) = 65 - 7.4 - 1.44 - \;0.288 = 55.872.$$

Table of the Backward difference

x	y	∇	∇^2	∇^3	∇^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{3.2-3}{1} = 0.2,$

$f(3.2) = 28 + (0.2)(19) + \frac{(0.2)(0.2+1)}{2}(12) + \frac{(0.2)(0.2+1)(0.2+2)}{6}(6)$.

$f(3.2) = 28 + 3.8 + 1.44 + 0.528 = 33.768.$

$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{2.8-3}{1} = -0.2,$

$f(2.8) = 28 + (-0.2)(19) + \frac{(-0.2)(-0.2+1)}{2}(12) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{6}(6)$.

$f(2.8) = 28 - 3.8 - 0.96 - 0.288 = 22.952.$

To find the function

$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{x-4}{1} = x - 4$

$f(x) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots.$

$f(x) = 65 + (x-4)(37) + \frac{(x-4)(x-3)}{2}(18) + \frac{(x-4)(x-3)(x-2)}{6}(6)$

$= 65 + 37x - 148 + 9x^2 - 63x + 108 + x^3 - 7x^2 + 12x - 2x^2 + 14x - 24 = x^3 + 1$

$f(x) = x^3 + 1$.

Homework

Find $f(20.3), f(24.8), f(23.2), f^{-1}(22), f^{-1}(30)$ of the following table use the Backward difference and find To find the function .

x	0	5	10	15	20	25
y	7	11	14	18	24	32

3- Central differences   الفروقات المركزية

To find a point image located in the center of a table or near the center, we use Central differences as follows:

للعثور على صورة نقطية تقع في وسط الجدول أو بالقرب من المركز، نستخدم الفروق المركزية كما يلي:

$$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \xrightarrow{\text{للزوجي} \quad Example} \delta y_3 = y_{\frac{7}{2}} - y_{\frac{5}{2}}$$

$$\delta y_{i+\frac{1}{2}} = y_{i+1} - y_{i-1} \xrightarrow{\text{للفردي} \quad Example} \delta y_{\frac{3}{2}} = y_2 - y_1$$

$$\delta^2 y_i = \delta(\delta y_i) = \delta\left(y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}\right) = \delta y_{i+\frac{1}{2}} - \delta y_{i-\frac{1}{2}} = (y_{i+1} - y_i) - (y_i - y_{i-1}).$$

$$\delta^2 y_i = y_{i+1} - 2y_i + y_{i-1}, \forall i = 0, \pm 1, \pm 2, \dots$$

$$\delta^3 y_{i+\frac{1}{2}} = \delta^2 y_{i+1} - \delta^2 y_i$$

:

:

$$\delta^k y_i = \delta(\delta^{k-1} y_i) = \delta^{k-1}(\delta y_i) = \delta^{k-2}((y_{i+1} - y_i) - (y_i - y_{i-1})) = \dots..$$

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		δy_0				
x_1	y_1		$\delta^2 y_0$			
		δy_1		$\delta^3 y_0$		
x_2	y_2		$\delta^2 y_1$		$\delta^4 y_0$	
		δy_2		$\delta^3 y_1$		$\delta^5 y_0$
x_3	y_3		$\delta^2 y_2$		$\delta^4 y_1$	
		δy_3		$\delta^3 y_2$		
x_4	y_4		$\delta^2 y_3$			
		δy_4				
x_5	y_5					

$$f(x) = f(x_0 + mh) = y_0 + m\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\delta^2 y_1 + \frac{m(m-1)(m-2)}{3!}\delta^3 y_{\frac{3}{2}} + \cdots.$$

There are two formulas for finding central point images:- هناك صيغتان للحصول على صور الفروقات المركزية

1-Besssl's formulas صيغ بسل المركزية

Its general formula :- صيغته العامة

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

2- Stirling's formulas **صيغ ستيرلينك المركزية**

Its general formula :- صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

Note:-

$\mu y_i = \frac{y_{i+1}+y_i}{2} \xrightarrow{Example} \mu y_1 = \frac{y_2+y_1}{2}$,

$\mu y_{i+\frac{1}{2}} = \frac{y_{i+1}+y_i}{2} \xrightarrow{Example} \mu y_{\frac{1}{2}} = \frac{y_1+y_0}{2}$, μ is the average

$\mu\delta y_i = \frac{\delta y_{i+\frac{1}{2}} + \delta y_{i-\frac{1}{2}}}{2}$ و $\mu\delta^2 y_i = \frac{\delta^2 y_{i+\frac{1}{2}} + \delta^2 y_{i-\frac{1}{2}}}{2}$.

Example 7:

Find $f(2.4)$ of the following table use the forward difference.

x	0	1	2	3	4	5
y	5	6	13	32	65	130

Solution:

Table of the forward difference. **Besssl's formulas** صيغ بسل المركزية
Its general formula :- صيغته العامة

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

x	y	δ	δ^2	δ^3	δ^4
0	5				
		1			
1	6		6		
		7		6	
2	13		12		0
		19		6	
3	32		18		0
		37		6	
4	65		24		
		61			
5	130				

$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{2.4-2}{1} = 0.4.$

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}}$$

$$y_{2.4} = \frac{13+32}{2} + (0.4-0.5)(19) + \frac{0.4(0.4-1)}{2}\left(\frac{12+18}{2}\right) + \frac{0.4(0.4-1)(0.4-0.5)}{6}(6).$$

$$y_{2.4} = 22.5 + (-1.9) + (-0.12)(15) + 0.024 = 22.5 - 1.9 - 1.8 + 0.024.$$

$$y_{2.4} = 18.824.$$

Stirling's formulas   **صيغ ستيرلينك المركزية**

Its general formula :-   صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

x	y	δ	δ^2	δ^3	δ^4
0	5				
		1			
1	6		6		
		7		6	
2	13		12		0
		19		6	
3	32		18		0
		37		6	
4	65		24		
		61			
5	130				

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{2.4} = 13 + (0.4)\left(\frac{7+19}{2}\right) + \frac{(0.4)^2}{2}(12) + \frac{0.4((0.4)^2-1)}{6}\left(\frac{6+6}{2}\right).$$

$$y_{2.4} = 13 + (0.4)(13) + (0.16)(6) + (0.4)(-0.84) = 13 + 5.2 + 0.96 - 0.336.$$

$$y_{2.4} = 18.824.$$

To find the function

$$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{x-2}{1} = x - 2$$

1) صيغ بسل المركزية

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$y_{x-2} = \frac{13+32}{2} + (x-2-0.5)(19) + \frac{(x-2)(x-3)}{2}\left(\frac{12+18}{2}\right) + \frac{(x-2)(x-3)(x-2-0.5)}{6}(6).$$

$$y_{x-2} = 22.5 + (x-2.5)(19) + (x-2)(x-3)(7.5) + (x-2)(x-3)(x-2.5).$$

$$y_{x-2} = 22.5 + 19x - 47.5 + 7.5x^2 - 37.5x + 45 + x^3 - 7.5x^2 + 18.5x - 15.$$

$f(x) = x^3 + 5$.

2) **صيغ ستيرلينك المركزية**

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{x-2} = 13 + (x-2)\left(\frac{7+19}{2}\right) + \frac{(x-2)^2}{2}(12) + \frac{(x-2)((x-2)^2-1)}{6}\left(\frac{6+6}{2}\right)$$

$$y_{x-2} = 13 + (x-2)(13) + 6(x^2-4x+4) + (x-2)(x^2-4x+4-1)$$

$$y_{x-2} = 13 + 13x - 26 + 6x^2 - 24x + 24 + x^3 - 6x^2 + 11x - 6.$$

$f(x) = x^3 + 5$.

Example 8:

Find $f(1.1)$ and $f(1.5)$of the following table use the forward difference.

x	-2	-1	0	1	2	3	4	5
y	15	0	-1	0	15	80	255	624

Solution:

Table of the forward difference. **Besssl's formulas**   **صيغ بسل المركزية**
Its general formula :-   صيغته العامة

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m-1)(m-2)(m-3)}{4!}\mu\delta^4 y_{\frac{1}{2}}.$$

x	y	δ	δ^2	δ^3	δ^4
-2	15				
		-15			
-1	0		14		
		-1		-12	
0	-1		2		24
		1		12	
1	0		14		24
		15		36	
2	15		50		24
		65		60	
3	80		110		24
		175		84	
4	255		194		
		369			
5	624				

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{1.1-1}{1} = 0.1$$,

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$y_{1.1} = \frac{0+15}{2} + (0.1-0.5)(15) + \frac{0.1(0.1-1)}{2}\left(\frac{14+50}{2}\right) + \frac{0.1(0.1-1)(0.1-0.5)}{6}(36) + \frac{(0.1)((0.1)^2-1)(0.1-2)}{24}\left(\frac{24+24}{2}\right).$$

$$y_{1.1} = 0.4641.$$

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{1.5-1}{1} = 0.5.$$

$$y_{1.5} = \frac{0+15}{2} + (0.5-0.5)(15) + \frac{0.5(0.5-1)}{2}\left(\frac{14+50}{2}\right) + \frac{0.5(0.5-1)(0.5-0.5)}{6}(36) + \frac{(0.5)((0.5)^2-1)(0.5-2)}{24}\left(\frac{24+24}{2}\right).$$

$$y_{1.5} = 4.0625.$$

Stirling's formulas **صيغ ستيرلينك المركزية**

Its general formula :- صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

x	y	δ	δ^2	δ^3	δ^4
-2	15				
		-15			
-1	0		14		
		-1		-12	
0	-1		2		24
		1		12	
1	0		14		24
		15		36	
2	15		50		24
		65		60	
3	80		110		24
		175		84	
4	255		194		
		369			
5	624				

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{1.1} = 0 + (0.1)\left(\frac{1+15}{2}\right) + \frac{(0.1)^2}{2}(14) + \frac{0.1((0.1)^2-1)}{6}\left(\frac{12+36}{2}\right) + \frac{(0.1)^2((0.1)^2-1)}{24}\ (24)$$

$$y_{1.1} = 0 + 0.8 + 0.07 - 0.396 - 0.0099 = 13 + 5.2 + 0.96 - 0.336$$

$$y_{1.1} = 0.4641$$

$$y_{1.5} = 0 + (0.5)\left(\frac{1+15}{2}\right) + \frac{(0.5)^2}{2}(14) + \frac{0.5((0.5)^2-1)}{6}\left(\frac{12+36}{2}\right) + \frac{(0.5)^2((0.5)^2-1)}{24}\ (24)$$

$$y_{1.5} = 4.0625.$$

To find the function

$$m=\frac{x_m-x_0}{h} \implies m=\frac{x-1}{1}=x-1$$

1) صيغ بسل المركزية

$$y_m=\mu y_{\frac{1}{2}}+\left(m-\frac{1}{2}\right)\delta y_{\frac{1}{2}}+\frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}}+\frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}}+\frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}}+\cdots$$

$$y_{x-1}=\frac{0+15}{2}+(x-1-0.5)(15)+\frac{(x-1)(x-2)}{2}\left(\frac{14+50}{2}\right)+\frac{(x-1)(x-2)(x-1-0.5)}{6}(36)+\frac{(x-1)((x-1)^2-1)(x-3)}{24}\left(\frac{24+24}{2}\right).$$

$$y_{x-1}=7.5+(x-1.5)(15)+(x-1)(x-2)(16)+(x-1)(x-2)(x-1.5)(6)+(x^3-3x^2+2x)(x-3).$$

$$y_{x-1}=7.5+15x-22.5+16x^2-48x+32+6x^3-18x^2+12x-9x^2+27x-18+x^4-6x^3+11x^2-6x.$$

$f(x)=x^4-1$.

2) صيغ ستيرلينك المركزية

$$y_m=y_0+m\mu\delta y_0+\frac{m^2}{2!}\delta^2 y_0+\frac{m(m^2-1)}{3!}\mu\delta^3 y_0+\frac{m^2(m^2-1)}{4!}\delta^4 y_0+\cdots$$

$$y_{x-1}=0+(x-1)\left(\frac{1+15}{2}\right)+\frac{(x-1)^2}{2}(14)+\frac{x-1((x-1)^2-1)}{6}\left(\frac{12+36}{2}\right)+\frac{(x-1)^2((x-1)^2-1)}{24}(24)$$

$$y_{x-1}=0+(x-1)(8)+\frac{x^2-2x+1}{2}(14)+\frac{x-1(x^2-2x+1-1)}{6}(24)+\frac{x^2-2x+1(x^2-2x+1-1)}{24}(24)$$

$$y_{x-1}=0+8x-8+7x^2-14x+7+\frac{(x-1)(x^2-2x)}{6}(24)+\frac{x^2-2x+1(x^2-2x)}{24}(24)$$

$$y_{x-1}=8x-8+7x^2-14x+7+4x^3-12x^2+8x+x^4-4x^3+5x^2-2x$$

$y_{x-1}=x^4-1$.

Homework

Find $f(20.3), f(24.8), f(23.2), f^{-1}(19), f^{-1}(13)$ of the following table use the forward difference and find To find the function .

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Homework

Find $(1.1), f(0.4), f(-0.1), f(3.8), f(1.5), f(4.1), f(5.3), f(-1.2), f(1.7), f^{-1}(10), f^{-1}(20)$ of the following table use the forward difference and find To find the function .

x	-1	0	1	2	3	4
y	0	-1	0	15	80	255

أسئلة إضافية محلولة شاملة

Example 1:
Find $f(0.3), f(0.8), f(1.2), f(3.8), f(4.2), f(3.2), f(2.4), f^{-1}(1), f^{-1}(15)$
of the following table use the differences.

x	0	1	2	3	4
y	0	2	6	12	20

Solution:
Table of the difference.

x	y	$\Delta=\nabla=\delta$	Δ^2	Δ^3	Δ^4
0	0				
		2			
1	2		2		
		4		0	
2	6		2		0
		6		0	
3	12		2		
		8			
4	20				

Find $f(0.3)$

$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{0.3-0}{1}=0.3,$

$f(x)=f(x_0+mh)=y_0+m\Delta y_0+\frac{m(m-1)}{2!}\Delta^2 y_0+\frac{m(m-1)(m-2)}{3!}\Delta^3 y_0+\frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0+\dots$

$f(0.3)=0+(0.3)(2)+\frac{(0.3)(0.3-1)}{2}(2)+\frac{(0.3)(0.3-1)(0.3-2)}{6}(0)=0+0.6-0.21+0=0.39.$

Find $f(0.8)$

$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{0.8-1}{1}=-0.2,$

$f(0.8)=2+(-0.2)(4)+\frac{(-0.2)(-0.2-1)}{2}(2)+\frac{(-0.2)(-0.2-1)(-0.2-2)}{6}(0)=2-0.8+0.24+0=1.44.$

Find $f(1.2)$

$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{1.2-1}{1}=0.2,$

$f(1.2)=2+(0.2)(4)+\frac{(0.2)(0.2-1)}{2}(2)+\frac{(0.2)(0.2-1)(0.2-2)}{6}(0)=2+0.8-0.16+0=2.64.$

To find the function

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{x-0}{1} = x$$

$$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \dots.$$

$$f(x) = 0 + (x)(2) + \frac{(x)(x-1)}{2}(2) + \frac{(x)(x-1)(x-2)}{6}(0) = 0 + 2x + x^2 - x = x^2 + x\,.$$

$$f(x) = x^2 + x$$

x	y	∇	∇^2	∇^3	∇^4
0	0				
		2			
1	2		2		
		4		0	
2	6		2		0
		6		0	
3	12		2		
		8			
4	20				

Find $f(3.8)$

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{3.8-4}{1} = -0.2,$$

$$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \dots.$$

$$f(3.8) = 20 + (-0.2)(8) + \frac{(-0.2)(-0.2+1)}{2}(2) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{6}(0) = 20 - 1.6 - 0.16 + 0 = 18.24.$$

Find $f(4.2)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{4.2-4}{1} = 0.2,$

$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots .$

$f(4.2) = 20 + (0.2)(8) + \frac{(0.2)(0.2+1)}{2}(2) + \frac{(0.2)(0.2+1)(0.2+2)}{6}(0) = 20 + 1.6 + 0.16 + 0 = 21.24.$

Find $f(3.2)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.2-3}{1} = 0.2,$

$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots .$

$f(3.2) = 12 + (0.2)(6) + \frac{(0.2)(0.2+1)}{2}(2) + \frac{(0.2)(0.2+1)(0.2+2)}{6}(0) = 12 + 1.2 + 0.24 + 0 = 13.44.$

To find the function

$$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{x-4}{1} = x - 4$$

$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots .$

$f(x) = 20 + (x-4)(8) + \frac{(x-4)(x-4+1)}{2}(2) + \frac{(x-4)(x-4+1)(x-4+2)}{6}(0)$.

$f(x) = 20 + 8x - 32 + x^2 - 7x + 12 + 0 = x^2 + x.$

x	y	δ	δ^2	δ^3	δ^4
0	0				
		2			
1	2		2		
		4		0	
2	6		2		0
		6		0	
3	12		2		
		8			
4	20				

Find $f(2.4)$

$m = \frac{x_m - x_0}{h} \implies m = \frac{2.4-2}{1} = 0.4.$

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}}$$

$$y_{2.4} = \frac{6+12}{2} + (0.4-0.5)(6) + \frac{0.4(0.4-1)}{2}\left(\frac{2+2}{2}\right) + \frac{0.4(0.4-1)(0.4-0.5)}{6}(0).$$

$$y_{2.4} = 9 + (-0.6) + (-0.24) + 0 = 8.16.$$

Stirling's formulas   **صيغ ستيرلينك المركزية**

x	y	δ	δ^2	δ^3	δ^4
0	0				
		2			
1	2		2		
		4		0	
2	6		2		0
		6		0	
3	12		2		
		8			
4	20				

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{2.4} = 6 + (0.4)\left(\frac{4+6}{2}\right) + \frac{(0.4)^2}{2}(2) + \frac{0.4((0.4)^2-1)}{6}\left(\frac{0+0}{2}\right).$$

$$y_{2.4} = 6 + (0.4)(5) + (0.16) + 0 = 6 + 2 + 0.16 = 8.16.$$

To find the function

$$m = \frac{x_m - x_0}{h} \implies m = \frac{x-2}{1} = x - 2$$

1) صيغ بسل المركزية

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$y_{x-2} = \frac{6+12}{2} + (x-2-0.5)(6) + \frac{(x-2)(x-3)}{2}\left(\frac{2+2}{2}\right) + \frac{(x-2)(x-3)(x-2-0.5)}{6}(0).$$

$$y_{x-2} = 9 + (x-2.5)(6) + (x-2)(x-3) + 0$$

$$= 9 + 6x - 15 + x^2 - 5x + 6 = x^2 + x.$$

$$f(x) = x^2 + x \ .$$

2) **صيغ ستيرلينك المركزية**

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{x-2} = 6 + (x-2)\left(\frac{4+6}{2}\right) + \frac{(x-2)^2}{2}(2) + \frac{(x-2)((x-2)^2-1)}{6}\left(\frac{0+0}{2}\right)$$

$$y_{x-2} = 6 + (x-2)(5) + (x^2-4x+4) + 0 = 6 + 5x - 10 + x^2 - 4x + 4 = x^2 + x.$$

$$f(x) = x^2 + x.$$

Find $f^{-1}(1)$ & $f^{-1}(15)$

$1 = x^2 + x \Rightarrow x^2 + x - 1 = 0 \xrightarrow{\text{بالدستور}} x = -1.618$ يهمل $\quad or \quad x = 0.618.$

$15 = x^2 + x \Rightarrow x^2 + x - 15 = 0 \xrightarrow{\text{بالدستور}} x = -4.4051$ يهمل $\quad or \quad x = 3.4051.$

Example 2:

Find $f(2.4), f(5.6), f(3.7), f(4.3), f^{-1}(998)$ of the following table use the differences.

x	2	3	4	5	6
y	6	25	62	123	214

Solution:

Table of the difference.

x	y	$\Delta = \nabla = \delta$	Δ^2	Δ^3	Δ^4
2	6				
		19			
3	25		18		
		37		6	
4	62		24		0
		61		6	
5	123		30		
		91			
6	214				

Find $f(2.4)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{2.4-2}{1} = 0.4,$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(2.4) = 6 + (0.4)(19) + \frac{(0.4)(0.4-1)}{2}(18) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(6) + 0 \ .$

$f(2.4) = 6 + 7.6 - 2.16 + 0.384 = 11.824.$

Find $\boldsymbol{f(5.6)}$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{5.6-6}{1} = -0.4,$

$$f(x) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 +$$

$$f(5.6) = 214 + (-0.4)(91) + \frac{(-0.4)(-0.4+1)}{2}(30) + \frac{(-0.4)(-0.4+1)(-0.4+2)}{6}(6) + 0$$

$f(5.6) = 214 - 36.4 - 3.6 - 0.384 = 173.616.$

Find $\boldsymbol{f(3.7)}$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.7-4}{1} = -0.3,$

Stirling 's formulas **صيغ ستيرلينك المركزية**

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$f(3.7) = 62 + (-0.3)\left(\frac{37+61}{2}\right) + \frac{(-0.3)^2}{2}(24) + \frac{(-0.3)((-0.3)^2-1)}{6}\left(\frac{6+6}{2}\right)$$

$f(3.7) = 62 + (-0.3)(49) + (0.045)(24) + (0.0455)(6) = 62 - 14.7 + 1.08 + 0.273 = 48.653.$

Find $\boldsymbol{f(4.3)}$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{4.3-4}{1} = 0.3,$

Besssl 's formulas **صيغ بسل المركزية**

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$f(4.3) = \frac{62+123}{2} + (0.3-0.5)(61) + \frac{(0.3)(0.3-1)}{2}\left(\frac{24+30}{2}\right) + \frac{(0.3)(0.3-1)(0.3-0.5)}{6}(6) + 0.$$

$$f(4.3) = 92.5 + (-0.2)(61) + (-0.105)(27) + (0.042).$$

$$f(4.3) = 92.5 - 12.2 - 2.835 + 0.042 = 77.507.$$

To find the function

$$m = \frac{x_m - x_0}{h} \implies m = \frac{x-2}{1} = (x-2)$$

$$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$$

$$f(x) = 6 + (x-2)(19) + \frac{(x-2)(x-3)}{2}(18) + \frac{(x-2)(x-3)(x-4)}{6}(6) + 0 \; .$$

$$f(x) = 6 + 19x - 38 + 9x^2 - 45x + 54 + x^3 - 9x^2 + 26x - 24 = x^3 - 2\,.$$

$$f(x) = x^3 - 2$$

Find $f^{-1}(998)$

$$998 = x^3 - 2 \Rightarrow x^3 = 1000 \Rightarrow x = 10.$$

Example 3:

Find $\boldsymbol{f(1.4), f(4.7), f(2.8), f(3.3), f^{-1}(2.125), f^{-1}(134.651)}$ of the following table use the differences.

x	0	1	2	3	4	5	6
y	2	3	10	29	66	127	218

Solution:

Table of the difference.

x	y	$\Delta = \nabla = \delta$	Δ^2	Δ^3	Δ^4
0	2				
		1			
1	3		6		
		7		6	
2	10		12		0
		19		6	
3	29		18		0
		37		6	
4	66		24		0
		61		6	
5	127		30		
		91			
6	218				

Find $\boldsymbol{f(1.4)}$

$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{1.4-1}{1} = 0.4,$

$f(x) = f(x_0 + mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(1.4) = 3 + (0.4)(7) + \frac{(0.4)(0.4-1)}{2}(12) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(6) = 3 + 2.8 - 1.44 + 0.384 = 4.744.$

Find $f(4.7)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{4.7-5}{1} = -0.3,$

$$f(x) = f(x_0 + mh)$$
$$= y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 +$$

$$f(4.7) = 127 + (-0.3)(61) + \frac{(-0.3)(-0.3+1)}{2}(24) + \frac{(-0.3)(-0.3+1)(-0.3+2)}{6}(6)$$
$$= 127 - 18.3 - 2.52 - 0.357 = 105.823 \ .$$

Find $f(2.8)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{2.8-3}{1} = -0.2,$

Stirling 's formulas **صيغ ستيرلينك المركزية**

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$f(2.8) = 29 + (-0.2)\left(\frac{19+37}{2}\right) + \frac{(-0.2)^2}{2}(18) + \frac{(-0.2)((-0.2)^2-1)}{6}\left(\frac{6+6}{2}\right) + 0$$
$$= 29 + (-0.2)(28) + (0.02)(18) + (-0.032)(6) = 23.952.$$

Find $f(3.3)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.3-3}{1} = 0.3,$

Besssl 's formulas   **صيغ بسل المركزية**

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}}$$

$$+ \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$f(3.3) = \frac{29+66}{2} + (0.3-0.5)(37) + \frac{(0.3)(0.3-1)}{2}\ \left(\frac{18+24}{2}\right) +$
$\frac{(0.3)(0.3-1)(0.3-0.5)}{6}\ (6) + 0.$

$f(3.3) = 47.5\ + (-0.2)(37) +\ \ (-0.105)(21) + 0.042 = 37.937.$

To find the function

$$m = \frac{x_m - x_0}{h}\ \ \Rightarrow m = \frac{x-0}{1} = x$$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 +$
$\frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$$f(x) = 2 + (x)(1) + \frac{(x)(x-1)}{2}(6) + \frac{x(x-1)(x-2)}{6}(6)$$

$$= 2 + x + 3\,x^2 - 3x + x^3 - 3\,x^2 + 2x = x^3 + 2$$

$f(x) = x^3 + 2$

Find $f^{-1}(2.125)$ & $f^{-1}(134.651)$

$2.125 = x^3 + 2 \Rightarrow x^3 = 0.125 \Rightarrow\ x = 0.5$.

$134.651 = x^3 + 2 \Rightarrow x^3 = 132.651 \Rightarrow\ x = 5.1.$

Example 4:

Find $f(0.2), f(2.7), f(1.2), f(1.6), f^{-1}(2.8385), f^{-1}(6.0093)$ of the following table use the differences.

x	0	1	2	3
y	2	5	3	7

Solution:

Table of the difference.

x	y	$\Delta = \nabla = \delta$	Δ^2	Δ^3
0	2			
		3		
1	5		-5	
		-2		11
2	3		6	
		4		
3	7			

Find $f(0.2)$

$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{0.2-0}{1} = 0.2,$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(0.2) = 2 + (0.2)(3) + \frac{(0.2)(0.2-1)}{2}(-5) + \frac{(0.2)(0.2-1)(0.2-2)}{6}(11) + 0$.

$f(0.2) = 2 + 0.6 + 0.4 + 0.528 = 3.528.$

Find $f(2.7)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{2.7-3}{1} = -0.3,$

$$f(x) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 +$$

$f(2.7) = 7 + (-0.3)(4) + \frac{(-0.3)(-0.3+1)}{2}(6) + \frac{(-0.3)(-0.3+1)(-0.3+2)}{6}(11) + 0$.

$f(2.7) = 7 - 1.2 - 0.63 - 0.6545 = 4.5155.$

Find $f(1.2)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{1.2-1}{1} = 0.2,$

Bessl 's formulas ***صيغ بسل المركزية***

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$f(1.2) = \frac{5+3}{2} + (0.2-0.5)(-2) + \frac{(0.2)(0.2-1)}{2}\left(\frac{-5+6}{2}\right) + \frac{(0.2)(0.2-1)(0.2-0.5)}{6}(11) + 0.$

$f(1.2) = 4 + (-0.3)(-2) + (-0.08)(0.5) + (0.008)(11).$

$f(1.2) = 4 + 0.6 - 0.04 + 0.088 = 4.648.$

Find $f(1.6)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{1.6-2}{1} = -0.4,$

Stirling 's formulas ***صيغ ستيرلينك المركزية***

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$f(1.6) = 3 + (-0.4)\left(\frac{-2+4}{2}\right) + \frac{(-0.4)^2}{2}(6) + 0$$

$$= 3 + (-0.4)(1) + (0.08)(6) = 3 - 0.4 + 0.48 = 3.88.$$

To find the function

$$m = \frac{x_m - x_0}{h} \implies m = \frac{x-0}{1} = x$$

$$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$$

$$f(x) = 2 + (x)(3) + \frac{(x)(x-1)}{2}(-5) + \frac{x(x-1)(x-2)}{6}(11)$$

$$= 2 + 3x - 2.5x^2 + 2.5x + 1.8333x^3 - 5.5x^2 + 3.6667x$$

$$= 1.8333x^3 - 8x^2 + 9.1667x + 2\ .$$

Find $\boldsymbol{f^{-1}(2.8385)}$ & $\boldsymbol{f^{-1}(6.0093)}$

$2.8385 = 1.8333x^3 - 8x^2 + 9.1667x + 2$

$$\Rightarrow 1.8333x^3 - 8x^2 + 9.1667x - 0.8385 = 0 \Rightarrow x = 0.1.$$

$6.0093 = 1.8333x^3 - 8x^2 + 9.1667x + 2 \Rightarrow 1.8333x^3 - 8x^2 + 9.1667x - 4.0093 = 0 \Rightarrow x = 2.9.$

Example 5:

Find $f(2.9), f(9.8), f(6.2), f(6.6), f^{-1}(70), f^{-1}(208)$ of the following table use the differences.

x	2	4	6	8	10
y	28	54	88	130	180

Solution:

Table of the difference.

x	y	$\Delta = \nabla = \delta$	Δ^2	Δ^3	Δ^4
2	28				
		26			
4	54		8		
		34		0	
6	88		8		0
		42		0	
8	130		8		
		50			
10	180				

Find $f(2.9)$

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{2.9-2}{2} = 0.45,$$

$$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$$

$$f(2.9) = 28 + (0.45)(26) + \frac{(0.45)(0.45-1)}{2}(8) + \frac{(0.45)(0.45-1)(0.45-2)}{6}(0) \ .$$

$$f(2.9) = 28 + 11.7 - 0.99 + 0 = 38.71.$$

Find $f(9.8)$

$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{9.8-10}{2}=-0.1,$

$$f(x)=y_0+m\nabla y_0+\frac{m(m+1)}{2!}\nabla^2 y_0+\frac{m(m+1)(m+2)}{3!}\nabla^3 y_0+\frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0+$$

$f(9.8)=180+(-0.1)(50)+\frac{(-0.1)(-0.1+1)}{2}(8)+\frac{(-0.1)(-0.1+1)(-0.1+2)}{6}(0)$.

$f(9.8)=180-5-0.36+0=174.64.$

Find $f(6.2)$

$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{6.2-6}{2}=0.1,$

Stirling 's formulas **صيغ ستيرلينك المركزية**

$$y_m=y_0+m\mu\delta y_0+\frac{m^2}{2!}\delta^2 y_0+\frac{m(m^2-1)}{3!}\mu\delta^3 y_0+\frac{m^2(m^2-1)}{4!}\delta^4 y_0+\cdots$$

$$f(6.2)=88+(0.1)\left(\frac{34+42}{2}\right)+\frac{(0.1)^2}{2}(8)+\frac{0.1((0.1)^2-1)}{6}\left(\frac{0+0}{2}\right)+0.$$

$f(6.2)=88+(0.1)(38)+(0.005)(8)+0=88+3.8+0.04=91.84.$

Find $f(6.6)$

$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{6.6-6}{2}=0.3,$

$$f(6.6)=88+(0.3)\left(\frac{34+42}{2}\right)+\frac{(0.3)^2}{2}(8)+\frac{0.3((0.3)^2-1)}{6}\left(\frac{0+0}{2}\right)+0.$$

$f(6.6)=88+(0.3)(38)+(0.045)(8)+0=88+11.4+0.36=99.76.$

To find the function

$$m=\frac{x_m-x_0}{h} \Rightarrow m=\frac{x-2}{2}=(0.5x-1)$$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots .$

$$f(x) = 28 + (0.5x-1)(26) + \frac{(0.5x-1)(0.5x-2)}{2}(8)$$

$$= 28 + 13x - 26 + x^2 - 6x + 8 = x^2 + 7x + 10 .$$

$f(x) = x^2 + 7x + 10$

Find $f^{-1}(70)$ & $f^{-1}(208)$

$70 = x^2 + 7x + 10 \Rightarrow x^2 + 7x - 60 = 0 \Rightarrow x = 5.$

$208 = x^2 + 7x + 10 \Rightarrow x^2 + 7x - 198 = 0 \Rightarrow x = 11.$

<u>Example 6:</u>

Find $f(2.6), f(10.8), f(6.2), f(7.6), f^{-1}(49), f^{-1}(1409)$ of the following table use the differences.

x	2	4	6	8	10	12
y	23	93	259	569	1071	1813

<u>Solution:</u>

Table of the difference.

x	y	$\Delta = \nabla = \delta$	Δ^2	Δ^3	Δ^4	Δ^5
2	23					
		70				
4	93		96			
		166		48		
6	259		144		0	
		310		48		0
8	569		192		0	
		502		48		
10	1071		240			
		742				
12	1813					

Find $f(2.6)$

$m = \frac{x_m - x_0}{h} \implies m = \frac{2.6-2}{2} = 0.3,$

$f(x) = f(x_0 + mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(2.6) = 23 + (0.3)(70) + \frac{(0.3)(0.3-1)}{2}(96) + \frac{(0.3)(0.3-1)(0.3-2)}{6}(48)$.

$f(2.6) = 23 + 21 - 10.08 + 2.856 = 36.776.$

Find $f(10.8)$

$m = \frac{x_m - x_0}{h} \implies m = \frac{10.8-10}{2} = 0.4,$

$$f(x) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 +$$

$f(10.8) = 1071 + (0.4)(502) + \frac{(0.4)(0.4+1)}{2}(192) + \frac{(0.4)(0.4+1)(0.4+2)}{6}(48)$.

$f(10.8) = 1071 + 200.8 + 53.76 + 10.752 = 1336.312.$

Find $f(6.2)$

$m = \frac{x_m - x_0}{h} \implies m = \frac{6.2-6}{2} = 0.1,$

Stirling 's formulas   صيغ ستيرلينك المركزية

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{6.2} = f(6.2)$$

$$= 259 + (0.1)\left(\frac{166+310}{2}\right) + \frac{(0.1)^2}{2}(144) + \frac{0.1((0.1)^2-1)}{6}\left(\frac{48+48}{2}\right) + 0.$$

$f(6.2) = 259 + (0.1)(238) + (0.005)(144) + (-0.0165)(48) = 259 + 23.8 + 0.72 - 0.792 = 282.728.$

Find $f(7.6)$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{7.6-8}{2} = -0.2,$

Bessl 's formulas **صيغ بسل المركزية**

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$y_{7.6} = f(7.6) = \frac{259+569}{2} + (-0.2-0.5)(310) + \frac{(-0.2)(-0.2-1)}{2}\left(\frac{144+192}{2}\right) + \frac{(-0.2)(-0.2-1)(-0.2-0.5)}{6}(48) + 0.$

$f(7.6) = 414 + (-0.7)(310) + (0.12)(168) + (-0.028)(48).$

$f(7.6) = 414 - 217 + 20.16 - 1.344 = 215.816.$

To find the function

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{x-2}{2} = (0.5x - 1)$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$$f(x) = 23 + (0.5x-1)(70) + \frac{(0.5x-1)\big((0.5x-1)-1\big)}{2}(96) + \frac{(0.5x-1)\big((0.5x-1)-1\big)\big((0.5x-1)-2\big)}{6}(48)$$

$= 23 + 35x - 70 + 12x^2 - 72x + 96 + x^3 - 12x^2 + 44x - 48 = x^3 + 7x + 1$

$f(x) = x^3 + 7x + 1$

Find $f^{-1}(49)$ & $f^{-1}(1409)$

$49 = x^3 + 7x + 1 \Rightarrow x^3 + 7x - 48 = 0 \Rightarrow x = 3.$

$1409 = x^3 + 7x + 1 \Rightarrow x^3 + 7x - 1408 = 0 \Rightarrow x = 11.$

Relative finite differences (Divided differences) : الفروقات المنتهية النسبية

Finite differences with points of unequal dimensions to each other

الفروقات المنتهية ذات النقاط غير المتساوية الأبعاد فيما بينها

When points are not equal in dimensions (h Different between values x) we use Divided differences

$$\partial y_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \quad , i = 1,2, \cdots m-1$$

$$\partial^2 y_i = \frac{\partial y_{i+1} - \partial y_i}{x_{i+2} - x_i} \quad , i = 1,2, \cdots m-2$$

$$\partial^3 y_i = \frac{\partial^2 y_{i+1} - \partial^2 y_i}{x_{i+3} - x_i} \quad , i = 1,2, \cdots m-3$$

$$\vdots$$

$$\partial^k y_i = \frac{\partial^{k-1} y_{i+1} - \partial^{k-1} y_i}{x_{i+k} - x_i} \quad , i = 1,2, \cdots m-k$$

x	y	∂	∂^2	∂^3	∂^4	∂^5
x_0	y_0					
		∂y_0				
x_1	y_1		$\partial^2 y_0$			
		∂y_1		$\partial^3 y_0$		
x_2	y_2		$\partial^2 y_1$		$\partial^4 y_0$	
		∂y_2		$\partial^3 y_1$		$\partial^5 y_0$
x_3	y_3		$\partial^2 y_2$		$\partial^4 y_1$	
		∂y_3		$\partial^3 y_2$		
x_4	y_4		$\partial^2 y_3$			
		∂y_4				
x_5	y_5					

Its general formula :-

$$f(x_m) = y_m = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$

Example 9:

Find $f(2), f(5), f(-1)$ of the following table use the Divided differences and find To find the function

x	-2	1	3	4	6
y	-3	0	22	57	205

Solution:

$\partial y_0 = \frac{0-(-3)}{1-(-2)}=1$, $\partial y_1 = \frac{22-0}{3-1}=11$, $\partial y_2 = \frac{57-22}{4-3}=35$, $\partial y_3 = \frac{205-57}{6-4}=74$.

$\delta^2 y_0 = \frac{11-1}{3-(-2)}=2$, $\delta^2 y_1 = \frac{35-11}{4-1}=8$, $\delta^2 y_2 = \frac{73-35}{6-3}=13$.

$\delta^3 y_0 = \frac{8-2}{4-(-2)}=1$, $\delta^3 y_1 = \frac{13-8}{6-1}=1$ and $\delta^4 y_0 = \frac{1-1}{6-(-2)}=0$.

x	y	δ	δ^2	δ^3	δ^4
-2	-3				
		1			
1	0		2		
		11		1	
3	22		8		0
		35		1	
4	57		13		
		74			
6	205				

$$f(x_m) = y_m = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$

$$f(2) = -3 + (2 - (-2))(1) + (2 - (-2))(2 - 1)(2) + (2 - (-2))(2 - 1)(2 - 3)(1).$$

$$f(2) = -3 + 4 + 8 - 4 = 5\ .$$

$$f(5) = -3 + (5 - (-2))(1) + (5 - (-2))(5 - 1)(2) + (5 - (-2))(5 - 1)(5 - 3)(1).$$

$$f(5) = -3 + 7 + 56 + 56 = 116\ .$$

$$f(-1) = -3 + ((-1) - (-2))(1) + ((-1) - (-2))((-1) - 1)(2) + ((-1) - (-2))((-1) - 1)((-1) - 3)(1).$$

$$f(5) = -3 + 1 - 4 + 8 = 2\ .$$

To find the function

$$f(x) = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$

$$f(x) = -3 + (x - (-2))(1) + (x - (-2))(x - 1)(2) + (x - (-2))(x - 1)(x - 3)(1).$$

$$f(x) = -3 + x + 2 + 2x^2 + 2x - 4 + x^3 - 2x^2 - 5x + 6.$$
$$f(x) = x^3 - 2x + 1.$$

Example 10:

Find $f(2), f(3)$ of the following table use the Divided differences and find To find the function

x	0	1	4	6
y	-10	20	14	30

Solution:

$\partial y_0 = \frac{20-(-10)}{1-0}$ =30 , $\partial y_1 = \frac{14-20}{4-1}$ =-2 , $\partial y_2 = \frac{30-14}{6-2}$ =8 .

$\delta^2 y_0 = \frac{(-2)-30}{4-0}$ =-8 , $\delta^2 y_1 = \frac{8-(-2)}{6-1}$ =2 , $\delta^3 y_0 = \frac{2-(-8)}{6-0} = \frac{10}{6}$ =1.6667 .

x	y	δ	δ^2	δ^3
0	-10			
		30		
1	20		-8	
		-2		$\frac{5}{3} = 1.6667$
4	14		2	
		8		
6	30			

$$f(x_m) = y_m = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$

$$f(2) = y_0 + (2 - x_0)\partial y_0 + (2 - x_0)(2 - x_1)\partial^2 y_0 + (2 - x_0)(2 - x_1)(2 - x_2)\partial^3 y_0 + \cdots$$

$$f(2) = -10 + (2 - 0)(30) + (2 - 0)(2 - 1)(-8) + (2 - 0)(2 - 1)(2 - 4)(1.6667) .$$

$$f(2) = -10 + 60 - 16 + (-4)(1.6667) = 34 - 6.6668 = 27.3332$$

$$f(3) = y_0 + (3 - x_0)\partial y_0 + (3 - x_0)(3 - x_1)\partial^2 y_0 + (3 - x_0)(3 - x_1)(3 - x_2)\partial^3 y_0 + \cdots$$

$$f(3) = -10 + (3 - 0)(30) + (3 - 0)(3 - 1)(-8) + (3 - 0)(3 - 1)(3 - 4)(1.6667) .$$

$$f(3) = -10 + 90 - 48 + (-6)(1.6667) = 32 - 10.0002 = 21.9998$$

To find the function

$f(x) = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$
$f(x) = -10 + (x - 0)(30) + (x - 0)(x - 1)(-8) + (x - 0)(x - 1)(x - 4)(1.6667)$.
$f(x) = -10 + 30x - 8x^2 + 8x + (1.6667)x^3 - 5(1.6667)x^2 + 4(1.6667)x$
$f(x) = (1.6667)x^3 - (16.3335)x^2 + (44.6667)x - 10$.

Homework

Find $f(32), f(43)$ of the following table use the Divided difference and find To find the function .

x	30	35	40
y	0.5	0.5736	0.6428

Homework

Find $f(0.8), f(4.1), f(5.7)$ of the following table use the Divided difference and find To find the function .

x	1	2	3	4	5	6
y	0	0.7	1.1	1.4	1.6	1.7

Example 11:

Find $f^{-1}(205), f'(2), f''(4)$ of the following table use the Divided differences .

x	1	3	4	5
y	0	22	57	116

Solution:

$\partial y_0 = \frac{22-0}{3-1} = 11$, $\partial y_1 = \frac{57-22}{4-3} = 35$, $\partial y_2 = \frac{116-57}{5-4} = 59$.

$\delta^2 y_0 = \frac{35-11}{4-1} = 8$, $\delta^2 y_1 = \frac{59-35}{5-3} = 12$. $\delta^3 y_0 = \frac{12-8}{5-1} = 1$.

x	y	δ	δ^2	δ^3
1	0			
		11		
3	22		8	
		35		1
4	57		12	
		59		
5	116			

$$f(x_m) = y_m = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$

$$f(x) = 0 + (x - 1)(11) + (x - 1)(x - 3)(8) + (x - 1)(x - 3)(x - 4)(1).$$

$$f(x) = 11x - 11 + 8x^2 - 32x + 24 + x^3 - 4x^2 + 3x - 4x^2 + 16x - 12.$$

$$f(x) = x^3 - 2x + 1.$$

$$f(x) = x^3 - 2x + 1 \Rightarrow \; f'(x) = 3x^2 - 2 \Rightarrow \; f''(x) = 6x.$$

$$\Rightarrow \; f'(2) = 3(2)^2 - 2 = 10 \; and \; f''(4) = 6(4) = 24.$$

$$f(x) = x^3 - 2x + 1 \Rightarrow \; f(x) = 205 \Rightarrow \; f^{-1}(205) = ? \Rightarrow \; 205 = x^3 - 2x + 1.$$

$$\Rightarrow \; x^3 - 2x - 204 = 0 \Rightarrow \; x = 6. \therefore f^{-1}(205) = 6.$$

Example 12:

Find $f^{-1}(1.64), f^{-1}(0.18)$ of the following table use the Divided differences .

x	1	2	3	4	5	6
y	0	0.7	1.1	1.4	1.6	1.7

Solution:

$$\partial y_0 = \frac{22-0}{3-1} = 11 \;, \partial y_1 = \frac{57-22}{4-3} = 35 \;, \partial y_2 = \frac{116-57}{5-4} = 59 \;.$$

$\delta^2 y_0 = \frac{35-11}{4-1} = 8, \delta^2 y_1 = \frac{59-35}{5-3} = 12$. $\delta^3 y_0 = \frac{12-8}{5-1} = 1$.

x	y	δ	δ^2	δ^3
1	0			
		11		
3	22		8	
		35		1
4	57		12	
		59		
5	116			

$$f(x_m) = y_m = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$

$$f(x) = 0 + (x-1)(11) + (x-1)(x-3)(8) + (x-1)(x-3)(x-4)(1).$$

$$f(x) = 11x - 11 + 8x^2 - 32x + 24 + x^3 - 4x^2 + 3x - 4x^2 + 16x - 12.$$

$f(x) = x^3 - 2x + 1.$

$$f(x) = x^3 - 2x + 1 \Longrightarrow \ f'(x) = 3x^2 - 2 \ \Longrightarrow \ f''(x) = 6x.$$

$$\Longrightarrow \ f'(2) = 3(2)^2 - 2 = 10 \ and \ \ f''(4) = 6(4) = 24.$$

$$f(x) = x^3 - 2x + 1 \Longrightarrow \ f(x) = 205 \Longrightarrow \ f^{-1}(205) = ? \Longrightarrow \ 205 = x^3 - 2x + 1.$$

$$\Longrightarrow \ x^3 - 2x - 204 = 0 \Longrightarrow \ x = 6. \therefore f^{-1}(205) = 6.$$

Homework

Find $f^{-1}(205), f'(2), f''(4)$ of the following table use the Divided differences .

x	1	2	3	4	5	6
y	0	0.7	1.1	1.4	1.6	1.7

Homework

Find $f(4.1), f(3.2), f^{-1}(1.5)$ of the following table use the Divided differences .

x	1	2	3	4	5	6
y	0	0.7	1.1	1.4	1.6	1.7

The general formula of polynomial $P_m(x)$ (Lagrange polynomial)

$$P_m(x) = \sum_{j=0}^{m} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{m} \frac{(x-x_i)}{(x_j-x_i)}$$

<u>Forward differences</u>

$$y_m = f(x_0+mh)$$
$$= y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \cdots$$

<u>Backward differences</u>

$$y_m = f(x_0+mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \cdots$$

<u>Central differences</u>

Bessl's formulas

$$y_m = \mu y_{\frac{1}{2}} + \left(m-\frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

Stirling's formulas **صيغ ستيرلينك المركزية**

Its general formula :- صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

Divided differences

Its general formula :-

$$f(x_m) = y_m = y_0 + (x - x_0)\partial y_0 + (x - x_0)(x - x_1)\partial^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\partial^3 y_0 + \cdots$$