Numerical Analysis

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Chapter Six

Numerical Solution of Ordinary Differential Equations

In this chapter we present several methods by which we can find numerical solutions for differential equations. These numerical solutions become necessary when we can not find the exact solution of the equations.

نقدم في هذا الفصل عدة طرق يمكننا من خلالها إيجاد الحلول العددية للمعادلات التفاضلية. تصبح هذه الحلول العددية ضرورية عندما لا نتمكن من إيجاد الحل الدقيق للمعادلات.

The general formula of the differential equation of the first order $y' = f(x,y), y(x_0) = y_0$ and The solution of this equation is $y = \int f(x,y)\,dx + c$.

الصيغة العامة للمعادلة التفاضلية من الدرجة الأولى $y' = f(x,y), y(x_0) = y_0$ وحل هذه المعادلة هو $y = \int f(x,y)\,dx + c$.

The equations of the second order and more will be converted into a system of first-order equations. سيتم تحويل المعادلات من الدرجة الثانية وأكثر إلى نظام معادلات من الدرجة الأولى

Numerical Methods for Solving Ordinary Differential Equations

الطرق العددية لحل المعادلات التفاضلية العادية

1. **Taylor Series Method طريقة سلسلة تايلور**
2. **Euler's Method طريقة اويلر**
3. **Runge -Kutta Method طريقة رونج -كوتا**

1. Taylor Series Method طريقة سلسلة تايلور

Let $x_i = x_0 + ih$, i $= 1,2,3,\dots,n$,The Taylor series about point x_0 is

لنفترض أن متسلسلة تايلور حول النقطة x_0 هي

$$f(x) = f(x_0) + (x-x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \dots \qquad \dots(1)$$

To find the solution of equation by using Taylor Series method at the point x_1. we put $x = x_1$ in (1) we get :

لإيجاد حل المعادلة باستخدام طريقة متسلسلة تايلور عند النقطة $x = x_1$. نضع في (1) نحصل على:

$f(x_1) = f(x_0 + h) = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{(x_1-x_0)^2}{2!}f''(x_0) + \frac{(x_1-x_0)^3}{3!}f'''(x_0) + \cdots + \frac{(x_1-x_0)^{(k-1)}}{(k-1)!}f^{(k-1)}(x_0) + \frac{(x_1-x_0)^{(k)}}{(k)!}f^{(k)}(x_0) + \cdots$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots + \frac{h^{k-1}}{(k-1)!}y_0^{(k-1)} + R_k$$

$$R_k = \frac{h^k}{k!}y_0^{(k)}(\theta) \ , \quad x_0 \le \theta \le x_1$$

And to find the solution of equation by using Taylor Series Method at the point x_2

وإيجاد حل المعادلة باستخدام طريقة متسلسلة تايلور عند النقطة x_2

(i) if $x_2 = x_0 + 2h$ then :

$$y_2 = y_0 + 2hy_0' + \frac{(2h)^2}{2!}y_0'' + \frac{(2h)^3}{3!}y_0''' + \ldots + \frac{(2h)^{k-1}}{(k-1)!}y_0^{(k-1)} + R_k$$

Note that the error is increasing

(ii) if $x_2 = x_1 + h$ then :

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots + \frac{h^{k-1}}{(k-1)!}y_1^{(k-1)} + R_k$$

Similarly to $x_3, x_4, \ldots$

In general $y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \ldots + \frac{h^{k-1}}{(k-1)!}y_n^{(k-1)} + R_k$

Example 1 :

Use the Taylor series method to obtain a solution of $y' = x - y$, $y(0) = 1$, $h = 0.1$ at points 0.1, 0.2, 0.3 (use the **Taylor series method** up to $y^{(4)}$)

Solution:-

$x_0 = 0\ , y_0 = 1\ , h = 0.1\ , x_1 = 0.1$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{(4)} + R_k$$

$y' = x - y \Rightarrow y_0' = x_0 - y_0 \Rightarrow y_0' = 0 - 1 \Rightarrow y_0' = -1$

$y'' = 1 - y' \Rightarrow y_0'' = 1 - y_0' \Rightarrow y_0'' = 1 - (-1) = 2$

$y''' = -y'' \Rightarrow y_0''' = -y_0'' \Rightarrow y_0''' = -(2) = -2$

$y^{(4)} = -y''' \Rightarrow y_0^{(4)} = -y''' \Rightarrow y_0^{(4)} = -(-2) \Rightarrow y_0^{(4)} = 2$

$y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(-2) + \frac{(0.1)^4}{4!}(2) \Rightarrow y_1 = 0.909675$

$x_0 = 0\,, y_0 = 1\,, h = 0.1\,, x_2 = 0.2$

(i) if $x_2 = x_0 + 2h$ then :

$y_2 = y_0 + 2hy_0' + \frac{(2h)^2}{2!}y_0'' + \frac{(2h)^3}{3!}y_0''' + \frac{(2h)^4}{(4)!}y_0^{(4)} + R_k$

$y_2 = 1 + (0.2)(-1) + \frac{(0.2)^2}{2!}(2) + \frac{(0.2)^3}{3!}(-2) + \frac{(0.2)^4}{4!}(2) \Rightarrow y_2 = 0.837463$

(ii) if $x_2 = x_1 + h$, $x_1 = 0.1$, $y_1 = 0.909675$, $h = 0.1$ then :

$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots + \frac{h^{k-1}}{(k-1)!}y_1^{(k-1)} + R_k$

$y' = x - y \Rightarrow y_1' = x_1 - y_1 \Rightarrow y_1' = 0.1 - 0.909675 \Rightarrow y_1' = -0.809675$

$y'' = 1 - y' \Rightarrow y_1'' = 1 - y_1' \Rightarrow y_1'' = 1 - (-0.809675) = 1.809675$

$y''' = -y'' \Rightarrow y_1''' = -y_1'' \Rightarrow y_1''' = -(1.809675) \Rightarrow y''' = -1.809675$

$y^{(4)} = -y''' \Rightarrow y_1^{(4)} = -y_1''' \Rightarrow y_1^{(4)} = -(-1.809675) \Rightarrow y_1^{(4)} = 1.809675$

$y_2 = 0.909675 + (0.1)(0.809675) + \frac{(0.1)^2}{2!}(1.809675) + \frac{(0.1)^3}{3!}(-1.809675)$

$+ \frac{(0.1)^4}{4!}(1.809675) + R^k \Rightarrow y_2 = 0.837462$

Homework $x_3 = 0.3$.

2. Euler's Method

We assume that the solution $x = f(y)$ is continuous and differentiable and that the starting point is (x_0, y_0) The solution is required when $x = x'$, so the interval must be divided into n of parts with width h (but small), $h = (x` - x)/n$, where

Then we apply the law

$y_{i+1} = y_i + h\,y'_i = y_i + h * f(x_i, y_i) + R_k, i = 0,1,..n-1.$

طريقة اويلر نفترض أن الحل $x = f(y)$ متصل وقابل للتفاضل وأن نقطة البداية هي (x_0, y_0) وأن الحل مطلوبا عند $x = x'$ لذلك يجب تجزئة الفترة إلى n من الاجزاء بعرض (w على أن تكون صغيرة $h = (x` - x)/n$ حيث ثم نطبق القانون) $y_{i+1} = y_i + h\,y'_i = y_i + h * f(x_i, y_i), i = 0,1,..n-1$

Example 2 :

Use the Taylor series method to obtain a solution of $y' = x - y, y(0) = 1, h = 0.1$ at points 0.1, 0.2, 0.3 (use the **Euler's method**)

Solution:-

$y_{i+1} = y_i + h\,y'_i = y_i + h * f(x_i, y_i), i = 0,1,..n-1$

$y_1 = y_0 + h * f(x_0, y_0) = 1 + (0.1)(0-1) = 1 - 0.1 = 0.9.$

$y_2 = y_1 + h * f(x_1, y_1) = 0.9 + (0.1)(0.1 - 0.9) = 0.82.$

$y_3 = y_2 + h * f(x_2, y_2) = 0.82 + (0.1)(0.2 - 0.82) = 0.758.$

$y_4 = y_3 + h * f(x_3, y_3) = 0.758 + (0.1)(0.3 - 0.758) = 0.7122.$

Euler Extended More method

Taking the fourth term of the Taylor series we obtain MEEM. In general

$y_{i+1} = y_i + h\ f(x_i, y_i) + \frac{h^2}{2!} * f'(x_i, y_i) + \frac{h^3}{3!} * f''(x_i, y_i) + R_k, i = 0,1,..n-1.$

طريقة اويلر الاكثر امتدادا: بأخذ الحد الرابع من متسلسلة تايلور نحصل على.

Method Euler Modified

In this method we exploit EM to find Predictor $f'(x_i, y_i)$ and then use it .

To calculate the average $\left(f(x_i, y_i) + f(x_i, y_i)\right)/2$ We calculate again (Corrector)

$y_{i+1} = y_i + \frac{h}{2}\ (f'(x_i, y_i) + f'(x_{i+1}, y_{i+1}) + R_k.$

طريقة اويلر المعدلة في هذه الطريقة نستغل EM لإيجاد Predictor $f'(x_i, y_i)$ ثم نستخدمها لحساب المتوسط .

Example 3 :

Use the Taylor series method to obtain a solution of $y' = x - y, y(0) = 1, h = 0.1$ at points 0.1, 0.2, 0.3 (use the **Method Euler Modified**)

Solution:-

$y_{i+1} = y_i + h\, y'_i = y_i + h * f(x_i, y_i), i = 0,1,..n-1$

$y_1 = y_0 + \frac{h}{2} * \left(f'(x_0, y_0) + f'(x_1, y_1)\right) = 1 + \left(\frac{0.1}{2}\right)\left((0-1) + (0.1 - 0.9)\right)$
$= 1 + (0.05)(-1 - 0.8) = 0.91.$

$y_1 = y_0 + \frac{h}{2} * \left(f'(x_0, y_0) + f'(x_1, y_1)\right) = 1 + \left(\frac{0.1}{2}\right)\left((0-1) + (0.1 - 0.91)\right)$
$= 1 + (0.05)(-1 - 0.81) = 0.9095.$

$y_1 = y_0 + \frac{h}{2} * \left(f'(x_0, y_0) + f'(x_1, y_1)\right) = 1 + \left(\frac{0.1}{2}\right)\left((0-1) + (0.1 - 0.9095)\right)$
$= 1 + (0.05)(-1 - 0.1905) = 0.909525.$

$y_1 = y_0 + \frac{h}{2} * \left(f'(x_0, y_0) + f'(x_1, y_1)\right) = 1 + \left(\frac{0.1}{2}\right)\left((0-1) + (0.1 - 0.909525)\right)$
$= 0.9095238.$

$y_1 = y_0 + \frac{h}{2} * \left(f'(x_0, y_0) + f'(x_1, y_1)\right) = 1 + \left(\frac{0.1}{2}\right)\left((0-1) + (0.1 - 0.9095238)\right)$
$= 0.9095238.$

Homework

Use the Taylor series method to obtain a solution of $y' = \frac{1}{\sqrt{x+y}}$, $y(0) = 1$, $h = 0.1$ at points $x_1 = 1$(use the **Taylor series method**).

3. Runge -Kutta Method- طريقة رونج -كوتا

The Runge Kutta method is the most widely used method of solving differential equations . It differs from the Taylor series method in that we use values of the first derivative of $f(x, y)$ at several points instead of the values of successive derivatives at a one point.

طريقة رونج كوتا هي الطريقة الأكثر استخدامًا لحل المعادلات التفاضلية. وهي تختلف عن طريقة متسلسلة تايلور في أننا نستخدم قيم المشتقة الأولى عند عدة نقاط بدلاً من قيم المشتقات $f(x, y)$ المتعاقبة عند نقطة واحدة.

Runge-Kutta Method of Order 2:طريقة رونج-كوتا من الأمر

$k_1 = hf(x_n, y_n)$
$k_2 = hf(x_n + h, y_n + k_1)$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) + R_k$$

A- **Runge-Kutta Method of Order 3:طريقة رونج-كوتا من الأمر**

$$I_1 = hf(x_n, y_n)$$
$$I_2 = hf(x_n + \frac{h}{2}, y_n + \frac{I_1}{2})$$
$$I_3 = hf(x_n + h, y_n + 2(I_2 - I_1))$$
$$y_{n+1} = y_n + \frac{1}{6}(I_1 + 4I_2 + I_3)$$

B- **Runge-Kutta Method of Order 4:طريقة رونج-كوتا من الأمر**

$$m_1 = hf(x_n, y_n)$$
$$m_2 = hf(x_n + \frac{h}{2}, y_n + \frac{m_1}{2})$$
$$m_3 = hf(x_n + \frac{h}{2}, y_n + \frac{m_2}{2})$$
$$m_4 = hf(x_n + h, y_n + m_3)$$
$$y_{n+1} = y_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

Example 4 :-

Find a solution equation $f(x, y) = y' = x - y$, $y(0) = 1$ by using **Runge-Kutta method** of order 2,3 and 4 with $h = 0.1$

Solution:-

A- Runge-Kutta Method of Order 2:

$x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = 0.1$

$$k_1 = hf(x_n, y_n) \Rightarrow k_1 = hf(x_0, y_0) \Rightarrow k_1 = (0.1)(x_0 - y_0) \Rightarrow k_1 = -0.1$$
$$k_2 = hf(x_n + h, y_n + k_1) \Rightarrow k_2 = hf(x_0 + h, y_0 + k_1)$$
$$k_2 = (0.1)f\big(0.1, 1 + (-0.1)\big) \Rightarrow k_2 = (0.1)(0.1 - 0.9) \Rightarrow k_2 = -0.08$$
$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) + R_k \Rightarrow y_1 = y_0 + \frac{1}{2}(k_1 + k_2) + R_k$$
$$y_1 = 1 + \tfrac{1}{2}(-0.1 - 0.08) + R_k \Rightarrow y_1 = 0.91 + R_k.$$

B-Runge-Kutta Method of Order 3:

$$I_1 = hf(x_n, y_n) \Rightarrow I_1 = hf(x_0, y_0) \Rightarrow I_1 = (0.1)(x_0 - y_0) \Rightarrow k_1 = -0.1$$

$$I_2 = hf(x_n + \frac{h}{2}, y_n + \frac{I_1}{2}) \Rightarrow I_2 = (0.1)f\left(x_0 + \frac{(0.1)}{2}, y_0 + \frac{(-0.1)}{2}\right)$$

$$I_2 = (0.1)f(0.05,0.95) \Rightarrow I_2 = (0.1)(0.05 - 0.95) \Rightarrow I_2 = -0.09$$

$$I_3 = hf(x_n + h, y_n + 2(I_2 - I_1)) \Rightarrow I_3 = (0.1)f(x_0 + 0.1, y_0 + 2(-0.09 - (-0.1)))$$

$$I_3 = (0.1)f(0.1,1.2) \Rightarrow I_3 = (0.1)(0.1 - 1.2) \Rightarrow I_3 = -0.092$$

$$y_{n+1} = y_n + \frac{1}{6}(I_1 + 4I_2 + I_3) \Rightarrow y_1 = y_0 + \frac{1}{6}(I_1 + 4I_2 + I_3)$$

$$y_1 = 1 + \tfrac{1}{6}\big(-0.1 + 4(-0.09) + (-0.092)\big) \Rightarrow y_1 = 0.908.$$

C-Runge-Kutta Method of Order 4:

$$m_1 = hf(x_n, y_n) \Rightarrow m_1 = hf(x_0, y_0) \Rightarrow m_1 = (0.1)(x_0 - y_0) \Rightarrow m_1 = -0.1$$

$$m_2 = hf(x_n + \frac{h}{2}, y_n + \frac{m_1}{2}) \Rightarrow m_2 = (0.1)f\left(x_0 + \frac{(0.1)}{2}, y_0 + \frac{(-0.1)}{2}\right)$$

$$m_2 = (0.1)f(0.05,0.95) \Rightarrow m_2 = (0.1)(0.05 - 0.95) \Rightarrow m_2 = -0.09$$

$$m_3 = hf(x_n + \frac{h}{2}, y_n + \frac{m_2}{2}) \Rightarrow m_3 = (0.1)f(x_0 + \frac{(0.1)}{2}, y_0 + \frac{(-0.09)}{2})$$

$$m_3 = (0.1)f(0.05,0.955) \Rightarrow m_3 = (0.1)(0.05 - 0.955) \Rightarrow m_3 = -0.0905$$

$$m_4 = hf(x_n + h, y_n + m_3) \Rightarrow m_4 = (0.1)f(x_0 + 0.1, y_0 + (-0.0905))$$

$$m_4 = (0.1)f(0.1,0.9095) \Rightarrow m_4 = (0.1)(0.1 - 0.9095) = -0.08095$$

$$y_{n+1} = y_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$y_1 = 1 + \frac{1}{6}\big(-0.1 + 2(-0.09) + 2(-0.0905) + (-0.08095)\big)$$

$$y_1 = 0.909675 + R^k.$$

Homework: -

Find a solution equation $f(x,y) = y' = -xy$, $y(0) = 1$ by using Runge-Kutta method of order 2,3 and 4 with $h = 0.1$ at point $x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$.

Equations of the higher order and system of equations of the first order

معادلات من الدرجة العليا ونظام المعادلات من الدرجة الأولى

The general formula for the equation of order (m) is $y^{(m)} = f(x, y, y', y'', \dots, y^{(m-1)})$ and the simplest way to solve them is to convert them to system of equations of the first order contains (m) from equations as follows:-

الصيغة العامة لمعادلة الرتبة (م) هي

وأبسط طريقة لحلها هو تحويلها إلى نظام معادلات من الدرجة الأولى يحتوي على (م) من المعادلات كما يلي:-

$y = y_1$, $y' = y_2$, $y'' = y_2$, $y^{(m-1)} = y_m$, So $y_m' = f(x, y_1, y_2, y_3, \dots, y_m)$.

Example :

$y'' = 2xy + y$ let $y' = z$ we get $z' = 2xz + y$

Example :

$y^{(4)} = 3x^2y'' - 2y'''$ let $y' = z$, $z' = h$, $h' = k$ we get $k' = 3x^2h - 2k$.

To solve a system of first-order equations, we will summarize the methods preceding a system consisting of two equations, In the same way if the system is composed (m) from equations.

لحل نظام من المعادلات من الدرجة الأولى سنلخص الطرق السابقة لنظام مكون من معادلتين، بنفس الطريقة إذا كان النظام مكون من (م) من المعادلات

$y' = f(x, y, z)$, $z' = g(x, y, z)$, $y(x_0) = y_0$, $z(x_0) = z_0$

Numerical Methods for Solving system of equations of the first order

الطرق العددية لحل نظام المعادلات من الدرجة الأولى

1.Taylor Series Method طريقة سلسلة تايلور

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \dots + \frac{h^{k-1}}{(k-1)!}y_n^{(k-1)} + R_k$$

$$R_k = \frac{h^k}{k!}y_0^{(k)}(\theta) \;, \quad x_0 \le \theta \le x_1.$$

$$z_{n+1} = z_n + hz_n' + \frac{h^2}{2!}z_n'' + \frac{h^3}{3!}z_n''' + \dots + \frac{h^{k-1}}{(k-1)!}z_n^{(k-1)} + R_k \,.$$

$$R_k = \frac{h^k}{k!}z_0^{(k)}(\theta) \;, \quad x_0 \le \theta \le x_1.$$

2- Runge -Kutta Method

2- طريقة رونج -كوتا

أ- طريقة رونج-كوتا من الأمر 2:

A- **Runge-Kutta Method of Order 2:**

$k_1 = hf(x_n, y_n, z_n)$

$L_1 = hg(x_n, y_n, z_n)$

$k_2 = hf(x_n + h\,, y_n + k_1\,, z_n + L_1)$

$L_2 = hg(x_n + h\,, y_n + k_1\,, z_n + L_1)$

$$y_{n+1} = y_n + \frac{1}{2}\,(k_1 + k_2)\, + R_k$$

$$z_{n+1} = z_n + \frac{1}{2}\,(L_1 + L_2)\, + R_k$$

B- Runge-Kutta Method of Order 3:

$I_1 = hf(x_n\,, y_n\,, z_n)$

$J_1 = hg(x_n\,, y_n\,, z_n)$

$$I_2 = hf(x_n + \frac{h}{2}\,, y_n + \frac{I_1}{2}\,, z_n + \frac{J_1}{2})$$

$$J_2 = hg(x_n + \frac{h}{2}\,, y_n + \frac{I_1}{2}\,, z_n + \frac{J_1}{2})$$

$I_3 = hf(x_n + h\,, y_n + 2(I_2 - I_1), z_n + 2(J_2 - J_1))$

$J_3 = hg(x_n + h\,, y_n + 2(I_2 - I_1), z_n + 2(J_2 - J_1))$

$$y_{n+1} = y_n + \frac{1}{6}(I_1 + 4I_2 + I_3)$$

$$z_{n+1} = z_n + \frac{1}{6}(J_1 + 4J_2 + J_3)$$

C- Runge-Kutta Method of Order 4:

$m_1 = hf(x_n\,, y_n\,, z_n)$

$p_1 = hg(x_n\,, y_n\,, z_n)$

$$m_2 = hf(x_n + \frac{h}{2}\,, y_n + \frac{m_1}{2}\,, z_n + \frac{p_1}{2})$$

$$p_2 = hg(x_n + \frac{h}{2}\,, y_n + \frac{m_1}{2}\,, z_n + \frac{p_1}{2})$$

$$m_3 = hf(x_n + \frac{h}{2}\,, y_n + \frac{m_2}{2}\,, z_n + \frac{p_2}{2})$$

$$p_3 = hg(x_n + \frac{h}{2}, y_n + \frac{m_2}{2}, z_n + \frac{p_2}{2})$$
$$m_4 = hf(x_n + h, y_n + m_3, z_n + p_3)$$
$$p_4 = hg(x_n + h, y_n + m_3, z_n + p_3)$$
$$y_{n+1} = y_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$
$$z_{n+1} = z_n + \frac{1}{6}(p_1 + 2p_2 + 2p_3 + p_4)$$

Example 5 :

Use the Taylor series method to obtain a solution of $y'' = x\ (y')^2 - y^2$ when $y(0) = 1$, $y'(0) = 0$, $h = 1$ (use the Taylor series method up to $y^{(3)}$ and $z^{(3)}$) .

Solution:-

Let $y' = z$, $x_0 = 0$, $y_0 = 1$, $z_0 = y'_0 = 0$, $h = 1$
$$y'' = x(y')^2 - y^2 \Rightarrow z' = xz^2 - y^2$$
$$y''_0 = z'_0 = x_0(y'_0)^2 - {y_0}^2 = 0 - 1 = -1$$
$$y''' = 2xy'y'' + (y')^2 - 2yy'$$
$$z'' = 2xzz' + z^2 - 2yz$$
$$y'''_0 = z''_0 = 2x_0y'_0y''_0 + (y'_0)^2 - 2y_0y'_0 = 0$$
$$y^{'''' \ ''' }(y'')^{2\ '\ '\ ''\ ''}\ (y')^2$$
$$z''' = 2x(zz'' + (z')^2) + 2zz' - 2yz' - 2z^2$$
$$y^{''''\ '''}{}_{0_0}$$
$$y_1 = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y_0^{(4)} + R_k$$
$$y_1 = 1 + 1(0) + \frac{(1)^2}{2!}(-1) + \frac{(1)^3}{3!}(0) + R_k \Rightarrow y_1 = \frac{1}{2} + R_k$$
$$z_1 = z_0 + hz'_0 + \frac{h^2}{2!}z''_0 + \frac{h^3}{3!}z'''_0 + \frac{h^4}{4!}z_0'''' + R_k$$
$$z_1 = 0 + (1)(-1) + \frac{1}{2!}(0) + \frac{h^3}{3!}(2) + R_k \Rightarrow z_1 = \frac{-2}{3} + R_k$$

$$\therefore (x_1, y_1, z_1) = \left(1, \frac{1}{2}, \frac{-2}{3}\right)$$

Homework :

Use the Taylor series method to obtain a solution of $y'' = xy$ when $y(0) = 0$, $y'(0) = 1$ at points 0.5 (use the Taylor series method up to y''' and z''')

Example 5 :-

Find a solution of system $\frac{dy}{dx} = -x + y + z$, $\frac{dz}{dx} = 2x + 3y - z$ when $y(0) = 1$, z(0) = 2 by using **Runge-Kutta method** of order 2,3 *and* 4.

Solution :-

$x_0 = 0$, $y_0 = 1$, $z_0 = 2$, $h = 0.1$

$y' = f(x, y, z) = -x + y + z$

$z' = g(x, y, z) = 2x + 3y - z$

A- Runge-Kutta Method of Order 2:

$k_1 = hf(x_n, y_n, z_n)$

$k_1 = (0.1)(-x_0 + y_0 + z_0) \Rightarrow k_1 = (0.1)(0 + 1 + 2) \Rightarrow k_1 = 0.3$

$L_1 = hg(x_n, y_n, z_n)$

$L_1 = (0.1)(2x_0 + 3y_0 - z_0) \Rightarrow L_1 = (0.1)(0 + 3_0 - 2) \Rightarrow L_1 = 0.1$

$k_2 = hf(x_n + h, y_n + k_1, z_n + L_1) \Rightarrow k_2 = (0.1)f(0 + 0.1, 1 + 0.3, 2 + 0.1)$

$k_2 = (0.1)f(0.1, 1.3, 2.1) \Rightarrow k_2 = (0.1)(-0.1 + 1.3 + 2.1) \Rightarrow k_2 = 3.3$

$L_2 = hg(x_n + h, y_n + k_1, z_n + L_1) \Rightarrow L_2 = (0.1)g(0 + 0.1, 1 + 0.3, 2 + 0.1)$

$L_2 = (0.1)g(0.1, 1.3, 2.1) \Rightarrow L_2 = (0.1)(2(0.1) + 3(1.3) - 2.1) \Rightarrow L_2 = 0.2$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) + R_k$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) + R_k \Rightarrow y_1 = 1 + \frac{1}{2}(0.3 + 3.3) + R_k \Rightarrow y_1 = 2.8 + R_k$$

$$z_{n+1} = z_n + \frac{1}{2}(L_1 + L_2) + R_k$$

$$z_1 = z_0 + \frac{1}{2}(L_1 + L_2) + R_k \Rightarrow z_1 = 2 + \frac{1}{2}(0.1 + 0.2) + R_k \Rightarrow z_1 = 2.15$$

B- Runge-Kutta Method of Order 3:

$I_1 = hf(x_n , y_n , z_n)$

$I_1 = (0.1)(-x_0 + y_0 + z_0) \Rightarrow I_1 = (0.1)(0 + 1 + 2) \Rightarrow I_1 = 0.3$

$J_1 = hg(x_n , y_n , z_n)$

$J_1 = (0.1)(2x_0 + 3y_0 - z_0) \Rightarrow J_1 = (0.1)(0 + 3_0 - 2) \Rightarrow J_1 = 0.1$

$I_2 = hf(x_n + \frac{h}{2} , y_n + \frac{I}{2} , z_n + \frac{J_1}{2})$

$I_2 = (0.1)f(0 + \frac{0.1}{2} , 1 + \frac{0.3}{2} , 2 + \frac{0.1}{2})$

$I_2 = (0.1)f(0.05 ,1.15 , 2.05)$

$I_2 = (0.1)(-0.05 +1.15 +2.05) \Rightarrow I_2 = 0.315$

$J_2 = hg(x_n + \frac{h}{2} , y_n + \frac{I_1}{2} , z_n + \frac{J_1}{2})$

$J_2 = (0.1)g(0 + \frac{0.1}{2} , 1 + \frac{0.3}{2} , 2 + \frac{0.1}{2})$

$J_2 = (0.1)g(0.05 ,1.15 , 2.05)$

$J_2 = (0.1)(2(0.05) +3(1.15) - 2.05) \Rightarrow J_2 = 0.15$

$I_3 = hf(x_n + h , y_n + 2(I_2 - I_1), z_n + 2(J_2 - J_1))$

$I_3 = (0.1)f(0 + 0.1 , 1 + 2(0.315 - 0.3), 2 + 2(0.15 - 0.1))$

$I_3 = (0.1)f(0.1 , 1.03, 2.1)$

$I_3 = (0.1)(-0.1 +1.03+2.1) \Rightarrow I_3 = 0.303$

$J_3 = hg(x_n + h , y_n + 2(I_2 - I_1), z_n + 2(J_2 - J_1))$

$J_3 = (0.1)g(0 + 0.1 , 1 + 2(0.315 - 0.3), 2 + 2(0.15 - 0.1))$

$J_3 = (0.1)g(0.1 , 1.03, 2.1)$

$J_3 = (0.1)(2(0.1) +3(1.03)-2.1) \Rightarrow J_3 = 0.119$

$y_{n+1} = y_n + \frac{1}{6}(I_1 + 4I_2 + I_3)$

$y_1 = 1 + \frac{1}{6}(0.3 + 4(0.315) + 0.303) \Rightarrow y_1 = 1.3105$

$z_{n+1} = z_n + \frac{1}{6}(J_1 + 4J_2 + J_3)$

$z_1 = 2 + \frac{1}{6}(0.1 + 4(0.15) + 0.119) \Rightarrow z_1 = 2.1365$

C- Runge-Kutta Method of Order 4:

$m_1 = hf(x_n, y_n, z_n) = 0.3$

$p_1 = hg(x_n, y_n, z_n) = 0.1$

$m_2 = hf(x_n + \frac{h}{2}, y_n + \frac{m_1}{2}, z_n + \frac{p_1}{2}) = 0.315$

$p_2 = hg(x_n + \frac{h}{2}, y_n + \frac{m_1}{2}, z_n + \frac{p_1}{2}) = 0.15$

$m_3 = hf(x_n + \frac{h}{2}, y_n + \frac{m_2}{2}, z_n + \frac{p_2}{2}) = 0.31825$

$p_3 = hg(x_n + \frac{h}{2}, y_n + \frac{m_2}{2}, z_n + \frac{p_2}{2}) = 0.14975$

$m_4 = hf(x_n + h, y_n + m_3, z_n + p_3) = 0.3368$

$p_4 = hg(x_n + h, y_n + m_3, z_n + p_3) = 0.2005$

$y_{n+1} = y_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) = 1.317217$

$z_{n+1} = z_n + \frac{1}{6}(p_1 + 2p_2 + 2p_3 + p_4) = 2.15.$

Homework :

Use the Taylor series method to obtain a solution of $y'' = x^2 y$ when $y(0) = 0, y'(0) = 1$ at points 0.5 (use the **Runge-Kutta method** up to y''' and z''') .