Numerical Analysis

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Chapter Five

Numerical Derivation and Numerical Integration

Numerical derivation:الاشتقاق العددي

أنواع الاشتقاقات : هناك ثلاثة أنواع رئيسية من الاشتقاقات

1-الاشتقاق العددي :(Numerical Derivative) يستخدم لإيجاد معدل التغيير في قيمة الدالة عند تغيير متغير الإدخال بشكل صغير جداً.

2-الاشتقاق الجزئي :(Partial Derivative) يستخدم للحصول على معدل التغير في الدالة التي تعتمد على أكثر من متغير.

3-الاشتقاق الذاتي :(Self Derivative) يستخدم للحصول على معدل التغير في دالة تعتمد على نفسها في عملية متكررة.

1.1 Numerical derivation when points are not equal in dimensions

1.1 الاشتقاق العددي عندما لا تكون النقاط متساوية في الأبعاد

If we have $m+1$points $x_0, x_1, x_2, \cdots, x_m$ that are unequal dimensions and let f have a real and continuous function and are derivable in interval $[a, b]$, the convergence the function f to Lagrange polynomial which The general formula,

إذا كان لدينا$m+1$ نقاط $x_0, x_1, x_2, \cdots, x_m$ ذات أبعاد غير متساوية ولتكن f دالة حقيقية ومستمرة وقابله للاشتقاق ويمكن اشتقاقها في الفترة $[a, b]$ فإننا الاقتراب من الدالة f على متعدد حدود لاكرانج الذي الصيغة العامة

$$P_m(x) = \sum_{j=0}^{m} f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{m} \frac{(x - x_i)}{(x_j - x_i)}$$

We can find derivative of Lagrange polynomial using known derivation rules and make up the points needed to find their derivative.

يمكننا إيجاد مشتقة متعددة حدود لاكرانج باستخدام قواعد الاشتقاق المعروفة وتكوين النقاط اللازمة للعثور على مشتقتها.

Example 1 :

find $f'(1.67)$, $f''(3.28)$, $f^{(4)}(6.5)$ of the following table by using Lagrange polynomial

x	-1	0	2
y	0	1	9

Solution:

$$P_2(x) = \sum_{j=0}^{2} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{2} \frac{(x - x_i)}{(x_j - x_i)}$$
$$P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$
$$P_2(x) = 0 + (1)\frac{(x+1)(x-2)}{(0+1)(0-2)} + (9)\frac{(x+1)(x-0)}{(2+1)(2-0)} \implies P_2(x) = \frac{x^2-x-2}{-2} + \frac{9(x^2+x)}{6}$$
$$P_2(x) = x^2 + 2x + 1 \Rightarrow f(x) = x^2 + 2x + 1 \Rightarrow f'(x) = 2x + 2$$
$$\Rightarrow f'(x) = 2x + 2 \Rightarrow f'(x) = 2(1.67) + 2 = 5.34.$$
$$\Rightarrow f''(x) = 2 \Rightarrow f''(x) = 2 \Rightarrow f^{(3)}(x) = 0 = f^{(4)}(x).$$
$$f''(3.28) = 2, f^{(4)}(6.5) = 0$$

1.2 Numerical derivation when points are equal in dimensions

1.2 الاشتقاق العددي عندما تكون النقاط متساوية في الأبعاد

If we have $m+1$ points $x_0, x_1, x_2, \cdots, x_m$ that are equal dimensions between them and let it be h such that $x_m = x_0 + mh$ If required find the derivative we calculate it

إذا كان لدينا $m+1$ نقاط متساوية الأبعاد بينهما وليكن h طول الفترة حيث ان $x_m = x_0 + mh$ وبذلك فان المشتقة نحسبها كما يلي:

$$m = (x - x_0)/h \Rightarrow x = x_0 + mh,\ dx = hdm \Rightarrow \frac{dx}{dm} = h \Rightarrow \frac{dm}{dx} = \frac{1}{h}.$$

$$\frac{dy}{dx} = \frac{dy}{dm}.\frac{dm}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{h}.\frac{dy}{dm}$$

The formula $\frac{dy}{dm}$ can be obtained from deriving the previous Interpolation formulas

يمكن الحصول على الصيغة $\frac{dy}{dm}$ من استخلاص صيغ الاستكمال السابقة

a- Forward differences الفروقات التقدمية

$$f(x) = f(x_0 + mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \cdots.$$

$$y = y_0 + m\Delta y_0 + (\frac{m^2}{2} - \frac{m}{2})\Delta^2 y_0 + \frac{m^3 - 3m^2 + 2m}{6}\Delta^3 y_0 + \frac{m^4 - 6m^3 + 11m^2 - 6m}{24}\Delta^4 y_0 + \cdots.$$

$$\frac{dy}{dm} = \Delta y_0 + (m - \frac{1}{2})\Delta^2 y_0 + (\frac{m^2}{2} - m + \frac{1}{3})\Delta^3 y_0 + (\frac{m^3}{6} - \frac{3m^2}{4} + \frac{11}{12}m - \frac{1}{4})\Delta^4 y_0 + \cdots.$$

$\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{dm}$.

$\frac{dy}{dx} = \frac{1}{h}.(\Delta y_0 + (m - \frac{1}{2})\Delta^2 y_0 + (\frac{m^2}{2} - m + \frac{1}{3})\Delta^3 y_0 + (\frac{m^3}{6} - \frac{3m^2}{4} + \frac{11}{12}m - \frac{1}{4})\Delta^4 y_0 + \cdots)$.......(1)

if $m = 0$,

$\frac{dy}{dx} = \frac{1}{h}.\frac{dy}{dm} = \frac{1}{h}.(\Delta y_0 + \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots)$. .... (10).

Example 3 :

Find $f'(18.4), f'(18.8)$ $f'(19.2)$ of the following table by using **Forward differences** .

x	18	19	20	21	22
y	10.8894	12.7032	14.7781	17.1489	19.8550

Solution:

Use the **Forward differences .**

x	y	Δ	Δ^2	Δ^3	Δ^4
18	10.8894				
		1.8138			
19	12.7032		0.2611		
		2.0749		-2.336	
20	14.7781		-2.0749		9.4878
		0		7.1518	
21	14.7781		5.0769		
		5.0769			
22	19.8550				

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{18.4 - 18}{1} = 0.4$$

$$\frac{dy}{dx} = \frac{1}{h}.(\Delta y_0 + (m - \frac{1}{2})\Delta^2 y_0 + (\frac{m^2}{2} - m + \frac{1}{3})\Delta^3 y_0 + (\frac{m^3}{6} - \frac{3m^2}{4} + \frac{11}{12}m - \frac{1}{4})\Delta^4 y_0 + \cdots).$$

$$\frac{dy}{dx_{(18.4)}} = \frac{1}{1}.(1.8138) + \left(0.4 - \frac{1}{2}\right)(0.2611) + \left(\frac{(0.4)^2}{2} - (0.4) + \frac{1}{3}\right)(-2.336) + (\frac{(0.4)^3}{6} - \frac{3(0.4)^2}{4} + \frac{11}{12}(0.4) - \frac{1}{4})(9.4878)).$$

$$\frac{dy}{dx}_{(18.4)} = (1.8138) + (-0.1)(0.2611) + (0.08 - 0.4 + 0.3333)(-2.336) + (0.0107 - 0.12 + 0.3667 - 0.25)(9.4878)).$$

$$\frac{dy}{dx}_{(18.4)} = (1.8138) - (0.02611) - (0.0304) + (0.0702) = 1.82749 .$$

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{18.8 - 19}{1} = -0.2$$

$$\frac{dy}{dx} = \frac{1}{h}.(\Delta y_0 + (m - \frac{1}{2})\Delta^2 y_0 + (\frac{m^2}{2} - m + \frac{1}{3})\Delta^3 y_0 + (\frac{m^3}{6} - \frac{3m^2}{4} + \frac{11}{12}m - \frac{1}{4})\Delta^4 y_0 + \cdots).$$

$$\frac{dy}{dx}_{(18.8)} = \frac{1}{1}.(2.0749) + \left((-0.2) - \frac{1}{2}\right)(-2.0749) + \left(\frac{(-0.2)^2}{2} - (-0.2) + \frac{1}{3}\right)(7.1518).$$

$$\frac{dy}{dx}_{(18.8)} = (2.0749) + (-0.2 - 0.5)(-2.0749) + (0.02 + 0.2 + 0.3333)(7.1518).$$

$$\frac{dy}{dx}_{(18.8)} = (2.0749) + (1.4524) + (3.9570) = 7.4843 .$$

$$m = \frac{x_m - x_0}{h} \Longrightarrow m = \frac{19.2 - 19}{1} = 0.2$$

$$\frac{dy}{dx} = \frac{1}{h}.(\Delta y_0 + (m - \frac{1}{2})\Delta^2 y_0 + (\frac{m^2}{2} - m + \frac{1}{3})\Delta^3 y_0 + (\frac{m^3}{6} - \frac{3m^2}{4} + \frac{11}{12}m - \frac{1}{4})\Delta^4 y_0 + \cdots).$$

$$\frac{dy}{dx}_{(19.2)} = \frac{1}{1}.(2.0749) + \left((0.2) - \frac{1}{2}\right)(-2.0749) + \left(\frac{(0.2)^2}{2} - (0.2) + \frac{1}{3}\right)(7.1518).$$

$$\frac{dy}{dx}_{(19.2)} = (2.0749) + (-0.3)(-2.0749) + (0.02 - 0.2 + 0.3333)(7.1518).$$

$$\frac{dy}{dx}_{(19.2)} = (2.0749) + (0.6225) + (1.0964) = 3.7938 .$$

Homework

Find the function $f'(0.820), f'(0.840), f'(0.860)$ of the following table use the **Forward differences .**

x	0.800	0.850	0.880	0.890	0.895	0.898	0.899
y	0.7174	0.7513	0.7707	0.7771	0.7802	0.7821	0.7827

***b*- Backward differences الفروقات التراجعية**

$$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \cdots.$$

$$y = y_0 + m\nabla y_0 + (\frac{m^2}{2} + \frac{m}{2})\nabla^2 y_0 + \frac{m^3 + 3m^2 + 2m}{6}\nabla^3 y_0 + \frac{m^4 + 6m^3 + 11m^2 + 6m}{24}\nabla^4 y_0 + \cdots.$$

$$\frac{dy}{dm} = \nabla y_0 + (m + \frac{1}{2})\nabla^2 y_0 + (\frac{m^2}{2} + m + \frac{1}{3})\nabla^3 y_0 + (\frac{m^3}{6} + \frac{3m^2}{4} + \frac{11}{12}m + \frac{1}{4})\nabla^4 y_0 + \cdots.$$

$$\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{dm}.$$

$$\frac{dy}{dx} = \frac{1}{h}.(\nabla y_0 + (m + \frac{1}{2})\nabla^2 y_0 + (\frac{m^2}{2} + m + \frac{1}{3})\nabla^3 y_0 + (\frac{m^3}{6} + \frac{3m^2}{4} + \frac{11}{12}m + \frac{1}{4})\nabla^4 y_0 + \cdots)\ldots\ldots(2)$$

if $m = 0$,

$$\frac{dy}{dx} = \frac{1}{h}.\frac{dy}{dm} = \frac{1}{h}.(\nabla y_0 + \frac{1}{2}\nabla^2 y_0 + \frac{1}{3}\nabla^3 y_0 + \frac{1}{4}\nabla^4 y_0 + \cdots). \quad \ldots\ldots (20).$$

Example 4 :

Find the function $f'(1.41), f'(1.38), f'(1.32)$of the following table use the **Backward differences .**

x	1.1	1.2	1.3	1.4
y	0.4860	0.8616	1.5975	3.7616

Solution:

Use the **Backward differences .**

x	y	∇	∇^2	∇^3
1.1	0.4860			
		0.3756		
1.2	0.8616		0.3603	
		0.7359		1.0679
1.3	1.5975		1.4282	
		2.1641		
1.4	3.7616			

$$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{1.41 - 1.4}{0.1} = \frac{0.01}{0.1} = 0.1$$

$$\frac{dy}{dx} = \frac{1}{h}.(\nabla y_0 + (m + \frac{1}{2})\nabla^2 y_0 + (\frac{m^2}{2} + m + \frac{1}{3})\nabla^3 y_0 + (\frac{m^3}{6} + \frac{3m^2}{4} + \frac{11}{12}m + \frac{1}{4})\nabla^4 y_0 + \cdots)$$

$$\frac{dy}{dx}_{(1.41)} = \frac{1}{0.1}.\left((2.1641) + \left(0.1 + \frac{1}{2}\right)(1.4282) + \left(\frac{(0.1)^2}{2} + (0.1) + \frac{1}{3}\right)(1.0679)\right).$$

$$\frac{dy}{dx_{(1.41)}} = \frac{1}{0.1}.((2.1641) + (0.6)(1.4282) + (\, 0.005 + 0.1 + 0.3333)(1.0679)).$$

$$\frac{dy}{dx_{(1.41)}} = \frac{1}{0.1}.(2.1641 + 0.8569 + 0.4681) = 34.891.$$

$$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{1.38 - 1.4}{0.1} = \frac{-0.02}{0.1} = -0.2$$

$$\frac{dy}{dx} = \frac{1}{h}.(\nabla y_0 + (m + \frac{1}{2})\nabla^2 y_0 + (\frac{m^2}{2} + m + \frac{1}{3})\nabla^3 y_0 + \ (\frac{m^3}{6} + \frac{3m^2}{4} + \frac{11}{12}m + \frac{1}{4})\nabla^4 y_0 + \cdots)$$

$$\frac{dy}{dx_{(1.38)}} = \frac{1}{0.1}.\left((2.1641) + \left((-0.2) + \frac{1}{2}\right)(1.4282) + \left(\frac{(-0.2)^2}{2} + (-0.2) + \frac{1}{3}\right)(1.0679)\right).$$

$$\frac{dy}{dx_{(1.38)}} = \frac{1}{0.1}.((2.1641) + (0.3)(1.4282) + (\, 0.02 - 0.2 + 0.3333)(1.0679)).$$

$$\frac{dy}{dx_{(1.38)}} = \frac{1}{0.1}.(2.1641 + 0.4285 + 0.1637) = 27.563.$$

$$m = \frac{x_m - x_0}{h} \quad \Rightarrow m = \frac{1.32 - 1.3}{0.1} = \frac{0.02}{0.1} = 0.2$$

$$\frac{dy}{dx} = \frac{1}{h}.(\nabla y_0 + (m + \frac{1}{2})\nabla^2 y_0 + (\frac{m^2}{2} + m + \frac{1}{3})\nabla^3 y_0 + \ (\frac{m^3}{6} + \frac{3m^2}{4} + \frac{11}{12}m + \frac{1}{4})\nabla^4 y_0 + \cdots)$$

$$\frac{dy}{dx_{(1.32)}} = \frac{1}{0.1}.\left((2.1641) + \left(0.2 + \frac{1}{2}\right)(1.4282) + \left(\frac{(0.2)^2}{2} + (0.2) + \frac{1}{3}\right)(1.0679)\right).$$

$$\frac{dy}{dx_{(1.32)}} = \frac{1}{0.1}.((2.1641) + (0.7)(1.4282) + (\, 0.02 + 0.2 + 0.3333)(1.0679)).$$

$$\frac{dy}{dx_{(1.32)}} = \frac{1}{0.1}.(2.1641 + 0.9997 + 0.5905) = 37.543.$$

Homework

Find the function $f'(0.085), f'(0.040), f'(0.090)$of the following table use the **Backward differences .**

x	0.001	0.002	0.005	0.010	0.020	0.050	0.100
y	0.6250	0.6225	0.6220	0.6215	0.6215	0.6214	0.6206

c- Central differences   الفروقات المركزية

i-Besssl's formulas   الفروقات بسل المركزية

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$y = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m^2 - m}{2}\mu\delta^2 y_{\frac{1}{2}} + \frac{m^3 - \frac{3}{2}m^2 - 0.5m}{6}\delta^3 y_{\frac{1}{2}} + \frac{m^4 - 2m^3 - m^2 - 2m}{24}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$\frac{dy}{dm} = \delta y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\mu\delta^2 y_{\frac{1}{2}} + \left(\frac{m^2}{2} - \frac{m}{2} + \frac{1}{12}\right)\delta^3 y_{\frac{1}{2}} + \left(\frac{m^3}{6} - \frac{m^2}{4} - \frac{m}{12} + \frac{1}{12}\right)\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$\frac{dy}{dx} = \frac{1}{h} \cdot \frac{dy}{dm}.$$

$$\frac{dy}{dx} = \frac{1}{h}.\left(\delta y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\mu\delta^2 y_{\frac{1}{2}} + \left(\frac{m^2}{2} - \frac{m}{2} + \frac{1}{12}\right)\delta^3 y_{\frac{1}{2}} + \left(\frac{m^3}{6} - \frac{m^2}{4} - \frac{m}{12} + \frac{1}{12}\right)\mu\delta^4 y_{\frac{1}{2}} + \cdots\right) \ldots\ldots(3)$$

if $m = 0$,

$$\frac{dy}{dx} = \frac{1}{h}.\frac{dy}{dm} = \frac{1}{h}.\left(\delta y_{\frac{1}{2}} - \frac{1}{2}\mu\delta^2 y_{\frac{1}{2}} + \frac{1}{12}{}^3\ \delta y_{\frac{1}{2}} + \frac{1}{12}\mu\delta^4 y_{\frac{1}{2}} + \cdots\right). \quad \ldots\ldots\ (30).$$

Example 5 :

Find the function $f'(4.5), f'(3.5), f'(4.9), f'(3.1)$of the following table use the **Besssl's formulas differences .**

x	2	4	6	8
y	56	126	268	530

Solution:

use the ***Besssl's formulas*** **.**

x	y	δ	δ^2	δ^3
2	56			
		70		
4	126		72	
		142		48
6	268		120	
		262		
8	530			

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{4.5-4}{2} = \frac{0.5}{2} = 0.25.$

$\frac{dy}{dx} = \frac{1}{h}.(\delta y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\mu\delta^2 y_{\frac{1}{2}} + (\frac{m^2}{2} - \frac{m}{2} + \frac{1}{12})\delta^3 y_{\frac{1}{2}} + (\frac{m^3}{6} - \frac{m^2}{4} - \frac{m}{12} + \frac{1}{12})\mu\delta^4 y_{\frac{1}{2}} + \cdots)$

$\frac{dy}{dx}_{(4.5)} = \frac{1}{2}.\left((142) + (0.25 - 0.5)\left(\frac{72+120}{2}\right) + \left(\frac{(0.25)^2}{2} - \frac{(0.25)}{2} + \frac{1}{12}\right)(48)\right)$

$\frac{dy}{dx}_{(4.5)} = \frac{1}{2}.(142 - (0.25)(96) + (\ 0.03125 - 0.125 + 0.0833)(48))$

$\frac{dy}{dx}_{(4.5)} = \frac{1}{2}.(142 - (0.25)(96) + (\ -0.01045)(48))$

$\frac{dy}{dx}_{(4.5)} = \frac{1}{2}.(142 - 24 - 0.5016) = \frac{1}{2}.(117.4984) = 58.7492.$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.5-4}{2} = \frac{-0.5}{2} = -0.25$

$\frac{dy}{dx}_{(3.5)} = \frac{1}{2}.\left((142) + ((-0.25) - 0.5)\left(\frac{72+120}{2}\right) + \left(\frac{(-0.25)^2}{2} - \frac{(-0.25)}{2} + \frac{1}{12}\right)(48)\right)$

$\frac{dy}{dx}_{(3.5)} = \frac{1}{2}.(142 - (0.75)(96) + (0.03125 + 0.125 + 0.0833)(48))$

$\frac{dy}{dx}_{(3.5)} = \frac{1}{2}.(142 - (0.75)(96) + (0.2396)(48))$

$\frac{dy}{dx}_{(3.5)} = \frac{1}{2}.(142 - 72 + 11.5008) = \frac{1}{2}.(81.5008) = 40.7504.$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{4.9-4}{2} = \frac{0.9}{2} = 0.45$

$\frac{dy}{dx}_{(4.9)} = \frac{1}{2}.\left((142) + ((0.45) - 0.5)\left(\frac{72+120}{2}\right) + \left(\frac{(0.45)^2}{2} - \frac{(0.45)}{2} + \frac{1}{12}\right)(48)\right)$

$\frac{dy}{dx}_{(4.9)} = \frac{1}{2}.(142 - (0.05)(96) + (0.10125 - 0.225 + 0.0833)(48))$

$\frac{dy}{dx}_{(4.9)} = \frac{1}{2}.(142 - (0.05)(96) + (-0.04045)(48))$

$\frac{dy}{dx}_{(4.9)} = \frac{1}{2}.(142 - 4.8 - 1.9416) = \frac{1}{2}.(135.2584) = 67.6292.$

$$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.1 - 4}{2} = \frac{-0.9}{2} = -0.45$$

$$\frac{dy}{dx}_{(3.1)} = \frac{1}{2}.\left((142) + ((-0.45) - 0.5)\left(\frac{72 + 120}{2}\right) + \left(\frac{(-0.45)^2}{2} - \frac{(-0.45)}{2} + \frac{1}{12}\right)(48)\right)$$

$$\frac{dy}{dx}_{(3.1)} = \frac{1}{2}.(142 - (0.95)(96) + (0.10125 + 0.225 + 0.0833)(48))$$

$$\frac{dy}{dx}_{(3.1)} = \frac{1}{2}.(142 - (0.95)(96) + (0.40955)(48))$$

$$\frac{dy}{dx}_{(3.1)} = \frac{1}{2}.(142 - 91.2 + 19.6584) = \frac{1}{2}.(70.4584) = 35.2292.$$

Homework

Find the function $f'(0.025), f'(0.045), f'(0.03) f'(0.04)$ of the following table use the **Besssl's formulas differences .**

x	0.01	0.02	0.05	0.10
y	0.6215	0.6215	0.6214	0.6206

ii- Stirling's formulas **الفروقات سترلنك المركزية**

$$y = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y = y_0 + m\mu\delta y_0 + \frac{m^2}{2}\delta^2 y_0 + \frac{m^2 - m}{6}\mu\delta^3 y_0 + \frac{m^4 - m^2}{24}\delta^4 y_0 + \cdots$$

$$\frac{dy}{dm} = \mu\delta y_0 + m\delta^2 y_0 + (\frac{m^2}{2} - \frac{1}{6})\mu\delta^3 y_0 + (\frac{m^3}{6} - \frac{m}{12})\delta^4 y_0 + \cdots$$

$$\frac{dy}{dx} = \frac{1}{h}.\frac{dy}{dm}.$$

$$\frac{dy}{dx} = \frac{1}{h}.(\mu\delta y_0 + m\delta^2 y_0 + (\frac{m^2}{2} - \frac{1}{6})\mu\delta^3 y_0 + (\frac{m^3}{6} - \frac{m}{12})\delta^4 y_0 + \cdots)\ldots\ldots(4)$$

if $m = 0$,

$$\frac{dy}{dx} = \frac{1}{h}.\frac{dy}{dm} = \frac{1}{h}.(\mu\delta y_0 - \frac{1}{6}\mu\delta^3 y_0 + \cdots). \quad \ldots\ldots. \ (40).$$

Example 6 :

Find the function $f'(1.022), f'(1.018)$of the following table use the **Stirling's formulas differences .**

x	1	1. 01	1. 02	1.03	1.04
y	3.10	3.12	3.14	3.18	3.24

Solution:

use the **Stirling's formulas differences.**

x	y	δ	δ^2	δ^3	δ^4
1	3.10				
		0.02			
1.01	3.12		0		
		0.02		0.02	
1.02	3.14		0.02		-0.02
		0.04		0	
1.03	3.18		0.02		
		0.06			
1.04	3.24				

$$m = \frac{x_m - x_0}{h} \implies m = \frac{1.022 - 1.02}{2} = \frac{0.002}{2} = 0.001.$$

$$\frac{dy}{dx} = \frac{1}{h}.(\mu\delta y_0 + m\delta^2 y_0 + (\frac{m^2}{2} - \frac{1}{6})\mu\delta^3 y_0 + (\frac{m^3}{6} - \frac{m}{12})\delta^4 y_0 + \cdots).$$

$$\frac{dy}{dx_{(1.022)}} = \frac{1}{0.01}.\left(\left(\frac{0.02+0.04}{2}\right) + (0.001)(0.02) + \left(\frac{(0.001)^2}{2} - \frac{1}{6}\right)\left(\frac{0.02+0}{2}\right) + \left(\frac{(0.001)^3}{6} - \frac{(0.001)}{12}\right)(-0.02)\right)$$

$$\frac{dy}{dx_{(1.022)}} = \frac{1}{0.01}.((0.03) + (0.001)(0.02) + (0.0000005 - 0.1667)(0.01) + (1.6667 * 10^{-10} - 0.000083)(-0.02))$$

$$\frac{dy}{dx_{(1.022)}} = \frac{1}{0.01}.(0.03 + 0.00002 + (-0.1666995)(0.01))$$

$$\frac{dy}{dx_{(1.022)}} = \frac{1}{0.01}.(0.03 + 0.00002 - 0.001666995) = \frac{1}{0.01}.(0.028353005) = 2.8353005.$$

$$m = \frac{x_m - x_0}{h} \implies m = \frac{1.018 - 1.02}{2} = \frac{-0.002}{2} = -0.001.$$

$$\frac{dy}{dx} = \frac{1}{h}.(\mu\delta y_0 + m\delta^2 y_0 + (\frac{m^2}{2} - \frac{1}{6})\mu\delta^3 y_0 + (\frac{m^3}{6} - \frac{m}{12})\delta^4 y_0 + \cdots).$$

$$\frac{dy}{dx_{(1.018)}} = \frac{1}{0.01}.\left(\left(\frac{0.02+0.04}{2}\right) + (-0.001)(0.02) + \left(\frac{(-0.001)^2}{2} - \frac{1}{6}\right)\left(\frac{0.02+0}{2}\right) + \left(\frac{(-0.001)^3}{6} - \frac{(-0.001)}{12}\right)(-0.02)\right)$$

$$\frac{dy}{dx_{(1.018)}} = \frac{1}{0.01}.\left((0.03) + (-0.001)(0.02) + (-0.0000005 + 0.1667)(0.01) + (-1.6667 * 10^{-10} + 0.000083)(-0.02)\right)$$

$$\frac{dy}{dx_{(1.018)}} = \frac{1}{0.01}.\left(0.03 - 0.00002 + (0.1666995)(0.01)\right)$$

$$\frac{dy}{dx_{(1.018)}} = \frac{1}{0.01}.(0.03 - 0.00002 + 0.001666995) = \frac{1}{0.01}.(0.031646995) = 3.1646995 .$$

Homework

Find the function $f'(0.63), f'(0.54), f'(0.69), f'(0.58)$of the following table use the **Stirling's formulas differences .**

x	0.2	0.4	0.6	0.8	1
y	0.9799	0.9178	0.8080	0.6386	0.3843

Numerical Integration

التكامل العددي:

التكامل العددي هو دراسة لكيفية العثور على تقديرات تقريبية لتكامل معين.

هناك أسباب عديدة للتكامل العددي:

1 -عدم العثور على القيمة الحقيقية للتكامل.

2 -عندما تكون عملية إيجاد قيمة التكامل الحقيقي ممكنة ولكن الدالة قد تكون معقدة وتحتاج إلى جهد ووقت طويل..

3 -يمكن تعريف دالة التكامل على أنها جدول القيم.

To find the value of Numerical integration, the function can be written in the form of a power series as follows:

لإيجاد قيمة التكامل العددي يمكن كتابة الدالة على شكل متسلسلة قوى كما يلي:

$$f(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \ldots + c_n(x-x_0)^n + \ldots$$

If the closed interval $[a,b]$is a continuous on the function and be within the convergence of the series, we get إذا كانت الفترة المغلقة [a,b] متصلة على الدالة وضمن تقارب المتسلسلة نحصل على

$$I_{f(x)} = \int_a^b f(x)dx = c_0\int_a^b dx + c_1\int_a^b (x-x_0)\,dx + c_2\int_a^b (x-x_0)^2\,dx$$
$$+ c_3\int_a^b (x-x_0)^3\,dx + \ldots$$
$$+ c_n\int_a^b (x-x_0)^n\,dx + \ldots = \sum_{i=0}^{\infty} c_i\int_a^b (x-x_0)^i\,dx$$

Example 1 :

Find the approximate value of integration $\int_0^1 \frac{cos(x)}{\sqrt{x}}dx$ for four term.

Solation :

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$I_{f(x)} = \int_0^1 \frac{cos(x)}{\sqrt{x}}dx = \int_0^1 \frac{1}{\sqrt{x}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)dx$$
$$= \int_0^1 \left(\frac{1}{\sqrt{x}} - \frac{x^{\frac{3}{2}}}{2!} + \frac{x^{\frac{7}{2}}}{4!} - \frac{x^{\frac{11}{2}}}{6!} + \cdots\right)dx$$
$$= \left(2x^{\frac{1}{2}} - \frac{x^{\frac{5}{2}}}{2!.\left(\frac{5}{2}\right)} + \frac{x^{\frac{9}{2}}}{4!.\left(\frac{11}{2}\right)} - \frac{x^{\frac{13}{2}}}{6!.\left(\frac{13}{2}\right)}\right)\Bigg|_0^1$$
$$= 2 - \frac{1}{5} + \frac{1}{108} - \frac{1}{4680} + \cdots = 1.809045584 \cong 1.8090$$

Example 2 :

Find the approximate value of the determinate integration of the interval [0.5,2.5], $\left(\int_{0.5}^{2.5} f(x)dx\right)$ on the function defined in the following table.

x	0.05	0.10	0.20	0.26
y	0.05	0.0999	0.1987	0.2571

Solation :

x	y	$\Delta = \nabla = \delta$	Δ^2	Δ^3
0.05	0.05			
		0.9980		
0.10	0.09999		-0.0667	
		0.9880		-0.1200
0.20	0.1987		-0.0919	
		0.9733		
0.26	0.2571			

$$f(x_m) = y_m = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 + \cdots$$

$$f(x) = 0.05 + (x - 0.05)(0.9980) + (x - 0.05)(x - 0.1)(-0.0667) + (x - 0.05)(x - 0.1)(x - 0.20)(-0.1200)$$

$$f(x) = -0.119x^3 - 0.025x^2 + 1.004x$$

$$I_{f(x)} = \int_{0.5}^{2.5} f(x)dx = \int_{0.5}^{2.5} (-0.119x^3 - 0.025x^2 + 1.004x)dx$$

$$= \frac{-0.119x^4}{4}\Big|_{0.5}^{2.5} - \frac{0.025x^3}{3}\Big|_{0.5}^{2.5} + \frac{1.004x^2}{2}\Big|_{0.5}^{2.5} = -0.0299.$$

Homework

1- Find the approximate value of integration $\int_1^{1.7} \frac{e^x}{x^2}dx$ for four .

2- Find the approximate value of the determinate integration of the interval [2,3], $\left(\int_2^3 f(x)dx\right)$ on the function defined in the following table.

x	0	1	2	3	4
y	1	2	9	28	65

Newton-Cotes Formulas صيغ نيوتن-كوتس

Newton-Coates formulas are the most important methods of numerical integration.

تعتبر صيغ نيوتن-كوتس من أهم طرق التكامل العددي

a- Trapezoidal Rule. القاعدة شبه المنحرف

Let n Number of partial interval (sub-interval) and $h = \frac{b-a}{n}$.

لتكن n عدد الفترات الجزئي (الفترات الفرعي) و $h = \frac{b-a}{n}$.

When n=1 then The trapezoidal rule of integration $\int_a^b f(x)dx$ is

عندما يكون n=1 فإن قاعدة التكامل شبه المنحرف هي

When n=1 , $\int_a^b f(x)dx \cong \frac{h}{2}(f(a)+f(b))$, $h=\frac{b-a}{1}$

When n=2 , $\int_a^b f(x)dx \cong \frac{h}{2}(f(a)+2f(a+h)+f(b))$, $h=\frac{b-a}{2}$

When n=3 , $\int_a^b f(x)dx \cong \frac{h}{2}(f(a)+2f(a+h)+2f(a+2h)+f(b))$, $h=\frac{b-a}{3}$

Then the general formula of the trapezoidal rule

ثم الصيغة العامة لقاعدة شبه المنحرف

$\int_a^b f(x)dx \cong \frac{h}{2}[f(a)+2f(a+h)+2f(a+2h)+2f(a+3h)+\cdots+2f(a+(n-1)h)+f(b)]$, $h=\frac{b-a}{n}$.

$\int_a^b f(x)dx \cong \frac{h}{2}[f(a)+2\sum_{i=1}^{n-1} f(a+ih)+f(b)]+E_T(h)$.

$E_T(h)$ is The error formula for the trapezoidal rule $E_T(h)=\frac{-h^3}{12}f''(\lambda)$, $a<\lambda<b$

هي صيغة الخطأ لقاعدة شبه المنحرف

Example 3 :

Find the approximate value of integration $\int_1^2 \frac{1}{x}dx$ by using the trapezoidal rule when n=4

Solation :

$\int_1^2 \frac{1}{x}dx = ln\,x|_1^2 = ln\,2 - ln\,1 = ln\,2 = 0.6931$,$h=\frac{b-a}{n}=\frac{2-1}{4}=0.25$.

x	1	1.25	1.5	1.75	2
$y=f(x)$	1	0.8	0.6667	0.5714	0.5

$\int_a^b f(x)dx \cong \frac{h}{2}\left[f(a)+2\sum_{i=1}^{3} f(a+ih)+f(b)\right]+E_T(h)$.

$$\int_a^b f(x)dx \cong \frac{h}{2}[f(a)+2f(a+h)+2f(a+2h)+2f(a+3h)+f(b)]$$

$$\int_1^2 \frac{1}{x}dx \cong \frac{0.25}{2}[1+2(0.8+0.6667+0.5714)+0.5]=0.6970$$

$e_x=|0.6931-0.6970|=0.0039$.

Example 4 :

Find the approximate value of integration $\int_0^{10} 3x^2\,dx$ by using the **trapezoidal rule** when n=10

Solation :

$\int_0^{10} 3x^2\,dx = x^3|_0^{10} = 1000 \quad ,h = \frac{b-a}{n} = \frac{10-0}{10} = 1.$

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	0	1	12	27	48	75	108	147	192	243	300

$\int_a^b f(x)dx \cong \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{9} f(a+ih) + f(b)\right] + E_T(h).$

$$\int_a^b f(x)dx \cong \frac{h}{2}[f(a) + 2(f(a+h) + f(a+2h) + f(a+3h) + \cdots f(a+9h)) + f(b)]$$

$$\int_0^{10} 3x^2dx \cong \frac{1}{2}[0 + 2(3 + 12 + 27 + 48 + 75 + 108 + 147 + 192 + 243) + 300] = 1005$$

$e_x = |1000 - 1005| = 5.$

Homework

Find the approximate value of integration $\int_1^6 \frac{1}{\sqrt{x+2}}dx$ by using the **trapezoidal rule** when n=10

b-Mid-point Rule قاعدة النقطة الوسطى.

Let n Number of partial interval (sub- interval) and $h = \frac{b-a}{n}$.

لتكن n عدد الفترات الجزئية (الفترات الفرعي) و $\int_a^b f(x)dx$ هو

When n=1 then The Mid- point rule of integration $\int_a^b f(x)dx$ is

عندما يكون n=1 فإن قاعدة النقطة الوسطى للتكامل هي

If n=1 $\Rightarrow \int_a^b f(x)dx \cong h\left(f(a+\frac{h}{2})\right)$, $h = \frac{b-a}{1}$,

If n=2 $\Rightarrow \int_a^b f(x)dx \cong h\left(f(a+\frac{1}{2}h) + f(a+\frac{3}{2}h)\right)$, $h = \frac{b-a}{2}$

If n=3 $\Rightarrow \int_a^b f(x)dx \cong h\left(f(a+\frac{1}{2}h) + f(a+\frac{3}{2}h) + f(a+\frac{5}{2}h)\right)$, $h = \frac{b-a}{3}$.

Then the general formula of the Mid- point rule:- الصيغة العامة لقاعدة النقطة الوسطى

$$\int_a^b f(x)dx \cong h\left(f\left(a+\frac{1}{2}h\right) + f\left(a+\frac{3}{2}h\right) + f\left(a+\frac{5}{2}h\right) + \cdots + f\left(a+\frac{(2n-1)}{2}h\right)\right) + E_S(h),$$

$\int_a^b f(x)dx \cong h\sum_{i=1}^{n} f(a+\frac{(2i-1)}{2}h) + E_M(h).$

$E_M(h)$ is The error formula for the Mid- point rule $E_M(h) = \frac{h^3}{6}f''(\beta)$ $, a < \beta < b$

هي صيغة الخطأ لقاعدة النقطة الوسطى

Example 5 :

Find the approximate value of integration $\int_1^2 \frac{1}{x}dx$ by using the **Mid- point rule** when n=4

Solation :

$\int_1^2 \frac{1}{x}dx = ln\, x|_1^2 = ln\, 2 - ln\, 1 = ln\, 2 = 0.6931$ $, h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25.$

x	1.125	1.375	1.625	1.875
$y = f(x)$	0.8889	0.7273	0.6154	0.5333

$$\int_a^b f(x)dx \cong h\sum_{i=1}^{4} f\left(a+\frac{(2i-1)}{2}h\right) + E_M(h).$$

$$\int_a^b f(x)dx \cong h\left[f(a+\frac{1}{2}h) + f(a+\frac{3}{2}h) + f(a+\frac{5}{2}h) + f(a+\frac{7}{2}h)\right]$$

$$\int_1^2 \frac{1}{x}dx \cong (0.25)[0.8889 + 0.7273 + 0.6154 + 0.5333] = 0.6912$$

$e_x = |\ 0.6931 - 0.6912| = 0.0019.$

Example 6 :

Find the approximate value of integration $\int_0^{10} 3x^2\, dx$ by using the **Mid- point rule** when n=6

Solation :

$\int_0^{10} 3x^2\, dx = x^3|_0^{10} = 1000$ $, h = \frac{b-a}{n} = \frac{10-0}{10} = 1.$

x	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2	17/2	19/2
$f(x)$	0.75	6.75	18.75	36.75	60.75	90.75	126.75	168.75	216.75	270.75

$$\int_a^b f(x)dx \cong h\sum_{i=1}^{10} f(a+\frac{(2i-1)}{2}h) + E_M(h)$$

$$\int_a^b f(x)dx \cong h\left[f(a+\frac{1}{2}h)+f(a+\frac{3}{2}h)+f(a+\frac{5}{2}h)+\cdots+f(a+\frac{19}{2}h)\right]$$

$$\int_0^{10} 3x^2dx \cong (1)[0.75+6.75+18.75+36.75+60.75+90.75+126.75+168.75$$
$$+216.75+270.75]=997.5$$

$e_x = |1000-997.5| = 2.5.$

Homework

Find the approximate value of integration $\int_1^6 \frac{1}{\sqrt{x+2}}dx$ by using the Mid -point rule when n=10

c- Simpson's Rule .قاعدة سمبسون

This rule deals with even interval (i.e n =2,4,6,8,…) When n=2 then The Simpsons'rule of integration $\int_a^b f(x)dx$,

تتعامل هذه القاعدة مع الفترات الزوجية (أي n = 2,4,6,8,...) عندما يكون n = 2 إذن قاعدة التكامل الخاصة بعائلة سمبسون

$$if \text{ n=2 then } h=\frac{b-a}{2} \Rightarrow \int_a^b f(x)dx \cong \frac{h}{3}(f(a)+4f(a+h)+f(b))$$

$$if\ n=4, then\ h=\frac{b-a}{4} \Rightarrow \int_a^b f(x)dx \cong \frac{h}{3}(f(a)+4f(a+h)+2f(a+2h)+f(b))$$

$$if \text{ n=6, then } h=\frac{b-a}{6}$$

$$\int_a^b f(x)dx \cong \frac{h}{3}(f(a)+4f(a+h)+2f(a+2h)+4f(a+3h)+2f(a+4h)+4f(a+5h)$$
$$+f(b)).$$

Then the general formula of the Simpson's rule

ثم الصيغة العامة لقاعدة سمبسون

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(a)+4f(a+h)+2f(a+2h)+\cdots+2f(a+(2n-2)h)+4f(a$$
$$+(2n-1)h)+f(b)]$$

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(a)+4(f(a+h)+f(a+3h)+\cdots+f(a+(2n-1)h))$$
$$+2(f(a+2h)+f(a+4h)+\cdots+f(a+(2n-2)h))+f(b)]+E_S(h)$$

$\int_a^b f(x)dx \cong \frac{h}{3}[f(a) + 4\sum_{i=1}^{n} f(a+(2i-1)h)\sum_{i=1}^{n-1} f(a+2ih) + f(b)] + E_S(h)$

$E_T(h)$ is The error formula for the Simpsons'rule be $E_S(h) = \frac{h^5}{90} f^{(4)}(\theta)\ , a < \theta < b$

هي صيغة الخطأ لقاعدة سمبسنز

Example 7 :

Find the approximate value of integration $\int_1^2 \frac{1}{x} dx$ by using the **Simpsons'rule** when n=4

Solution:-

$\int_1^2 \frac{1}{x} dx = \ln x|_1^2 = \ln 2 - \ln 1 = \ln 2 = 0.6931$,$h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25.$

x	1	1.25	1.5	1.75	2
y=f(x)	1	0.8	0.6667	0.5714	0.5

$$\int_a^b f(x)dx \cong \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{n} f(a+(2i-1)h) + 2\sum_{i=1}^{n-1} f(a+2ih) + f(b)\right] + E_S(h)$$

$$\int_a^b f(x)dx \cong \frac{h}{3}\left[f(a) + 4\sum_{i=1}^{2} f(a+(2i-1)h) + 2f(a+2h) + f(b)\right] + E_S(h)$$

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(a) + 4(f(a+h) + 2f(a+3h)) + 2f(a+2h) + f(b)]$$

$$\int_1^2 \frac{1}{x} dx \cong \frac{0.25}{3}[1 + 4(0.8 + 0.5714) + 2(0.6667) + 0.5] = 0.6933$$

$e_x|0.6933 - 0.6933| = 0.0003.$

Example 8 :

Find the approximate value of integration $\int_0^{10} 3x^2\, dx$ by using the **Simpsons'rule** when n=10

Solution:-

$\int_0^{10} 3x^2\, dx = x^3|_0^{10} = 1000$,$h = \frac{b-a}{n} = \frac{10-0}{10} = 1$

x	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	0	1	12	27	0.548	75	108	147	192	243	300

$$\int_a^b f(x)dx \cong \frac{h}{3}\left(f(a) + 4\sum_{i=1}^{n} f(a+(2i-1)h) + 2\sum_{i=1}^{n-1} f(a+2ih) + f(b)\right) + E_S(h)$$

$\int_a^b f(x)dx \cong \frac{h}{3}[f(a) + 4(f(a+h) + f(a+3h) + f(a+5h) + f(a+7h) + f(a+9h))+2(f(a+2h) + f(a+4h) + f(a+6h) + f(a+8h)) + f(b)]$.

$$\int_0^{10} 3x^2dx \cong \frac{1}{3}[0 + 4(3 + 27 + 75 + 147 + 243) + 2(12 + 48 + 108 + 192) + 300] = 1000$$

Homework

Find the approximate value of integration $\int_1^6 \frac{1}{\sqrt{x+2}}dx$ by using the **Simpsons'rule** when n=10