



Numerical Analysis

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Chapter Two

Solutions of Nonlinear Equations

In the next series of lectures, we shall discuss the problem of identifying the roots of equations in one Variable.

The basic formulation of the problem in the simplest case is this given a function $f(x)$ of one variable, find all x such that $f(x) = 0$.

وفي سلسلة المحاضرات القادمة سنناقش مشكلة تحديد جذور المعادلات في متغير واحد.
الصيغة الأساسية للمسألة في أبسط الحالات هي إعطاء دالة $f(x)$ لمتغير واحد، أوجد كل x بحيث يكون $f(x) = 0$.

We will review two methods for determining first approximations of possible positions for the equation $f(x) = 0$, where f is a continuous real function.

سنستعرض أسلوبين لتعيين تقريبات أولية لمواقع الجذور الحقيقية للمعادلة $f(x) = 0$ ، حيث f دالة حقيقية مستمرة.

1- (Method Graphic) تعيين مواقع الجذور بالرسم البياني

If we draw the graph of the function $f(x) = y$, the points of intersection of the function curve with the x axis represent the roots of the equation. If the equation is cut, then the function graph can be considered the axis at the points $x_1, x_2, x_3, \dots, x_n$. Each of these values represents a root. These values are first approximations to the roots.

إذا رسمنا مخطط الدالة $f(x) = y$ فإن نقاط تقاطع منحنى الدالة مع محور x تمثل جذور المعادلة، فإذا قطع للمعادلة وبالتالي يمكن اعتبار مخطط الدالة المحور في النقاط $x_1, x_2, x_3, \dots, x_n$ فإن كل من هذه القيم تمثل جذراً هذه القيم كتقريبات أولية للجذور.

Sometimes it is convenient to write the equation in the form $f_1(x) = f_2(x)$, where f_1, f_2 are functions that are easy to draw. If the two curves intersect at the point (x^*, y^*) , then x^* is considered the root of the equation.

في بعض الأحيان يكون من الملائم كتابة المعادلة بالصيغة $f_1(x) = f_2(x)$ حيث f_1, f_2 دالتين يسهل رسمهما فإذا تقاطع المنحنيان في النقطة (x^*, y^*) فإن (x^*, y^*) تعتبر جذراً للمعادلة.

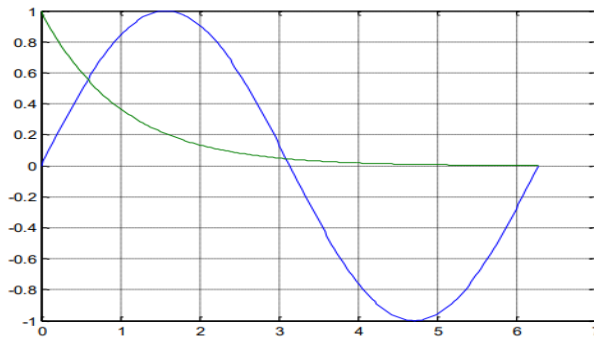
Example 1:

Determine the locations of the roots of the equation $e^x \sin(x) - 1 = 0$.

مثال: عين مواقع جذور المعادلة $e^x \sin(x) - 1 = 0$

Solution:

يمكن كتابة المعادلة السابقة بالصيغة المكافئة $\sin(x) = e^{-x}$
يمكن رسم الدالتين بنظام الماتلاب لتحديد مواقع الجذور



من الشكل الناتج سوف نلاحظ وجود تقاطع بين الدالتين الأول عند $x = 0.6$ والثاني عند $x = 3.1$ ويوجد عدد لا نهائي من التقاطعات الأخرى عندما تكون قريبة من $n\pi$.

2- Locating the position of roots (programming method):

تعيين مواقع الجذور بطريقة البرمجة

To locate the position of roots of the function (equation) $f(x) = 0$ by using programming method, we use $f(x)$ be continuous function on the interval $[a,b]$. We divide the interval $[a, b]$ into n subintervals

لتحديد مواقع الجذور للدالة (المعادلة) $f(x) = 0$ باستخدام طريقة البرمجة، نستخدم دالة مستمرة على الفترة $[a, b]$ ، نقسم الفترة $[a,b]$ إلى n فترات جزئية،

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b \text{ where } x_i = a + ih, i = 0, 1, \dots, n; h = \frac{b-a}{n}.$$

If $f(x_i) \times f(x_{i+1}) < 0$ for any $0 \leq i \leq n$, then there exists c , $a < c < b$ for which $f(c) = 0$.

Example 2:

Find the approximate location of the function

$$f(x) = x^4 - 7x^3 + 3x^2 + 26x - 10 = 0 \text{ on the interval } [-8,8] \text{ with } n = 4 \text{ and } n = 8.$$

Solution:

If $n = 4, h = \frac{b-a}{n} = \frac{8-(-8)}{4} = \frac{16}{4} = 4.$

x	-8	-4	0	4	8
$f(x)$	+	+	-	-	+

There is a root between $(-4,0)$ and $(4,8)$.

If $n=8, h = \frac{b-a}{n} = \frac{8-(-8)}{8} = \frac{16}{8} = 2.$

x	-8	-6	-4	-2	0	2	4	6	8
$f(x)$	+	+	+	+	-	+	-	+	+

There is a root between $(-2,0), (0,2), (2,4)$ and $(4,6)$.

Example 3:

Find the approximate location of the function

$f(x) = x^3 + 4x^2 - 10 = 0$ on the interval $[1,2]$ with $n = 5.$

Solution:

$n = 5, h = \frac{b-a}{n} = \frac{2-(1)}{5} = \frac{1}{5} = \frac{2}{10} = 0.2.$

x	1	1.2	1.4	1.6	1.8	2
$f(x)$	-	-	+	+	+	+

There is a root between $(1.2,1.4)$.

Example 4:

Find the approximate location of the function

$f(x) = x^2 - 1 = 0$ on the interval $[-1,1]$ with $n = 10.$

Solution:

$n = 10, h = \frac{b-a}{n} = \frac{1-(-1)}{10} = \frac{2}{10} = 0.2.$

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
$f(x)$	+	-	-	-	-	-	-	-	-	-	+

There is a root between $(-1, -0.8)$ and $(0.8,1)$.

Homework

Find the approximate location of the function

- 1- $f(x) = x^3 - 8x^2 + 7 = 0$ on the interval $[-3,3]$ with $n = 6$.
- 2- $f(x) = x^4 - 5x^3 - 3x^2 + x + 1 = 0$ on the interval $[-5,5]$ with $n = 5$.
- 3- $f(x) = 3x^4 + 2x^3 - x^2 + x + 1 = 0$ on the interval $[-8,8]$ with $n = 8$.

Numerical methods: الطرق العددية

Numerical Methods: are methods for solving problems numerically on a computer or calculator or by hand.

الطرق العددية: هي طرق حل المسائل عددياً على الحاسوب أو الآلة الحاسبة أو باليد.

* The steps from a given solution to the final answer are usually as follows:

خطوات حل أي مسألة عددية هي كالاتي

1- Modeling: set up a mathematical model of the problem such as an integral, a system of linear equation or differential equation.

النمذجة: إعداد نموذج رياضي للمشكلة مثل التكامل أو نظام المعادلة الخطية أو المعادلة التفاضلية.

2- Choice of mathematical method. اختيار طريقة الحل العددية

3- Programming: from an algorithm we write program, say as FORTRAN; C; C++ or we may decide to use a computing environment such as MATLAB and MATHEMATICA .

البرمجة: من خوارزمية نكتب برنامجاً، مثلاً FORTRAN؛ C++ أو قد نقرر لنا بيئة حوسبة مثل MATLAB وMATHEMATICA. اختيار أي برنامج جاهز أو كتابة برنامج.

4- Doing the computations. القيام بالعمليات الحسابية

5 - Interpolation the results ترتيب النتائج التي نحصل عليها على شكل جدول أو رسومات

الطرق العددية: هي طرق حل المسائل عددياً على الحاسوب أو الآلة الحاسبة أو باليد.

(i) Bisection Method: طريقة تنصيف الفترات

Suppose a continuous function f defined on the interval $[a,b]$ is given with $f(a)$ and $f(b)$ of different sign (i.e. $f(a) \times f(b) < 0$). Then by Intermediate value theorem ((If $f \in C[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ for which $f(c) = k$) there exists a point $c \in (a, b)$ such that $f(c) = 0$. If we choose the midpoint $c = \frac{b+a}{2}$, then three possibilities arise:

لنفترض أن دالة مستمرة f معرفة على الفترة المغلقة $[a,b]$ تعطي $f(a)$ و $f(b)$ بإشارات مختلفة أي $(f(a) \times f(b) < 0)$. فإنه بواسطة نظرية القيمة المتوسطة. نحصل على إذا كان $f \in C[a, b]$ و k هو أي عدد يقع بين $f(a)$ و $f(b)$ ، فهناك $c \in [a, b]$ بحيث $(f(c) = k)$ بحيث يكون $f(c) = 0$. إذا اخترنا نقطة المنتصف $c = (b + a)/2$ ، فستظهر ثلاثة احتمالات:

If $f(a) \times f(c) \begin{cases} < 0 & \text{there is a root between } a, c \Rightarrow d = \frac{a+c}{2} \\ > 0 & \text{there is a root between } b, c \Rightarrow d = \frac{b+c}{2} \\ = 0 & c \text{ is exact root ((Stop)).} \end{cases}$

The two conditions stop are التوقف بشرطين $|x_{i+1} - x_i| \leq \epsilon \vee |f(x_{i+1})| \leq \epsilon$ for any i

Example 5:

Find an approximate root of $f(x) = x^2 - 3$ in the interval $[1,2]$ by using Bisection method if error $\epsilon = 0.005$.

Solution:

$f(0) = 1^2 - 3 = -2$, $f(1) = 2^2 - 3 = 1$.

x	1	2	1.5	1.75	1.625	1.6875	1.7188	1.7344	1.7266
$y = f(x)$	-2	1	-0.75	0.0625	-0.3594	-0.1523	-0.0457	0.0081	-0.0189

$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$, $f(1.5) = (1.5)^2 - 3 = -0.75$.

$|x_1 - b| = |1.5 - 2| = 0.5 > \epsilon \vee |f(x_1)| = |-0.75| = 0.75 > \epsilon$

$x_2 = \frac{x_1 + b}{2} = \frac{1.5 + 2}{2} = 1.75$, $f(1.75) = (1.75)^2 - 3 = 0.0625$

$|x_2 - x_1| = |1.5 - 1.75| = 0.25 > \epsilon \vee |f(1.75)| = 0.0625 > \epsilon$

$$x_3 = \frac{x_2 + x_1}{2} = \frac{1.75 + 1.5}{2} = 1.625, f(1.625) = (1.625)^2 - 3 = -0.3594$$

$$|x_3 - x_1| = |1.625 - 1.75| = 0.125 > \epsilon \vee |f(1.625)| = |-0.3594| = 0.3594 > \epsilon$$

$$x_4 = \frac{x_3 + x_2}{2} = \frac{1.75 + 1.625}{2} = 1.6875, f(1.6875) = (1.6875)^2 - 3 = -0.1523$$

$$|x_4 - x_3| = |1.6875 - 1.75| = 0.0625 > \epsilon \vee |f(1.6875)| = |-0.1523| = 0.1523 > \epsilon$$

$$x_5 = \frac{x_2 + x_4}{2} = \frac{1.75 + 1.6875}{2} = 1.7188, f(1.7188) = (1.7188)^2 - 3 = -0.0457$$

$$|x_5 - x_4| = |1.7188 - 1.6875| = 0.0313 > \epsilon \vee |f(1.7188)| = |-0.0457| = 0.0457 > \epsilon$$

$$x_6 = \frac{x_4 + x_5}{2} = \frac{1.75 + 1.7188}{2} = 1.7344, f(1.7344) = (1.7344)^2 - 3 = 0.0081$$

$$|x_6 - x_5| = |1.7344 - 1.7188| = 0.0156 > \epsilon \vee |f(1.7344)| = |0.0081| = 0.0081 < \epsilon$$

$$x_7 = \frac{x_6 + x_5}{2} = \frac{1.7344 + 1.7188}{2} = 1.7266, f(1.7266) = (1.7266)^2 - 3 = -0.0189$$

$$|x_6 - x_5| = |1.7344 - 1.7266| = 0.0078 < \epsilon \vee |f(1.7266)| = |-0.0189| = 0.0189 > \epsilon$$

Then $x_7 = 1.7266$ is the approximate root.

Example 6:

Find an approximate root of $f(x) = x^2 - \frac{1}{2}$ in the interval (0,1) by using Bisection method if error $\epsilon = 0.01$.

Solution:

$$f(0) = 0^2 - \frac{1}{2} = -0.5, f(1) = 1^2 - \frac{1}{2} = 0.5.$$

x	0	1	0.5	0.75	0.625	0.6875	0.7188	0.7032
$y = f(x)$	-0.5	0.5	-0.25	0.0625	-0.1094	-0.0273	0.0167	-0.0055

$$x_1 = \frac{a+b}{2} = \frac{0+1}{2} = 0.5, f(0.5) = (0.5)^2 - \frac{1}{2} = -0.25.$$

$$|x_1 - b| = |0.5 - 1| = 0.5 > \epsilon \vee |f(x_1)| = |-0.25| = 0.25 > \epsilon$$

$$x_2 = \frac{x_1 + b}{2} = \frac{0.5 + 1}{2} = 0.75, f(0.75) = (0.75)^2 - \frac{1}{2} = 0.0625$$

$$|x_2 - x_1| = |0.5 - 0.75| = 0.25 > \epsilon \vee |f(0.75)| = 0.0625 > \epsilon$$

$$x_3 = \frac{x_2 + x_1}{2} = \frac{0.75 + 0.5}{2} = 0.625, f(0.625) = (0.625)^2 - \frac{1}{2} = -0.1094$$

$$|x_3 - x_1| = |0.625 - 0.75| = 0.125 > \epsilon \vee |f(0.625)| = |-0.1094| = 0.1094 > \epsilon$$

$$x_4 = \frac{x_3 + x_2}{2} = \frac{0.75 + 0.625}{2} = 0.6875, f(0.6875) = (0.6875)^2 - \frac{1}{2} = -0.0273$$

$$|x_4 - x_3| = |0.6875 - 0.75| = 0.0625 > \epsilon \vee |f(0.6875)| = |-0.0273| = 0.0273 > \epsilon$$

$$x_5 = \frac{x_2 + x_4}{2} = \frac{0.75 + 0.6875}{2} = 0.7188, f(0.7188) = (0.7188)^2 - \frac{1}{2} = -0.0167$$

$$|x_5 - x_4| = |0.7188 - 0.6875| = 0.0313 > \epsilon \vee |f(0.7188)| = |-0.0167| = 0.0167 > \epsilon$$

$$x_6 = \frac{x_4 + x_5}{2} = \frac{0.6875 + 0.7188}{2} = 0.7032, f(0.7032) = (0.7032)^2 - \frac{1}{2} = -0.0055$$

$$|x_6 - x_5| = |0.7032 - 0.7188| = 0.0156 > \epsilon \vee |f(0.7032)| = |-0.0055| = 0.0055 < \epsilon$$

Then $x_6 = 0.7032$ is the approximate root.

Example 7:

Find an approximate root of $f(x) = xe^x - 2 = 0$ in the interval $[0.5, 0.9]$ by using Bisection method if error $\epsilon = 0.007$.

Solution:

$$f(0.5) = 0.5 e^{(0.5)} - 2 = -1.1756, f(0.9) = 0.9 e^{(0.9)} - 2 = 0.2136$$

x	0.5	0.9	0.7	0.8	0.85	0.875	0.8625	0.8562
$y = f(x)$	-1.1756	0.2136	-0.5904	-0.2196	-0.0113	0.0990	0.0433	0.0157

$$x_1 = \frac{a+b}{2} = \frac{0.5+0.9}{2} = 0.7, f(0.7) = 0.7e^{(0.7)} - 2 = -0.5904$$

$$|x_1 - b| = |0.7 - 0.9| = 0.2 > \epsilon \vee |f(x_1)| = |f(-0.5904)| = |-0.5904| = 0.5904 > \epsilon$$

$$x_2 = \frac{x_1 + b}{2} = \frac{0.7 + 0.9}{2} = 0.8, \quad f(0.8) = 0.8 e^{(0.8)} - 2 = -0.2196$$

$$|x_2 - x_1| = |0.8 - 0.7| = 0.1 > \epsilon \vee |f(x_2)| = |-0.2196| = 0.2196 > \epsilon$$

$$x_3 = \frac{x_2 + b}{2} = \frac{0.8 + 0.9}{2} = 0.85, \quad f(0.85) = 0.85 e^{(0.85)} - 2 = -0.0113$$

$$|x_3 - x_2| = |0.85 - 0.8| = 0.05 > \epsilon \vee |f(x_3)| = |-0.0113| = 0.0113 > \epsilon$$

$$x_4 = \frac{x_3 + b}{2} = \frac{0.85 + 0.9}{2} = 0.875, \quad f(0.875) = 0.875 e^{(0.875)} - 2 = 0.0990$$

$$|x_4 - x_3| = |0.875 - 0.85| = 0.025 > \epsilon \vee |f(x_4)| = |0.0990| = 0.0990 > \epsilon$$

$$x_5 = \frac{x_3 + x_4}{2} = \frac{0.85 + 0.875}{2} = 0.8625, \quad f(0.8625) = 0.8625 e^{(0.8625)} - 2 = 0.0433$$

$$|x_5 - x_4| = |0.8625 - 0.875| = 0.0125 > \epsilon \vee |f(x_4)| = |0.0433| = 0.0433 > \epsilon$$

$$x_6 = \frac{x_5 + x_3}{2} = \frac{0.8625 + 0.85}{2} = 0.8562,$$

$$f(0.8562) = (0.8562) e^{(0.8562)} - 2 = 0.0157$$

$$|x_6 - x_5| = |0.8562 - 0.8625| = 0.0063 < \epsilon$$

Then $x_6 = 0.8562$ is the approximate root.

Example 8:

Find an approximate root of $f(x) = x \cdot \log x - 1 = 0$ in the interval $[2,3]$ by using Bisection method if error $\epsilon = 9 \times 10^{-3}$.

Solution:

$$, f(2) = 2 \cdot \log(2) - 1 = -0.3979 \quad \& \quad f(3) = 3 \cdot \log(3) - 1 = 0.4314$$

x	2	3	2.5	2.75	2.625	2.5625	2.5313	2.5157
$y = f(x)$	-0.3979	0.4314	-0.0051	0.2081	0.1002	0.0472	0.0210	0.0079

$$x_1 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5, \quad f(2.5) = (2.5) \cdot \log(2.5) - 1 = -0.0051$$

$$|x_1 - b| = |3 - 2.5| = 0.5 > \epsilon \vee |f(x_1)| = |f(2.5)| = |-0.0051| = 0.0051 > \epsilon$$

$$x_2 = \frac{x_1 + b}{2} = \frac{3 + 2.5}{2} = 2.75, \quad f(2.75) = (2.75) \cdot \log(2.75) - 1 = 0.2081$$

$$|x_2 - x_1| = |2.75 - 2.5| = 0.25 > \epsilon \vee |f(x_2)| = |0.2081| = 0.2081 > \epsilon$$

$$x_3 = \frac{x_2 + x_1}{2} = \frac{2.75 + 2.5}{2} = 2.625, f(2.625) = (2.625). \log(2.625) - 1 = 0.1002$$

$$|x_3 - x_2| = |2.625 - 2.5| = 0.125 > \epsilon \vee |f(x_3)| = |0.1002| = 0.1002 > \epsilon$$

$$x_4 = \frac{x_3 + x_1}{2} = \frac{2.625 + 2.5}{2} = 2.5625, f(2.5625) = (2.5625). \log(2.5625) - 1 = 0.0472$$

$$|x_4 - x_3| = |2.5625 - 2.625| = 0.0625 > \epsilon \vee |f(x_4)| = |0.0472| = 0.0472 < \epsilon$$

$$x_5 = \frac{x_2 + x_4}{2} = \frac{2.5 + 2.5625}{2} = 2.5313,$$

$$f(2.5313) = (2.5313). \log(2.5313) - 1 = 0.0210$$

$$|x_5 - x_4| = |2.5313 - 2.5625| = 0.0312 > \epsilon \vee |f(x_4)| = |0.0210| = 0.0210 > \epsilon$$

$$x_6 = \frac{x_5 + x_1}{2} = \frac{2.5313 + 2.5}{2} = 2.5157,$$

$$f(2.5157) = (2.5157). \log(2.5157) - 1 = 0.0079$$

$$|x_6 - x_5| = |2.5157 - 2.5313| = 0.0156 > \epsilon,$$

$$|f(2.5157)| = |0.0079| = 0.0079 < \epsilon$$

Then $x_6 = 2.5157$ is the approximate root.

Homework

1- Find an approximate root of $f(x) = x^3 - 5$ in the interval $[1,2]$ by using Bisection method if error $\epsilon = 0.003$.

2- Find an approximate root of $f(x) = x - 2^{-x} = 0$ in the interval $(0,1)$ by using Bisection method with error $\epsilon = 0.04$.

3- Find an approximate root of $f(x) = x^3 + 4x^2 - 10 = 0$ in the interval $[1,2]$ by using Bisection method with error $\epsilon = 0.04$.

(ii) The False-Position method : طريقة الموضع الكاذب

Suppose a continuous function f defined on the interval $[x_1, x_2]$ is given with $f(x_1)$ and $f(x_2)$ of opposite sign (i.e. $f(x_1) \cdot f(x_2) < 0$). To derive a formula for false-position method, approximate the graph of f by a straight line on $[x_1, x_2]$ connecting $(x_1, f(x_1))$ and $(x_2, f(x_2))$ which intersect x-axis at $(x_3, 0)$ where x_3 is more approximate to the exact root than x_1 and x_2 . To obtain a formula for x_3 we use the slop equality:

لنفترض أن دالة مستمرة ومعرفة على الفترة $[x_1, x_2]$ تعطي $f(x_1)$ و $f(x_2)$ بإشارات مختلفة ($f(x_1) \cdot f(x_2) < 0$). لاشتقاق صيغة لطريقة الموضع الكاذب، قم بتقريب الرسم البياني لـ f بخط مستقيم $[x_1, x_2]$ وتوصيل $(x_1, f(x_1))$ و $(x_2, f(x_2))$ التي يتقاطع مع المحور السيني عند $(x_3, 0)$ حيث يكون x_3 أكثر تقريباً للجذر الدقيق من x_1 و x_2 . للحصول على صيغة x_3 نستخدم المتطابقة ادناه:

$$\frac{y - y_2}{x - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow (x - x_2)(y_2 - y_1) = (y - y_2)(x_2 - x_1)$$

we put $y = 0$

$$(x - x_2)(y_2 - y_1) = -y_2(x_2 - x_1)$$

$$\Rightarrow ((x - x_2)(y_2 - y_1) = -y_2(x_2 - x_1)) \div (y_2 - y_1)$$

$$\Rightarrow (x - x_2) = -y_2 \frac{(x_2 - x_1)}{(y_2 - y_1)} \Rightarrow x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)}$$

To find another approximation: لإيجاد قيم أخرى

If $f(x_1) \times f(x_2) < 0$ there is a root between x_1, x_3

If $f(x_1) \times f(x_2) > 0$ there is not a root between x_2, x_3

If $f(x_1) \times f(x_2) = 0$ c is exact root ((Stop)).

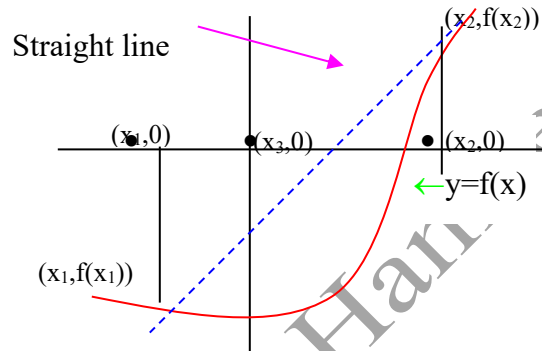
$$\Rightarrow x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)}$$

⋮

$$\Rightarrow x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1} - x_i)}{(y_{i+1} - y_i)}$$

And Stop condition is $|x_{i+2} - x_{i+1}| \leq \epsilon$ for any i.

and Stop condition is $|x_{i+2} - x_{i+1}| \leq \epsilon$ for any i .



Example 9:

Find an approximate root of $f(x) = x^2 - 3$ in the interval $[1,2]$ by using False-Position method if error $\epsilon = 0.005$.

Solution:

$$x_1 = 1, f(1) = 1^2 - 3 = -2, \quad \&$$

$$x_2 = 2, f(2) = 2^2 - 3 = 1 \Rightarrow (f(x_1) \cdot f(x_2) < 0).$$

x	1	2	1.6667	1.7273	1.7321
$y = f(x)$	-2	1	-0.2221	-0.0164	0.0002

$$x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1} - x_i)}{(y_{i+1} - y_i)}$$

$$x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)} \Rightarrow x_3 = 2 - \frac{(1)(2 - 1)}{(1 - (-2))} = 2 - \frac{1}{3} = 1.6667$$

$$f(1.6667) = (1.6667)^2 - 3 = -0.2221$$

$$|x_3 - x_2| = |1.6667 - 2| = 0.3333 > \epsilon = 0.005.$$

$$x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)} \Rightarrow x_4 = (1.6667) - \frac{(-0.2221)(1.6667 - 2)}{(-0.2221 - 1)} = 1.7273$$

$$f(1.7273) = (1.7273)^2 - 3 = -0.0164$$

$$|x_4 - x_3| = |1.7273 - 1.6667| = |-0.0164| = 0.0164 > \epsilon = 0.005.$$

$$x_5 = x_4 - \frac{y_4(x_4 - x_3)}{(y_4 - y_3)} \Rightarrow x_5 = (1.7273) - \frac{(-0.0164)(1.7273 - 1.6667)}{(-0.0164 - (-0.2221))}$$

$$= 1.7321$$

$$f(1.7321) = (1.7321)^2 - 3 = 0.0002$$

$$|x_5 - x_4| = |1.7321 - 1.7273| = 0.0048 < \epsilon = 0.005.$$

Then $x_5 = 1.7321$ is the approximate root.

Example 10:

Find an approximate root of $f(x) = x + e^{-2x} - 1 = 0$ in interval $[0.5, 1]$ by using False-Position method with error $\epsilon = 0.007$.

Solution:

$$x_1 = 0.5, f(0.5) = 0.5 + e^{-2(0.5)} - 1 = -0.1321 \quad \&$$

$$x_2 = 1, f(1) = 1 + e^{-2(1)} - 1 = 0.1353 \quad \Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	0.5	1	0.7471	0.7910	0.7961
$y = f(x)$	-0.1321	0.1353	-0.0285	-0.0034	-0.0004

$$x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1} - x_i)}{(y_{i+1} - y_i)}$$

$$x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)} \Rightarrow x_3 = 1 - \frac{(0.1353)(1 - 0.5)}{(0.1353 - (-0.1321))} = 0.7471$$

$$f(0.7471) = 0.7471 + e^{-2(0.7471)} - 1 = -0.0285$$

$$|x_3 - x_2| = |0.7471 - 1| = 0.2529 > \epsilon = 0.007$$

$$x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)} \Rightarrow x_4 = 0.7471 - \frac{(-0.0285)(0.7471 - 1)}{(-0.0285 - 0.1353)} = 0.7910$$

$$f(0.7910) = 0.7910 + e^{-2(0.7910)} - 1 = -0.0034$$

$$|x_4 - x_3| = |0.7910 - 0.7471| = 0.0439 > \epsilon$$

$$x_5 = x_4 - \frac{y_4(x_4 - x_3)}{(y_4 - y_3)} \Rightarrow x_5 = 0.7910 - \frac{(-0.0034)(0.7910 - 1)}{(-0.0034 - 0.1353)} = 0.7961$$

$$f(0.7961) = 0.7961 + e^{-2(0.7961)} - 1 = -0.0004$$

$$|x_5 - x_4| = |0.7961 - 0.7910| = 0.005 < \epsilon \vee |f(x_5)| = |-0.0004| = 0.0004 < \epsilon$$

Then $x_5 = 0.7961$ is the approximate root.

Example 11:

Find an approximate root of $f(x) = xe^x - 2 = 0$ in the interval $[0.5, 0.9]$ by using False-Position method if error $\epsilon = 0.005$.

Solution:

$$x_1 = 0.5, f(0.5) = 0.5 e^{(0.5)} - 2 = -1.1756 \text{ \&}$$

$$x_2 = 0.9, f(0.9) = 0.9 e^{(0.9)} - 2 = 0.2136, \Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	0.5	0.9	0.8693	0.8536
$y = f(x)$	-1.1756	0.2136	0.0734	0.0043

$$x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1} - x_i)}{(y_{i+1} - y_i)}$$

$$x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)} \Rightarrow x_3 = 0.9 - \frac{(0.2136)(0.9 - 0.5)}{(0.2136 - (-1.1756))}$$

$$= 0.9 - \frac{0.0427}{1.3892} = 0.9 - 0.0307 = 0.8693$$

$$f(0.8693) = (0.8693) \cdot e^{(0.8693)} - 2 = 0.0734$$

$$|x_3 - x_2| = |0.8693 - 0.9| = |-0.0307| = 0.0307 > \epsilon = 0.005$$

$$x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)} \Rightarrow x_4 = (0.8693) - \frac{(0.0734)(0.8693 - 0.9)}{(0.0734 - (0.2136))}$$

$$= (0.8693) - \frac{(0.0734)(-0.0307)}{(-0.1402)} = (0.8693) - \frac{-0.0022}{-0.1402} = 0.8693 - 0.0157$$

$$= 0.8536$$

$$f(0.8536) = (0.8536) \cdot e^{(0.8536)} - 2 = 0.0043$$

$$|x_4 - x_3| = |0.8536 - 0.8693| = 0.0157 > \epsilon = 0.005$$

$$|f(0.8536)| = |0.0043| = 0.0043 < \epsilon = 0.005$$

Then $x_4 = 0.8536$ is the approximate root.

Example 12:

Find an approximate root of $f(x) = x \cdot \log x - 1 = 0$ in the interval $[2,3]$ by using False-Position method if error $\epsilon = 9 \times 10^{-3}$.

Solution:

$$x_1 = 2, f(2) = 2 \cdot \log(2) - 1 = -0.3979 \quad \&$$

$$x_2 = 3, f(3) = 3 \cdot \log(3) - 1 = 0.4314 \Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	2	3	2.4798	2.5049	2.625
$y = f(x)$	-0.3979	0.4314	-0.0219	-0.0011	0.1002

$$x_{i+2} = x_{i+1} - \frac{y_{i+1}(x_{i+1} - x_i)}{(y_{i+1} - y_i)}$$

$$x_3 = x_2 - \frac{y_2(x_2 - x_1)}{(y_2 - y_1)} \Rightarrow x_3 = 3 - \frac{(0.4314)(3 - 2)}{(0.4314 - (-0.3979))} = 2.4798$$

$$f(2.4798) = (2.4798) \cdot \log(2.4798) - 1 = -0.0219$$

$$|x_3 - x_2| = |2.4798 - 3| = 0.5202 > \epsilon = 0.0700$$

$$x_4 = x_3 - \frac{y_3(x_3 - x_2)}{(y_3 - y_2)} \Rightarrow x_4 = (2.4798) - \frac{(-0.0219)(2.4798 - 3)}{(-0.0219 - 0.4314)} = 2.5049$$

$$f(2.5049) = (2.5049) \cdot \log(2.5049) - 1 = -0.0011$$

$$|x_4 - x_3| = |2.5049 - 2.4798| = 0.0251 < \epsilon = 0.0700$$

Then $x_4 = 2.5049$ is the approximate root.

Homework

1- Find an approximate root of $f(x) = x^3 - 5$ in the interval $[1,2]$ by using False-Position method if error $\epsilon = 0.003$.

2- Find an approximate root of $f(x) = x - 2^{-x} = 0$ in the interval $(0,1)$ by using False-Position method with error $\epsilon = 0.04$.

3- Find an approximate root of $f(x) = x^3 + 4x^2 - 10 = 0$ in the interval $[1,2]$ by False-Position method method with error $\epsilon = 0.04$.

4- Find an approximate root of $f(x) = e^x - 3x = 0$ in interval $[1,2]$ by using False-Position method with error $\epsilon = 5 \times 10^{-2}$.

(iii) Secant Method: طريقة القاطع

Suppose a continuous function f defined on the interval $[a, b]$ is given, the graph of the function f is approximated by a secant line, we get

لنفترض أن دالة مستمرة f معرفة على الفترة المغلقة $[a, b]$ معطاة، ويتم تقريب الرسم البياني للدالة f بواسطة خط قاطع، نحصل على

$$\frac{f(b)-f(a)}{b-a} = \frac{f(b)-y}{b-x} \Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{f(b)-0}{b-x_1}$$

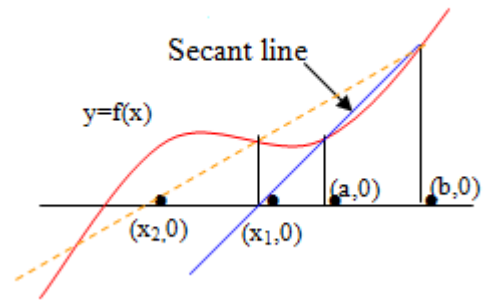
$$\Rightarrow x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$$

$$\text{Similarly, } x_2 = \frac{bf(x_1)-x_1f(b)}{f(x_1)-f(b)}$$

$$\text{In general, } x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)},$$

$i=0, 1, \dots, x_{i+2}$,

and Stop condition is $|x_{i+2} - x_{i+1}| < \epsilon$ for any i .



Example 13:

Find an approximate root of $f(x) = x^3 - 2x^2 - 5$ in interval $[2,3]$ by using Secant method with error $\epsilon = 0.005$.

Solution:

$$x_1 = 2, f(2) = (2)^3 - 2(2)^2 - 5 = -5 \quad \&$$

$$x_2 = 3, f(3) = (3)^3 - 2(3)^2 - 5 = 4, \quad \Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	2	3	2.5556	2.6691	2.6924	2.6903
$y = f(x)$	-5	4	-1.3714	-0.2333	0.0192	-0.0038

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)},$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \frac{2(4) - (3)(-5)}{((4) - (-5))} = \frac{8 + 15}{9} = \frac{23}{9} = 2.5556$$

$$f(2.5556) = (2.5556)^3 - 2(2.5556)^2 - 5 = 16.6908 - 13.0622 - 5 = -1.3714$$

$$|x_3 - x_2| = |2.5556 - 3| = 0.4444 > \epsilon = 0.007 .$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \Rightarrow x_4 = \frac{((3)(-1.3714) - (2.5556)(4))}{(-1.3714 - 4)}$$

$$= \frac{(-4.1142) - (10.2224)}{(-5.3714)} = \frac{-14.3366}{-5.3714} = 2.6691.$$

$$f(2.6691) = (2.6691)^3 - 2(2.6691)^2 - 5 = 19.0149 - 14.2482 - 5 = -0.2333$$

$$|x_4 - x_3| = |2.6691 - 2.5556| = 0.1135 > \epsilon.$$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} \Rightarrow x_5 = \frac{((2.5556)(-0.2333) - (2.6691)(-1.3714))}{(-0.2333 - (-1.3714))}$$

$$= \frac{-0.5962 + 3.6604}{1.1381} = \frac{3.0642}{1.1381} = 2.6924.$$

$$f(2.6923) = (2.6924)^3 - 2(2.6924)^2 - 5 = 19.5172 - 14.4980 - 5 = 0.0192$$

$$|x_5 - x_4| = |2.6924 - 2.6691| = 0.0233 > \epsilon.$$

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} \Rightarrow x_6 = \frac{((2.6691)(0.0192) - (2.6924)(-0.2333))}{(0.0192 - (-0.2333))}$$

$$= \frac{0.0512 + 0.6281}{0.2525} = \frac{0.6793}{0.2525} = 2.6903.$$

$$f(2.6903) = (2.6903)^3 - 2(2.6903)^2 - 5 = 19.4716 - 14.4754 - 5 = -0.0038$$

$$|x_6 - x_5| = |2.6903 - 2.6924| = |-0.0021| = 0.0021 < \epsilon = 0.005 \quad \checkmark$$

$$|f(x_6)| = |-0.0038| = 0.0038 < \epsilon = 0.005$$

Then $x_6 = 2.6903$ is the approximate root.

Example 14:

Find an approximate root of $f(x) = x^2 - 3$ in the interval $[1,2]$ by using Secant method if error $\epsilon = 0.005$.

Solution:

$$x_1 = 1, f(1) = (1)^2 - 3 = -2, \quad x_2 = 2, f(2) = (2)^2 - 3 = 1$$

$$\Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	1	2	1.6667	1.7273	1.9976	1.7321
$y = f(x)$	-2	1	-0.2221	-0.0164	0.9904	0.0002

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)},$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \frac{1(1) - (2)(-2)}{(1) - (-2)} = \frac{1 + 4}{3} = \frac{5}{3} = 1.6667$$

$$f(1.6667) = (1.6667)^2 - 3 = 2.7779 - 3 = -0.2221$$

$$|x_3 - x_2| = |1.6667 - 2| = -0.3333 > \epsilon = 0.005 \quad \vee$$

$$|f(x_3)| = |-0.2221| = 0.2221 > \epsilon = 0.005$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \Rightarrow x_4 = \frac{2(-0.2221) - (1.6667)(1)}{(-0.2221) - 1} = \frac{-2.1109}{-1.2221} = 1.7273$$

$$f(1.7273) = (1.7273)^2 - 3 = 2.9836 - 3 = -0.0164$$

$$|x_4 - x_3| = |1.7273 - 1.6667| = |-0.0606| = 0.0606 > \epsilon = 0.005 \quad \vee$$

$$|f(x_4)| = |-0.0164| = 0.0164 > \epsilon = 0.005$$

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} \Rightarrow x_5 = \frac{(1.6667)(-0.0164) - (0.2221)(1.7273)}{(-0.0164) - (-0.2221)} = \frac{-0.4109}{0.2057} = 1.9976$$

$$f(1.9976) = (1.9976)^2 - 3 = 3.9904 - 3 = 0.9904$$

$$|x_5 - x_4| = |1.9976 - 1.7273| = |0.2703| = 0.2703 > \epsilon = 0.005 .$$

$$|f(x_5)| = |0.9904| = 0.9904 > \epsilon = 0.005 .$$

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} \Rightarrow x_6 = \frac{(1.7273)(0.9904) - (-0.0164)(1.9976)}{(0.9904) - (-0.0164)} = \frac{1.7435}{1.0068} = 1.7317$$

$$f(1.7317) = (1.7317)^2 - 3 = 2.9988 - 3 = -0.0012$$

$$|x_6 - x_5| = |1.7317 - 1.9976| = |-0.2659| = 0.2659 > \epsilon = 0.005 .$$

$$|f(x_6)| = |-0.0012| = 0.0012 < \epsilon = 0.005$$

Then $x_6 = 1.7317$ is the approximate root.

Example 15:

Find an approximate root of $f(x) = x + e^{-2x} - 1 = 0$ in interval $[0.5,1]$ by using Secant method with error $\epsilon = 0.007$.

Solution:

$$x_1 = 0.5, f(0.5) = 0.5 + e^{-2 \cdot (0.5)} - 1 = -0.1321 \text{ \&}$$

$$x_2 = 1, f(1) = 1 + e^{-2 \cdot (1)} - 1 = 0.1353 \Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	0.5	1	0.7468	0.7907
$y = f(x)$	-0.1321	0.1353	-0.0286	-0.0036

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)},$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \frac{(0.5)(0.1353) - (1)(-0.1321)}{((-0.1321) - 0.1353)} = \frac{0.1997}{0.2674} = 0.7468$$

$$f(0.7468) = (0.7468) + e^{-2 \cdot (0.7468)} - 1 = -0.0286$$

$$|x_3 - x_2| = |0.7468 - 1| = -0.2532 > \epsilon = 0.007 \quad \vee$$

$$|f(x_3)| = |-0.0286| = 0.0286 > \epsilon = 0.005$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \Rightarrow x_4 = \frac{1(-0.0286) - (0.7468)(0.1353)}{(-0.0286 - 0.1353)} = \frac{-0.1296}{-0.1639} = 0.7907$$

$$f(0.7907) = (0.7907) + e^{-2 \cdot (0.7907)} - 1 = -0.0036$$

$$|x_4 - x_3| = |0.7907 - 0.7468| = |0.0439| = 0.0439 > \epsilon = 0.005 \quad \vee$$

$$|f(x_4)| = |-0.0036| = 0.0036 < \epsilon = 0.005$$

Then $x_4 = 0.7907$ is the approximate root.

Example 16:

Find an approximate root of $f(x) = xe^x - 2 = 0$ in the interval $[0.5,0.9]$ by using Secant method if error $\epsilon = 0.005$.

Solution:

$$x_1 = 0.5, f(0.5) = 0.5 e^{(0.5)} - 2 = -1.1756 \text{ \&}$$

$$x_2 = 0.9, f(0.9) = 0.9 e^{(0.9)} - 2 = 0.2136 \Rightarrow (f(x_1) \cdot f(x_2) < 0)$$

x	0.5	0.9	0.8385	0.8848	0.8536
$y = f(x)$	-1.1756	0.2136	-0.0606	0.1434	0.0043

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)},$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \frac{(0.5)(0.2136) - (0.9)(-1.1756)}{((0.2136) - (-1.1756))} = \frac{1.1648}{1.3892} = 0.8385$$

$$f(0.8385) = 0.5 e^{(0.8385)} - 2 = -0.9746$$

$$|x_3 - x_2| = |0.8385 - 0.9| = |-0.0615| = 0.0615 > \epsilon = 0.005 \vee$$

$$|f(x_3)| = |0.8385| = 0.8385 > \epsilon = 0.005$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \Rightarrow x_4 = \frac{(0.9)(-0.0606) - (0.8848)(0.2136)}{(-0.0606 - 0.2136)}$$

$$= \frac{-0.2435}{-0.2742} = 0.8880$$

$$f(0.8880) = (0.8880) e^{(0.8880)} - 2 = 0.1581$$

$$|x_4 - x_3| = |0.8880 - 0.8848| = |0.0032| = 0.0032 < \epsilon = 0.005 \vee$$

Then $x_4 = 0.8880$ is the approximate root.

Example 16:

Find an approximate root of $f(x) = x \cdot \log x - 1 = 0$ in the interval $[2,3]$ by using Secant method if error $\epsilon = 9 \times 10^{-3}$.

Solution:

$$, f(2) = 2 \cdot \log(2) - 1 = -0.3979 \text{ \& } f(3) = 3 \cdot \log(3) - 1 = 0.4314$$

x	2	3	2.4798	2.5050
$y = f(x)$	-0.3979	0.4314	-0.0219	-0.0009

$$x_{i+2} = \frac{x_i f(x_{i+1}) - x_{i+1} f(x_i)}{f(x_{i+1}) - f(x_i)}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \Rightarrow x_3 = \frac{2(0.4314) - (3)(-0.3979)}{(0.4314 - (-0.3979))}$$

$$= \frac{0.8628 + 1.1937}{0.8293} = \frac{2.0565}{0.8293} = 2.4798$$

$$f(2.4798) = (2.4798) \log(2.4798) - 1 = -0.0219$$

$$|x_3 - x_2| = |2.4798 - 3| = |-0.5202| = 0.5202 > \epsilon = 0.009$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} \Rightarrow x_4 = \frac{(3)(-0.0219) - (2.4798)(0.4314)}{((-0.0219) - 0.4314)}$$

$$= \frac{-0.0657 - 1.0698}{-0.4533} = \frac{-1.1355}{-0.4533} = 2.5050$$

$$f(2.5050) = (2.5050) \log(2.5050) - 1 = -0.0009$$

$$|x_4 - x_3| = |2.5050 - 2.4798| = |0.0252| = 0.0252 > \epsilon = 0.009$$

$$|f(2.5050)| = |-0.0009| = 0.0009 < \epsilon = 0.009$$

Then $x_4 = 2.5050$ is the approximate root.

Homework

By using Secant Method, Find an approximate root of

1- $f(x) = x^3 - 5$ in the interval $[1,2]$ with error $\epsilon = 0.003$.

2- $f(x) = x - 2^{-x} = 0$ in the interval $(0,1)$ with error $\epsilon = 0.04$.

3- $f(x) = x^3 + 4x^2 - 10 = 0$ in the interval $[1,2]$ with error $\epsilon = 0.04$.

4- $f(x) = e^x - 3x = 0$ in interval $[1,2]$ with error $\epsilon = 5 \times 10^{-2}$.

5- $f(x) = x^2 - 2x - 1 = 0$ between $[-1,0]$ with error $\epsilon = 10^{-3}$.

6- $f(x) = x^4 - x - 10$ in interval $[1,2]$ with error $\epsilon = 0.0001$.

طريقة نيوتن - رافسون: Newton-Raphson method (IV)

Suppose function f is continuous and differentiable on the interval $[a, b]$, Let x_0 be an approximation to the exact root x_1 such that $f'(x_1) \neq 0$ and h is error between x_1, x_0 such that $x_1 = x_0 + h$. Using the Taylor series to function f about point x_0 we get:
 لنفترض أن الدالة f مستمرة وقابلة للاشتقاق على الفترة $[a, b]$ ، فليكن x_0 هو جذر تقريبيًا للجذر المضبوط x_1 بحيث ان $f'(x_1) \neq 0$ و h يكون الخطأ بين x_0 و x_1 بحيث ان $x_1 = x_0 + h$. باستخدام متسلسلة تايلور للدالة f حول النقطة x_0 نحصل على:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \frac{(x-x_0)^3}{3!}f'''(x_0) + \dots \quad \dots(1)$$

put $x = x_1$ in formula (1) we get:-

$$f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) + \frac{(x_1 - x_0)^2}{2!}f''(x_0) + \frac{(x_1 - x_0)^3}{3!}f'''(x_0)$$

$$f(x_1) = f(x_0 + h) = f(x_0) + (h)f'(x_0) + \frac{h^2}{2!}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots$$

... (2)

h is Small then h^2, h^3, \dots are very small can be deleted.

h صغير فان h^2, h^3, \dots صغير جدًا ويمكن حذفه.

$$\Rightarrow f(x_1) = f(x_0 + h) = f(x_0) + (h)f'(x_0)$$

$$\because x_1 \text{ is exact root then } f(x_1) = 0 \Rightarrow 0 = f(x_0) + hf'(x_0)$$

$$\Rightarrow h = -\frac{f(x_0)}{f'(x_0)}, f'(x_0) \neq 0$$

We offset the value of h in formula $x_1 = x_0 + h$, we get:-

نقوم بتعويض قيمة h في الصيغة $x_1 = x_0 + h$ ، فنحصل على:-

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, f'(x_0) \neq 0, \quad x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, f'(x_1) \neq 0$$

Thus, the general formula of the **Newton-Raphson method** is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, f'(x_i) \neq 0, i = 1, 2, 3, \dots$$

The two conditions stop are $|x_{i+1} - x_i| \leq \epsilon \vee |f(x_{i+1})| \leq \epsilon$ for any i .

Note:

Sometime **Newton-Raphson method** does not converge to the required root:

1. If there is no real root of the function $f(x)$.
2. If the initial value so far from the exact root λ .
3. If $f'(x_i) = 0$.

ملاحظة:

في بعض الأحيان لا تتقارب طريقة نيوتن-رافسون مع الجذر المطلوب:

1. إذا لم يكن هناك جذر حقيقي للدالة f .
2. إذا كانت القيمة الأولية بعيدة عن الجذر المطلوب λ .
3. إذا كان $f'(x_i) = 0$.

Example 17:

Using **Newton-Raphson method**, find the solution to the equation $f(x) = x^3 + 5x - 3 = 0$ in interval $[0,1]$ with error $\epsilon = 0.0002$.

Solution:

$$f(0) = (0)^3 + 5(0) - 3 = -3, \quad f(1) = (1)^3 + 5(1) - 3 = 3$$

$$f(x) = x^3 + 5x - 3 = 0, \quad f'(x) = 3x^2 + 5 = 0, \quad \boxed{x_0 = 0.5}$$

x	0.5	0.5652	0.5641
$y = f(x)$	-0.375	0.0066	0.00002
$f'(x)$	5.75	5.9584	5.9546

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad f'(x_i) \neq 0, \quad i = 1, 2, 3, \dots$$

$$f(0.5) = (0.5)^3 + 5(0.5) - 3 = 0.125 + 2.5 - 3 = -0.375,$$

$$f'(0.5) = 3(0.5)^2 + 5 = 0.75 + 5 = 5.75.$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \quad f'(x_0) \neq 0 \Rightarrow x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$\Rightarrow x_1 = 0.5 - \frac{(-0.375)}{5.75} = 0.5 + \frac{0.375}{5.75} = 0.5 + 0.0652 \Rightarrow x_1 = 0.5652$$

$$f(0.5652) = (0.5652)^3 + 5(0.5652) - 3 = 0.1806 + 2.826 - 3 = 0.0066,$$

$$f'(0.5652) = 3(0.5652)^2 + 5 = 0.9584 + 5 = 5.9584$$

$$|x_1 - x_0| = |0.5652 - 0.5| = 0.0652 > \epsilon \vee |f(x_1)| = |0.0066| > \epsilon 0.0002.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, f'(x_1) \neq 0 \Rightarrow x_2 = 0.5652 - \frac{(0.0066)}{(5.9584)} \Rightarrow x_2 = 0.5641$$

$$f(0.5641) = (0.5641)^3 + 5(0.5641) - 3 = 0.1795 + 2.8205 - 3 = 0.000002 ,$$

$$f'(0.5641) = 3(0.5641)^2 + 5 = 0.9546 + 5 = 5.9546.$$

$$|x_2 - x_1| = |0.5641 - 0.5652| = |-0.0011| = 0.0011 > \epsilon = 0.0002 \quad \vee$$

$$|f(x_2)| = |0.000002| < \epsilon = 0.0002.$$

Then $x_2 = 0.5641$ is the approximate root.

Example 18:

Using **Newton-Raphson method**, find an approximate value of the solution to the equation $f(x) = x^2 - 5$ when $x = 2$, $\epsilon = 0.00007$.

Solution:

$$f(x) = x^2 - 5 \Rightarrow f'(x) = 2x.$$

x	2	2.25	2.2361	2.23607
$y = f(x)$	-1	0.0625	0.00143	0.000009
$f'(x)$	4	4.5	4.4722	4.47214

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, f'(x_i) \neq 0, i = 1, 2, 3, \dots$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, f'(x_0) \neq 0 \Rightarrow x_1 = 2 - \frac{(-1)}{4} = 2.25$$

$$f(2.25) = (2.25)^2 - 5 = 0.0625, f'(2.25) = 2(2.25) = 4.5.$$

$$|x_1 - x_0| = |2.25 - 2| = 0.25 > \epsilon \quad \vee \quad |f(x_1)| = |0.0625| > \epsilon.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, f'(x_1) \neq 0 \Rightarrow x_2 = 2.25 - \frac{(0.0625)}{(4.5)} \Rightarrow x_2 = 2.2361$$

$$f(2.2361) = (2.2361)^2 - 5 = 0.000143, f'(2.2361) = 2(2.2361) = 4.4722.$$

$$|x_2 - x_1| = |2.2361 - 2.25| = |-0.0139| = 0.0139 > \epsilon \quad \vee$$

$$|f(x_2)| = |0.000143| = 0.000143 > \epsilon = 0.00007$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}, f'(x_2) \neq 0 \Rightarrow x_3 = 2.2361 - \frac{0.000143}{(4.4722)} \Rightarrow x_3 = 2.23607$$

$$f(2.23607) = (2.23607)^2 - 5 = 0.000009 ,$$

$$f'(2.23607) = 2(2.23607) = 4.47214.$$

$$|x_3 - x_2| = |2.23607 - 2.2361| = |-0.00003| = 0.00003 < \epsilon = 0.00007 \quad \forall$$

$$|f(x_2)| = |0.000009| = 0.000009 < \epsilon = 0.00007.$$

Then $x_3 = 2.23607$ is the approximate root.

Homework

1- By using Newton-Raphson Method, Find an approximate root of

a) If $f(x) = x^3 - x + 1 = 0$ and $x_0 = 1$, find x_1 and x_2 .

b) $f(x) = x^2 - 8$ with error $\epsilon = 3 \times 10^{-3}$.

c) $f(x) = x^3 - 5$ in the interval $[1,2]$ with error $\epsilon = 0.003$.

d) $f(x) = e^x - 3x = 0$ in interval $[1,2]$ with error $\epsilon = 5 \times 10^{-2}$.

e) $f(x) = x^4 - x - 10$ in interval $[1,2]$ with error $\epsilon = 0.0001$.

2- Find an approximate root of $f(x) = x^3 - x - 1 = 0$ in $[1,2]$ with 10^{-5} first by method and then be the secant method.

(V) Fixed-point method: طريقة النقطة الصامدة

In this method, the equation $f(x) = 0$ is written in the form $x = g(x)$ and it is said to solve such equation the Fixed-point of the function $g(x)$, That is, the Fixed-point of the function g is the root of the equation $f(x) = 0$. To find this root let x_0 approximate value then $x_1 = g(x_0)$ and $x_2 = g(x_1)$.

Similarly, $x_3 = g(x_2), \dots$

in general, $x_{n+1} = g(x_n)$, $n = 1, 2, 3, \dots$ and the enough condition for convergence of the formula above $|g'(x_0)| < 1$ and Stop condition is $|x_{n+1} - x_n| \leq \epsilon$ for any n .

ويقال لحل هذه المعادلة النقطة $x = g(x)$ على الصورة $f(x) = 0$ في هذه الطريقة، تتم كتابة المعادلة للعثور على هذا الجذر لتكن $f(x) = 0$ هو جذر المعادلة g ، أي النقطة الثابتة للدالة $g(x)$ الصامدة للدالة، $x_1 = g(x_0)$ و $x_2 = g(x_1)$ ، القيمة التقريبية فان x_0 ،

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والشرط الكافي لتقارب الصيغة أعلاه $|g'(x_0)| < 1$ والشكل عام $x_{n+1} = g(x_n)$, $n = 1,2,3, \dots$ لأي n $|x_{n+1} - x_n| \leq \epsilon$ وشرط التوقف

Example 19:

Find an approximate solution of $f(x) = x^3 + 4x^2 - 10 = 0$ in interval $[1,2]$ by using Fixed-point method where $\epsilon = 9 \times 10^{-9}$.

Solution:

$$f(1) = (1)^3 + 4(1)^2 - 10 = -5, f(2) = (2)^3 + 4(2)^2 - 10 = 14$$

$$x_{n+1} = g(x_n) , n = 1,2,3, \dots , x_0 = 1.5, x = g(x) = \pm \left(\frac{1}{4}\right)(10 - x^3)^{1/2},$$

$$x = g_1(x) = x - x^3 - 4x^2 + 10,$$

$$x = g_2(x) = \left(\frac{10}{x} - 4x\right)^{1/2},$$

$$x = g_3(x) = \frac{1}{2}(10 - x^3)^{1/2},$$

$$x = g_4(x) = \left(\frac{10}{4+x}\right)^{1/2},$$

$$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x},$$

x	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	$g_5(x)$
0	1.5	1.5	1.5	1.5	1.5
1	-0.875	0.8165	1.286953768	1.348399725	1.373333333
2	6.732	2.9969	1.402540804	1.367376372	1.365262015
3	-469.7	$(-8.65)^{1/2}$	1.345458374	1.364957015	1.365230014
4	1.03×10^8		1.375170253	1.365264748	1.365230013
5			1.360094193	1.365225594	
6			1.67846968	1.365230576	
7			1.363887004	1.365229942	
8			1.365916734	1.365230022	
9			1.364878217	1.365230012	
10			1.365410062	1.365230014	
15			1.365223680	1.365230013	
20			1.365230236		
25			1.365230006		
30			1.365230013		

$$4x^2 = 10 - x^3 \Rightarrow x = \pm\left(\frac{1}{4}\right)(10 - x^3)^{1/2} \Rightarrow g'(x) = 2x \Rightarrow |g'(x_0)| = |2x_0| = |2(1.5)| = 3 > 1$$

So the formula $x = \mp\left(\frac{1}{4}\right)(10 - x^3)^{1/2}$ does not give a solution

Then $x = 1.365230013$ be root required.

Example 20:

Find an approximate solution of $f(x) = x^2 - x - 1 = 0$ in interval $[1,2]$ by using Fixed-point method where $\epsilon = 0.0009$.

Solution:

$$f(1) = (1)^2 - (1) - 1 = -1, f(2) = (2)^2 - (2) - 1 = 1$$

$$x_{n+1} = g(x_n), n = 1,2,3, \dots, x_0 = 1.5$$

$$x = g(x) = x^2 - 1 \Rightarrow g'(x) = 2x \Rightarrow |g'(x_0)| = |2x_0| = |2(1.5)| = 3 > 1$$

So the formula $x = x^2 - 1$ does not give a solution

$$x^2 = x + 1 \Rightarrow x = \sqrt{x+1}, g'(x) = \frac{1}{2\sqrt{x+1}}$$

$$\Rightarrow |g'(x_0)| = \left| \frac{1}{2\sqrt{x_0+1}} \right| = 0.3162 = 0.3 < 1.$$

The formula $x = \sqrt{x+1}$ gives a solution

n	0	1	2	3	4	5
x	1.5	1.5811	1.6066	1.6145	1.6169	1.6169

$$x_1 = \sqrt{x_0+1} = \sqrt{1.5+1} = 1.5811, |x_1 - x_0| = |1.5811 - 1.5| = 0.0811 > \epsilon$$

$$x_2 = \sqrt{x_1+1} = \sqrt{1.5811+1} = 1.6066, |x_2 - x_1| = |1.6066 - 1.5811| = 0.0255 > \epsilon$$

$$x_3 = \sqrt{x_2+1} = \sqrt{1.6066+1} = 1.6145, |x_3 - x_2| = |1.6145 - 1.6066| = 0.0079 > \epsilon$$

$$x_4 = \sqrt{x_3+1} = \sqrt{1.6145+1} = 1.6169, |x_4 - x_3| = |1.6169 - 1.6145| = 0.0024 > \epsilon$$

$x_5 = \sqrt{x_4+1} = \sqrt{1.6169+1} = 1.6177$, $|x_5 - x_4| = |1.6177 - 1.6169| = 0.0008 < \epsilon$
 Then $x_5 = 1.6177$ be root required.

Homework

By using *Fixed-point method*

1- Find an approximate solution of $f(x) = x^2 - x - 3 = 0$ in interval $[2,3]$ where $\epsilon = 0.005$.

2- Find an approximate solution of $f(x) = x^2 - 2x - 7 = 0$ in interval $[2,5]$ where $\epsilon = 0.001$.

(VI) Newton-Raphson method 2 :

Let $f(x) = a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \dots + a_{m-1}x + a_m$... (1) when dividing $f(x)$ on $(x - x_0)$ we get:-

$$\frac{a_0x^{m-1} + (a_0x_0 + a_1)x^{m-2} + \dots + b_{m-1}}{(x - x_0) \left(a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \dots + a_{m-1}x + a_m \right) \mp a_0x^m \pm a_0x_0x^{m-1}}$$

$$\frac{(a_0x_0 + a_1)x^{m-1} + a_2x^{m-2}}{\mp(a_0x_0 + a_1)x^{m-1} \pm (a_0x_0 + a_1)x_0x^{m-2}}$$

$$\frac{\vdots}{((a_0x_0 + a_1)x_0 + a_2)x^{m-2} + a_3x^{m-3}}$$

$$\frac{\vdots}{b_m}$$

$$\frac{f(x)}{x - x_0} = (b_0x^{m-1} + b_1x^{m-2} + b_2x^{m-3} + \dots + b_{m-1}) + \frac{b_m}{(x - x_0)}$$

$$f(x) = (x - x_0)(b_0x^{m-1} + b_1x^{m-2} + b_2x^{m-3} + \dots + b_{m-1}) + b_m \quad \dots (2)$$

$$f(x) = (b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \dots + b_{m-1}x) + (-b_0x_0x^{m-1} - b_1x_0x^{m-2} - b_2x_0x^{m-3} - \dots)$$

$$-b_{m-2}x_0x - b_{m-1}x_0) + b_m f(x) = b_0x^m + (b_1 - b_0x_0)x^{m-1} + (b_2 - b_1x_0)x^{m-2} + \dots + (b_{m-1} - b_{m-2}x_0)x + b_m \quad \dots(3) \text{ From (2) ,}$$

we get $f(x_0) = b_m$ and from (1) and (3) we get:-

$$a_0 = b_0$$

$$a_1 = b_1 - b_0x_0 \Rightarrow b_1 = a_1 + b_0x_0$$

$$a_2 = b_2 - b_1x_0 \Rightarrow b_2 = a_2 + b_1x_0$$

⋮

$$b_k = a_k + b_{k-1}x_0, k = 1, 2, 3, \dots, m$$

To find $f'(x_0)$ Suppose $g(x) = b_0x^{m-1} + b_1x^{m-2} + b_2x^{m-3} + \dots + b_{m-1}$

Thus, equation (2) become $f(x) = (x - x_0)g(x) + b_m$

$f'(x) = (x - x_0)g'(x) + g(x)$ then $f'(x) = g(x_0)$, to find $g(x)$ use the derivation above to find value c_{m-1}

$$c_0 = b_0, c_k = b_k + c_{k-1}x_0, k = 1, 2, 3, \dots, m - 1$$

Then we find the approximate value of the root from $x_1 = x_0 - \frac{b_m}{c_{m-1}}$.

The above operations can be completed by configuring the following table

	a_0	a_1	a_2	a_{m-1}	a_m
	+	+	+		+	
x_0		b_0x_0	b_1x_0	$b_{m-2}x_0$	
	b_0	b_1	b_2	b_{m-1}	b_m
			$= f(x_0)$			
x_0		+	+		+	
		c_0x_0	c_1x_0	$b_{m-2}x_0$	
	c_0	c_1	c_2	$c_{m-1} = f'(x_0)$	

The two conditions stop are $|x_{i+1} - x_i| \leq \epsilon \quad \forall \quad |f(x_{i+1})| = |b_m| \leq \epsilon$ for any i.
 This method is called to find $f(x^*), f'(x^*)$ a synthetic division

Example 21:

Using **Newton-Raphson method**, find the solution to the equation

$$f(x) = x^3 + 2x^2 - 20x + 12 = 0 \text{ in interval } [0,2] \text{ with error } \epsilon = 0.003.$$

Solution:

$$f(0) = (0)^3 + 2(0)^2 - 20(0) + 12 = 12,$$

$$f(2) = (2)^3 + 2(2)^2 - 20(2) + 12 = -12, \quad x_0 = 1$$

	1	2	-20	12
$x_0 = 1$		1	3	-17
	1	3	-17	-5 = f(1)
$x_0 = 1$		1	4	
	1	4	-13 = f'(1)	

$$x_1 = x_0 - \frac{f(1)}{f'(1)} = (1) - \frac{-5}{-13} = 1 - 0.3846 = 0.6154.$$

	1	2	-20	12
$x_1 = 0.6154$		0.6154	1.6095	-11.3175
	1	2.6154	-18.3905	0.6825 = f(0.6154)
$x_1 = 0.6154$		0.6154	1.9882	
	1	3.2308	-16.4023 = f'(0.6154)	

$$|x_1 - x_0| = |0.6154 - 1| = |-0.3846| = 0.3846 > \epsilon = 0.003 \quad \vee$$

$$|f(x_1)| = |0.6825| = 0.6825 > \epsilon = 0.003.$$

$$x_2 = x_1 - \frac{f(0.6154)}{f'(0.6154)} = (0.6154) - \frac{0.6825}{-16.4023} = 0.6154 + 0.0375 = 0.6529$$

	1	2	- 20	12
$x_2 = 0.6529$		0.6529	1.7321	- 11.9271
	1	2.6529	- 18.2679	0.0729 = $f(0.6529)$
$x_2 = 0.6529$		0.6529	2.1584	
	1	3.3058	- 16.1095 = $f'(0.6529)$	

$$|x_2 - x_1| = |0.6529 - 0.6154| = |0.0375| = 0.0375 > \epsilon = 0.003 \quad \vee$$

$$|f(x_2)| = |0.0729| = 0.0729 > \epsilon = 0.003.$$

$$x_3 = x_2 - \frac{f(0.6529)}{f'(0.6529)} = (0.6529) - \frac{0.0729}{-16.1095} = 0.6529 + 0.0045 = 0.6574$$

	1	2	- 20	12
$x_3 = 0.6574$		0.6574	1.7470	- 11.9995
	1	2.6574	- 18.253	0.0005 = $f(0.6574)$
$x_3 = 0.6574$		0.6574	2.1791	
	1	3.3148	- 16.0739 = $f'(0.6574)$	

$$|x_3 - x_2| = |0.6574 - 0.6529| = |0.0045| = 0.0045 > \epsilon = 0.003 \quad \vee$$

$$|f(x_3)| = |0.0005| = 0.0005 < \epsilon = 0.003.$$

Then $x_3 = 0.6574$ is the approximate root.

Example 22:

Using **Newton-Raphson method**, find the solution to the equation $f(x) = 4x^4 - 3x^2 + 3x - 4 = 0$, if $x_0 = -2$ with error $\epsilon = 0.07$.

Solution:

	4	0	-3	3	-4
$x_0 = -2$	-8	16	-26	46	
	4	-8	13	-23	$42 = f(-2)$
$x_0 = -2$	-8	32	-90		
	4	-16	45	-113	$= f'(-2)$

$$x_1 = x_0 - \frac{f(-2)}{f'(-2)} = (-2) - \frac{42}{-113} = -2 + 0.3717 = -1.6283.$$

	4	0	-3	3	-4
$x_1 = -1.6283$	-6.5132	10.6054	-12.3839	15.2798	
	4	-6.5132	7.6054	-9.3839	$11.2798 = f(-1.6283)$
$x_1 = -1.6283$	-6.5132	21.2109	-46.9216		
	4	-13.0264	28.8163	-56.3055	$= f'(-1.6283)$

$$|x_1 - x_0| = |(-1.6283) - (-2)| = |0.3717| = 0.3717 > \epsilon = 0.07 \quad \vee$$

$$|f(x_1)| = |11.2798| = 11.2798 > \epsilon = 0.07.$$

$$x_2 = x_1 - \frac{f(-1.6283)}{f'(-1.6283)} = (-1.6283) - \frac{11.2798}{-56.3055} = -1.6283 + 0.2003 = -1.428$$

	4	0	-3	3	-4
$x_2 = -1.428$		-5.712	8.1567	-7.3638	6.2315
	4	-5.712	5.1567	-4.3638	2.2315 = $f(-1.428)$
$x_2 = -1.428$		-5.712	16.3135	-30.6594	
	4	-11.424	21.4702	-35.0232 = $f'(-1.428)$	

$$|x_2 - x_1| = |(-1.428) - (-1.6283)| = |0.2003| = 0.2003 > \epsilon = 0.07 \quad \vee$$

$$|f(x_2)| = |2.2315| = 2.2315 > \epsilon = 0.07.$$

$$x_3 = x_2 - \frac{f(-1.428)}{f'(-1.428)} = (-1.428) - \frac{2.2315}{-35.0232} = -1.428 + 0.0637 = -1.3643$$

	4	0	-3	3	-4
$x_3 = -1.3643$		-5.4572	7.4452	-7.3638	5.9535
	4	-5.4572	4.4452	-4.3638	1.9535 = $f(-1.3643)$
$x_3 = -1.3643$		-5.4572	14.8905	-27.1366	
	4	-10.9144	19.3357	-31.5004 = $f'(-1.3643)$	

$$|x_3 - x_2| = |(-1.3643) - (-1.428)| = |0.0637| = 0.0637 < \epsilon = 0.0700 \quad \vee$$

$$|f(x_3)| = |1.9535| = 1.9535 > \epsilon = 0.07.$$

Then $x_3 = -1.3643$ is the approximate root.

Homework

By using **Newton-Raphson method** , find an approximate root

- 1- $f(x) = x^2 - x - 3 = 0$ in interval $[2,3]$ where $\epsilon = 0.005$.
- 2- $f(x) = x^4 - 4x^2 - 3x + 5 = 0$ in interval $[2,5]$ where $\epsilon = 1 \times 10^{-5}$.
- 3- $f(x) = x^3 + 2x^2 - 20x + 12 = 0$ in interval $[0,2]$ with error $\epsilon = 0.003$.