Numerical Analysis

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Chapter Four

Interpolation and Extrapolation

الاندراج والاستكمال

إذا كان لدينا مجموعة محدودة من قيم الدالة غير المعروفة $f(x_0), f(x_2), f(x_2), \cdots, f(x_m)$ ونريد تخمين قيمة الدالة عند النقطة x^*، إذا كانت النقطةx^* تقع ضمن نطاق النقاط فتسمى العملية الاستيفاء والعكس يسمى الاستقراء.

هناك العديد من الطرق الرياضية المستخدمة في هذا المجال

1- Lagrange polynomial

تكن f دالة حقيقية ومستمرة في الفترة $[a, b]$ وقيمها معروفة في النقاط $x_0, x_1, x_2, \cdots, x_m$ المرقمة$m + 1$.
تخمين قيمة دالة f في نقطة واحدة أو عدة نقاط جديدة، نقوم بإغلاق الدالة إلى كثيرة حدود من فئة متوافقة مع لدالة في النقاطm المعطاة وبالتالي العثور على كثيرة حدود جديدة وتكوين النقاط اللازمة للعثور على صورها .

The general formula of polynomial $P_m(x)$ (Lagrange polynomial)

لصيغة العامة لكثيرة الحدود (متعدد الحدود لاكرانج للاندراج)

$$P_m(x) = \sum_{j=0}^{m} f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{m} \frac{(x - x_i)}{(x_j - x_i)}$$

$$f(x) \approx P_m(x) = f(x_0)\frac{(x - x_1)(x - x_2)\cdots(x - x_m)}{(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_m)} + f(x_1)\frac{(x - x_0)(x - x_2)\cdots(x - x_m)}{(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_m)}$$
$$+ \cdots + f(x_m)\frac{(x - x_0)(x - x_1)\cdots(x - x_{m-1})}{(x_m - x_1)(x_m - x_2)\cdots(x_m - x_{m-1})}$$

And to find $f(x^*)$ without knowing form of the function we use the following law

$$P_m(x^*) = \sum_{j=0}^{m} f(x_j) \prod_{\substack{i=0 \\ i \neq j}}^{m} \frac{(x^* - x_i)}{(x_j - x_i)}$$

$$P_m(x^*) = f(x_0)\frac{(x^*-x_1)(x^*-x_2)\cdots(x^*-x_m)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_m)} + f(x_1)\frac{(x^*-x_0)(x^*-x_2)\cdots(x^*-x_m)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_m)} + \cdots + f(x_m)\frac{(x^*-x_0)(x^*-x_1)\cdots(x^*-x_{m-1})}{(x_m-x_1)(x_m-x_2)\cdots(x_m-x_{m-1})}$$

ملاحظة:-

1- متعددة حدود لاكرانج يعطي نتيجة دقيقة عند أي نقطة من نقاط الجدول لأن الدالة متوافقة مع متعددة الحدود في نقاط الجدول.

2- نستخدم متعددة حدود لاكرانج في كلتا الحالتين الاندراج والاستكمال.

Example 1:

Find the form of function $f(x)$ and then find $f(1.5), f(3.7), f(0.65)$ of the following table and using Lagrange polynomial

x	1	1.2	2
y	0	0.1823	0.6931

Solution:

$$P_2(x) = \sum_{j=0}^{2} f(x_j) \prod_{\substack{i=0 \\ i\neq j}}^{2} \frac{(x-x_i)}{(x_j-x_i)}$$

$$P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = 0 + 0.1823\frac{(x-1)(x-2)}{(1.2-1)(1.2-2)} + 0.6931\frac{(x-1)(x-1.2)}{(2-1)(2-1.2)}$$

$$P_2(x) = -0.273x^2 + 1.5121x - 1.2391 \Rightarrow$$

$$f(1.5) \approx P_2(1.5) = -0.273(1.5)^2 + 1.5121(1.5) - 1.2391 = 0.4148.$$

$$f(3.7) \approx P_2(3.7) = -0.273(3.7)^2 + 1.5121(3.7) - 1.2391 = -0.6183.$$

$$f(0.65) \approx P_2(0.65) = -0.273(0.65)^2 + 1.5121(0.65) - 1.2391 = -0.3715.$$

Homework

Find $f(1.7), f(5), f(2.5)$?

To find the function

$$f\,(x) \approx P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= f(1.1)\frac{(x-1.7)(x-3)}{(1.1-1.7)(1.1-3)} + f(1.7)\frac{(x-1.1)(x-3)}{(1.7-1.1)(1.7-3)} + f(3)\frac{(x-1.1)(x-1.7)}{(3-1.1)(3-1.7)}$$

$$= (0)\frac{(x-1.2)(x-2)}{(1-1.2)(1-2)} + (0.1823)\frac{(x-1)(x-2)}{(1.2-1)(1.2-2)} + (0.6931)\frac{(x-1)(x-1.2)}{(2-1)(2-1.2)}$$

$$= (0) + \frac{(0.1823)}{(1.2-1)(1.2-2)}(x-1)(x-2) + \frac{(0.6931)}{(2-1)(2-1.2)}(x-1)(x-1.2)$$

$$= \frac{(0.1823)}{0.16}(x^2 - 3\mathrm{x} + 2) + \frac{(0.6931)}{0.8}(x^2 - 2.2\mathrm{x} + 1.2)$$

$$f\,(x) = -0.273\,x^2 + 1.5121\,\mathrm{x} - 1.2391.$$

Finite differences:-   الفروق المنتهية:-

Let f be a real function whose values are known in $m+1$ points of equal dimensions $x_0, x_1, x_2, \cdots, x_m$ and that have known images $f(x_0), f(x_1), f(x_2), \cdots, f(x_m)$ or $y_0, y_1, y_2, \cdots, y_m$ definite the first difference at the point x_0 .

تكن f دالة حقيقية تكون قيمها معروفة في $m+1$ نقاط متساوية الأبعاد $x_0, x_1, x_2, \cdots, x_m$ ولها صور معروفة $y_0, y_1, y_2, \cdots, y_m$ أو تحديد الفرق الأول في هذه النقطة x_0 .

$\Delta f(x_0) = f(x_1) - f(x_0) \Rightarrow \Delta y_0 = y_1 - y_0$

$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \Delta y_2 = y_3 - y_2, \cdots, \Delta y_{m-1} = y_m - y_{m-1}$

defined the second difference as follows:-

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - y_1 - y_1 + y_0$$

$$\Rightarrow \Delta^2 y_0 = y_2 - 2y_1 + y_0 \Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1)$$

$$= y_3 - y_2 - y_2 + y_1 \Rightarrow \Delta^2 y_1 = y_3 - 2y_2 + y_1$$

$$\vdots$$

$$\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i \quad , i = 1,2,\cdots,n-2.$$

In general $\Delta^k y_i = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y_{i+k-j}$

Table of differences

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

Divided Finite differences to three types:-   تنقسم الفروق المنتهية إلى ثلاث أنواع:-

1- Forward differences   **الفروقات التقدمية**
2- Backward differences   **الفروقات التراجعية**
3- Central differences   **الفروقات المركزية**
4- Divided differences: **الفروقات المنتهية النسبية**

1- Forward differences   الفروقات التقدمية

To find an image point located at the beginning of the table we use the Forward difference. نلاحظ ان .للعثور على نقطة الصورة الموجودة في بداية الجدول نستخدم الفروقات التقدمية

$$\Delta y_0 = y_1 - y_0 \Rightarrow y_1 = y_0 + \Delta y_0 \Rightarrow y_1 = (1+\Delta) y_0$$

$$y_2 = y_1 + \Delta y_1 \Rightarrow y_2 = (1+\Delta) y_1 \Rightarrow y_2 = (1+\Delta)^2 y_0$$

$$\vdots$$

$$y_m = (1+\Delta)^m y_0 \quad .$$

We get $y_m = \sum_{k=0}^{m} \binom{m}{k} \Delta^k y_0$

$$y_m = f(x_0 + mh)$$

$$= y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \cdots$$

$m = \frac{x_m - x_0}{h}$.

Example 3:

Find $f(0.3), f(0.8), f(1.2)$ of the following table use the forward difference.

x	0	1	2	3	4
y	1	2	9	28	65

Solution:

differences.Table of the forward difference

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{0.3-0}{1} = 0.3,$

$$f(x) = f(x_0 + mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$$

$$f(0.3) = 1 + (0.3)(1) + \frac{(0.3)(0.3-1)}{2}(1) + \frac{(0.3)(0.3-1)(0.3-2)}{6}(6)\ .$$

$$f(0.3) = 1 + 0.3 - 0.63 + 0.357 = 1.027.$$

Table of the forward difference

x	y	Δ	Δ^2	Δ^3	Δ^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$$m = \frac{x_m - x_0}{h} \implies m = \frac{0.8-1}{1} = -0.2,$$

$$f(0.8) = 2 + (-0.2)(7) + \frac{(-0.2)(-0.2-1)}{2}(12) + \frac{(-0.2)(-0.2-1)(-0.2-2)}{6}(6)\ .$$

$$f(0.8) = 2 - 1.4 + 1.44 - 0.528 = 1.512.$$

$$m = \frac{x_m - x_0}{h} \implies m = \frac{1.2-1}{1} = 0.2,$$

$$f(1.2) = 2 + (0.2)(7) + \frac{(0.2)(0.2-1)}{2}(12) + \frac{(0.2)(0.2-1)(0.2-2)}{6}(6)\ .$$

$$f(1.2) = 2 + 1.4 - 0.96 + 0.288 = 2.728.$$

To find the function

$$m = \frac{x_m - x_0}{h} \implies m = \frac{x-0}{1} = x$$

$f(x) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(x) = 1 + (x)(1) + \frac{(x)(x-1)}{2}(1) + \frac{(x)(x-1)(x-2)}{6}(6) = 1 + x + 3x^2 - 3x + x^3 - 3x^2 + 2x = x^3 + 1$. $f(x) = x^3 + 1$

Example 4:

Find $f^{-1}(1.64), f^{-1}(0.18)$ of the following table use the forward differences .

x	1	2	3	4	5	6
y	0	0.7	1.1	1.4	1.6	1.7

Solution:

differences.Table of the forward difference

x	y	Δ	Δ^2	Δ^3	Δ^4
1	0				
		0.7			
2	0.7		-0.3		
		0.4		0.2	
3	1.1		-0.1		-0.2
		0.3		0	يهمل
4	1.4		-0.1		
		0.2		0	
5	1.6		-0.1		
		0.1			
6	1.7				

$m = \frac{x - x_0}{h} \Rightarrow m = \frac{x-1}{1} = x - 1,$

$f(x) = f(x_0 + mh) = y_0 + m\Delta y_0 + \frac{m(m-1)}{2!}\Delta^2 y_0 + \frac{m(m-1)(m-2)}{3!}\Delta^3 y_0 + \frac{m(m-1)(m-2)(m-3)}{4!}\Delta^4 y_0 + \ldots.$

$f(x) = 0 + (x-1)(0.7) + \frac{(x-1)(x-2)}{2}(-0.3) + \frac{(x-1)(x-2)(x-3)}{6}(0.2) + \frac{(x-1)(x-2)(x-3)(x-4)}{24}(-0.2^{\text{يهمل}})$

$f(x) = 0 + 0.7x + 0.7 - 0.15x^2 + 0.45x - 0.3 + 0.0333x^3 - 0.2x^2 + 0.3667x - 0.1999.$

$f(x) = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001.$

$f(x) = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001 \Longrightarrow \; f(x) = 1.64 \Longrightarrow \; f^{-1}(1.64) = ?$
$\Longrightarrow \; 1.64 = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001.$
$\Longrightarrow \; 0.0333x^3 - 0.35x^2 + 1.5167x - 1.4399 = 0$
$\Longrightarrow$ بالدستور $. \therefore f^{-1}(1.64) = 5.296.$

$f(x) = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001 \Longrightarrow \; f(x) = 0.18 \Longrightarrow \; f^{-1}(0.18) = ?$
$\Longrightarrow \; 0.18 = 0.0333x^3 - 0.35x^2 + 1.5167x + 0.2001.$
$\Longrightarrow \; 0.0333x^3 - 0.35x^2 + 1.5167x + 0.0201 = 0$
$\Longrightarrow$ بالدستور $\therefore f^{-1}(0.18) = 1.2.$

Homework

Find $f(0.7), f(2.3), f(4.2), f^{-1}(10), f^{-1}(30)$ of the following table use the forward difference and find To find the function .

x	0	5	10	15	20	25
y	7	11	14	18	24	32

2-Backward differences   الفروقات التراجعية

To find an image point located at the end of a table we use the Backward differences as follows

لعثور على نقطة صورة تقع في نهاية الجدول، نستخدم الفروقات التراجعية على النحو التالي .

$\nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \nabla y_3 = y_3 - y_2, \cdots, \nabla y_m = y_m - y_{m-1}$
$\nabla^2 y_m = \nabla(\nabla y_m) = \nabla y_m - \nabla y_{m-1} = (y_m - y_{m-1}) - (y_{m-1} - y_{m-2})$
$\Rightarrow \nabla^2 y_m = y_m - 2y_{m-1} + y_{m-2}.$
$\therefore \nabla^k y_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y_{i-j}$.

$\nabla^2 y_2 = \sum_{j=0}^{2} \binom{2}{j} (-1)^j y_{2-j} = \binom{2}{0}(-1)^0 y_2 + \binom{2}{1}(-1) y_1 + \binom{2}{2}(-1)^2 y_0 = y_2 - 2y_1 + y_0$.

From the definition we notice:-

$\Delta y_0 = \nabla y_{-1}, \Delta^2 y_0 = \nabla^2 y_{-1}, \Delta^3 y_0 = \nabla^3 y_{-1}$

$\nabla y_i = y_i - y_{i-1} \Rightarrow y_{i-1} = y_i - \nabla y_i = (1-\nabla) y_i$

$y_{i-1} = (1-\nabla) y_i \Rightarrow \; y_i = (1-\nabla)^{-1} y_{i-1}$

And $\Delta y_{i-1} = y_i - y_{i-1} \Rightarrow y_i = y_{i-1} + \Delta y_{i-1} = (1+\Delta) y_{i-1} \Rightarrow y_i = (1+\Delta) y_{i-1}$

So $\boxed{(1-\nabla)^{-1} = (1+\Delta)}$

In general $\Delta^k y_i = \sum_{j=0}^{k} \binom{k}{j} (-1)^j y_{i+k-j}$

Table of differences

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_0	y_0					
		∇y_0				
x_1	y_1		$\nabla^2 y_0$			
		∇y_1		$\nabla^3 y_0$		
x_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$	
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$
x_3	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$	
		∇y_3		$\nabla^3 y_2$		
x_4	y_4		$\nabla^2 y_3$			
		∇y_4				
x_5	y_5					

$$y_m = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \cdots$$

Example 5:

Find $f(3.8), f(3.2), f(2.8)$ of the following table use the forward difference.

x	0	1	2	3	4
y	1	2	9	28	65

Solution:

Table of the forward difference.differences

x	y	∇	∇^2	∇^3	∇^4

0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.8-4}{1} = -0.2,$

$f(x) = f(x_0 + mh) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 +$
$\frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots.$

$f(3.8) = 65 + (-0.2)(37) + \frac{(-0.2)(-0.2+1)}{2}(18) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{6}(6)$.

$f(3.8) = 65 - 7.4 - 1.44 - 0.288 = 55.872.$

Table of the forward difference

x	y	∇	∇^2	∇^3	∇^4
0	1				
		1			
1	2		6		
		7		6	
2	9		12		0
		19		6	
3	28		18		
		37			
4	65				

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{3.2-3}{1} = 0.2,$

$f(3.2) = 28 + (0.2)(19) + \frac{(0.2)(0.2+1)}{2}(12) + \frac{(0.2)(0.2+1)(0.2+2)}{6}(6)$.

$f(3.2) = 28 + 3.8 + 1.44 + 0.528 = 33.768.$

$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{2.8-3}{1} = -0.2,$

$f(2.8) = 28 + (-0.2)(19) + \frac{(-0.2)(-0.2+1)}{2}(12) + \frac{(-0.2)(-0.2+1)(-0.2+2)}{6}(6)$.

$f(2.8) = 28 - 3.8 - 0.96 - 0.288 = 22.952.$

To find the function

$$m = \frac{x_m - x_0}{h} \Rightarrow m = \frac{x-4}{1} = x - 4$$

$f(x) = y_0 + m\nabla y_0 + \frac{m(m+1)}{2!}\nabla^2 y_0 + \frac{m(m+1)(m+2)}{3!}\nabla^3 y_0 + \frac{m(m+1)(m+2)(m+3)}{4!}\nabla^4 y_0 + \ldots.$

$$f(x) = 65 + (x-4)(37) + \frac{(x-4)(x-3)}{2}(18) + \frac{(x-4)(x-3)(x-2)}{6}(6)$$

$$= 65 + 37x - 148 + 9x^2 - 63x + 108 + x^3 - 7x^2 + 12x - 2x^2 + 14x - 24 = x^3 + 1 \ .$$

$f(x) = x^3 + 1$.

Homework

Find $f(20.3), f(24.8), f(23.2), f^{-1}(22), f^{-1}(30)$ of the following table use the forward difference and find To find the function .

x	0	5	10	15	20	25
y	7	11	14	18	24	32

3- Central differences الفروقات المركزية

To find a point image located in the center of a table or near the center, we use Central differences as follows:

لعثور على صورة نقطية تقع في وسط الجدول أو بالقرب من المركز، نستخدم الفروق المركزية كما يلي:

$$\delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}} \xrightarrow{\text{للزوجي} \quad Example} \delta y_3 = y_{\frac{7}{2}} - y_{\frac{5}{2}}$$

$$\delta y_{i+\frac{1}{2}} = y_{i+1} - y_{i-1} \xrightarrow{\text{للفردي} \quad Example} \delta y_{\frac{3}{2}} = y_2 - y_1$$

$$\delta^2 y_i = \delta(\delta y_i) = \delta\left(y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}\right) = \delta y_{i+\frac{1}{2}} - \delta y_{i-\frac{1}{2}} = (y_{i+1} - y_i) - (y_i - y_{i-1}).$$

$$\delta^2 y_i = y_{i+1} - 2y_i + y_{i-1}, \forall i = 0, \pm 1, \pm 2, \ldots$$

$$\delta^3 y_{i+\frac{1}{2}} = \delta^2 y_{i+1} - \delta^2 y_i$$

:

:

$$\delta^k y_i = \delta(\delta^{k-1} y_i) = \delta^{k-1}(\delta y_i) = \delta^{k-2}((y_{i+1} - y_i) - (y_i - y_{i-1})) = \ldots..$$

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_0	y_0					
		δy_0				
x_1	y_1		$\delta^2 y_0$			
		δy_1		$\delta^3 y_0$		
x_2	y_2		$\delta^2 y_1$		$\delta^4 y_0$	
		δy_2		$\delta^3 y_1$		$\delta^5 y_0$
x_3	y_3		$\delta^2 y_2$		$\delta^4 y_1$	
		δy_3		$\delta^3 y_2$		
x_4	y_4		$\delta^2 y_3$			
		δy_4				
x_5	y_5					

$$f(x) = f(x_0 + mh) = y_0 + m\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\delta^2 y_1 + \frac{m(m-1)(m-2)}{3!}\delta^3 y_{\frac{3}{2}} + \cdots.$$

There are two formulas for finding central point images:-   هناك صيغتان للحصول على صور الفروقات المركزية

1-**Besssl's formulas**   صيغ بسل المركزية

Its general formula :-   صيغته العامة

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

2-**Stirling's formulas**   **صيغ ستيرلينك المركزية**

Its general formula :-   صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

Note:-

$\mu y_i = \frac{y_{i+1}+y_i}{2} \xrightarrow{Example} \mu y_1 = \frac{y_2+y_1}{2}$,

$\mu y_{i+\frac{1}{2}} = \frac{y_{i+1}+y_i}{2} \xrightarrow{Example} \mu y_{\frac{1}{2}} = \frac{y_1+y_0}{2}$, μ is the average

$\mu\delta y_i = \frac{\delta y_{i+\frac{1}{2}}+\delta y_{i-\frac{1}{2}}}{2}$ و $\mu\delta^2 y_i = \frac{\delta^2 y_{i+\frac{1}{2}}+\delta^2 y_{i-\frac{1}{2}}}{2}$.

Example 6:

Find $f(2.4)$ of the following table use the forward difference.

x	0	1	2	3	4	5
y	5	6	13	32	65	130

Solution:

Table of the forward difference. **Besssl's formulas**   صيغ بسل المركزية

Its general formula :-   صيغته العامة

$$y_m = \mu y_{\frac{1}{2}} + \left(m-\frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

x	y	δ	δ^2	δ^3	δ^4
0	5				
		1			
1	6		6		
		7		6	
2	13		12		0
		19		6	
3	32		18		0
		37		6	
4	65		24		
		61			
5	130				

$$m = \frac{x_m - x_0}{h} \implies m = \frac{2.4-2}{1} = 0.4.$$

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}}$$

$$y_{2.4} = \frac{13+32}{2} + (0.4-0.5)(19) + \frac{0.4(0.4-1)}{2}\left(\frac{12+18}{2}\right) + \frac{0.4(0.4-1)(0.4-0.5)}{6}(6).$$

$$y_{2.4} = 22.5 + (-1.9) + (-0.12)(15) + 0.024 = 22.5 - 1.9 - 1.8 + 0.024.$$

$$y_{2.4} = 18.824.$$

Stirling's formulas **صيغ ستيرلينك المركزية**

Its general formula :- صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

x	y	δ	δ^2	δ^3	δ^4
0	5				
		1			
1	6		6		
		7		6	
2	13		12		0
		19		6	
3	32		18		0
		37		6	
4	65		24		
		61			
5	130				

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{2.4} = 13 + (0.4)\left(\frac{7+19}{2}\right) + \frac{(0.4)^2}{2}(12) + \frac{0.4((0.4)^2-1)}{6}\left(\frac{6+6}{2}\right).$$

$$y_{2.4} = 13 + (0.4)(13) + (0.16)(6) + (0.4)(-0.84) = 13 + 5.2 + 0.96 - 0.336.$$

$$y_{2.4} = 18.824.$$

To find the function

$$m = \frac{x_m - x_0}{h} \implies m = \frac{x-2}{1} = x - 2$$

1) صيغ بسل المركزية

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$y_{x-2} = \frac{13+32}{2} + (x-2-0.5)(19) + \frac{(x-2)(x-3)}{2}\left(\frac{12+18}{2}\right) + \frac{(x-2)(x-3)(x-2-0.5)}{6}(6).$$

$$y_{x-2} = 22.5 + (x-2.5)(19) + (x-2)(x-3)(7.5) + (x-2)(x-3)(x-2.5).$$

$$y_{x-2} = 22.5 + 19x - 47.5 + 7.5x^2 - 37.5x + 45 + x^3 - 7.5x^2 + 18.5x - 15.$$

$f(x) = x^3 + 5$.

2) **صيغ ستيرلينك المركزية**

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{x-2} = 13 + (x-2)\left(\frac{7+19}{2}\right) + \frac{(x-2)^2}{2}(12) + \frac{(x-2)((x-2)^2-1)}{6}\left(\frac{6+6}{2}\right)$$

$$y_{x-2} = 13 + (x-2)(13) + 6(x^2-4x+4) + (x-2)(x^2-4x+4-1)$$

$$y_{x-2} = 13 + 13x - 26 + 6x^2 - 24x + 24 + x^3 - 6x^2 + 11x - 6.$$

$f(x) = x^3 + 5$.

Example 7:

Find $f(1.1)$ and $f(1.5)$of the following table use the forward difference.

x	-2	-1	0	1	2	3	4	5
y	15	0	-1	0	15	80	255	624

Solution:

Table of the forward difference. **Besssl's formulas**   **صيغ بسل المركزية**

Its general formula :-   صيغته العامة

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m-1)(m-2)(m-3)}{4!}\mu\delta^4 y_{\frac{1}{2}}.$$

x	y	δ	δ^2	δ^3	δ^4
-2	15				
		-15			
-1	0		14		
		-1		-12	
0	-1		2		24
		1		12	
1	0		14		24
		15		36	
2	15		50		24
		65		60	
3	80		110		24
		175		84	
4	255		194		
		369			
5	624				

$$m=\frac{x_m-x_0}{h}\implies m=\frac{1.1-1}{1}=0.1$$,

$$y_m=\mu y_{\frac{1}{2}}+\left(m-\frac{1}{2}\right)\delta y_{\frac{1}{2}}+\frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}}+\frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}}+\frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}}+\cdots$$

$$y_{1.1}=\frac{0+15}{2}+(0.1-0.5)(15)+\frac{0.1(0.1-1)}{2}\left(\frac{14+50}{2}\right)+\frac{0.1(0.1-1)(0.1-0.5)}{6}(36)+\frac{(0.1)((0.1)^2-1)(0.1-2)}{24}\left(\frac{24+24}{2}\right).$$

$y_{1.1}=0.4641.$

$$m=\frac{x_m-x_0}{h}\implies m=\frac{1.5-1}{1}=0.5.$$

$$y_{1.5}=\frac{0+15}{2}+(0.5-0.5)(15)+\frac{0.5(0.5-1)}{2}\left(\frac{14+50}{2}\right)+\frac{0.5(0.5-1)(0.5-0.5)}{6}(36)+\frac{(0.5)((0.5)^2-1)(0.5-2)}{24}\left(\frac{24+24}{2}\right).$$

$y_{1.5}=4.0625.$

Stirling's formulas   **صيغ ستيرلينك المركزية**

Its general formula :-   صيغته العامة

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

x	y	δ	δ^2	δ^3	δ^4
-2	15				
		-15			
-1	0		14		
		-1		-12	
0	-1		2		24
		1		12	
1	0		14		24
		15		36	
2	15		50		24
		65		60	
3	80		110		24
		175		84	
4	255		194		
		369			
5	624				

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{1.1} = 0 + (0.1)\left(\frac{1+15}{2}\right) + \frac{(0.1)^2}{2}(14) + \frac{0.1((0.1)^2-1)}{6}\left(\frac{12+36}{2}\right) + \frac{(0.1)^2((0.1)^2-1)}{24}\ (24)$$

$$y_{1.1} = 0 + 0.8 + 0.07 - 0.396 - 0.0099 = 13 + 5.2 + 0.96 - 0.336$$

$$y_{1.1} = 0.4641$$

$$y_{1.5} = 0 + (0.5)\left(\frac{1+15}{2}\right) + \frac{(0.5)^2}{2}(14) + \frac{0.5((0.5)^2-1)}{6}\left(\frac{12+36}{2}\right) + \frac{(0.5)^2((0.5)^2-1)}{24}\ (24)$$

$$y_{1.5} = 4.0625$$

To find the function

$$m = \frac{x_m - x_0}{h} \implies m = \frac{x-1}{1} = x - 1$$

1) صيغ بسل المركزية

$$y_m = \mu y_{\frac{1}{2}} + \left(m - \frac{1}{2}\right)\delta y_{\frac{1}{2}} + \frac{m(m-1)}{2!}\mu\delta^2 y_{\frac{1}{2}} + \frac{m(m-1)(m-0.5)}{3!}\delta^3 y_{\frac{1}{2}} + \frac{m(m^2-1)(m-2)}{4!}\mu\delta^4 y_{\frac{1}{2}} + \cdots$$

$$y_{x-1} = \frac{0+15}{2} + (x-1-0.5)(15) + \frac{(x-1)(x-2)}{2}\left(\frac{14+50}{2}\right) + \frac{(x-1)(x-2)(x-1-0.5)}{6}(36) + \frac{(x-1)((x-1)^2-1)(x-3)}{24}\left(\frac{24+24}{2}\right).$$

$$y_{x-1} = 7.5 + (x-1.5)(15) + (x-1)(x-2)(16) + (x-1)(x-2)(x-1.5)(6) + (x^3-3x^2+2x)(x-3).$$

$$y_{x-1} = 7.5 + 15x - 22.5 + 16x^2 - 48x + 32 + 6x^3 - 18x^2 + 12x - 9x^2 + 27x - 18 + x^4 - 6x^3 + 11x^2 - 6x.$$

$f(x) = x^4 - 1$.

2) **صيغ ستيرلينك المركزية**

$$y_m = y_0 + m\mu\delta y_0 + \frac{m^2}{2!}\delta^2 y_0 + \frac{m(m^2-1)}{3!}\mu\delta^3 y_0 + \frac{m^2(m^2-1)}{4!}\delta^4 y_0 + \cdots$$

$$y_{x-1} = 0 + (x-1)\left(\frac{1+15}{2}\right) + \frac{(x-1)^2}{2}(14) + \frac{x-1((x-1)^2-1)}{6}\left(\frac{12+36}{2}\right) + \frac{(x-1)^2((x-1)^2-1)}{24}(24)$$

$$y_{x-1} = 0 + (x-1)(8) + \frac{x^2-2x+1}{2}(14) + \frac{x-1(x^2-2x+1-1)}{6}(24) + \frac{x^2-2x+1(x^2-2x+1-1)}{24}(24)$$

$$y_{x-1} = 0 + 8x - 8 + 7x^2 - 14x + 7 + \frac{(x-1)(x^2-2x)}{6}(24) + \frac{x^2-2x+1(x^2-2x)}{24}(24)$$

$$y_{x-1} = 8x - 8 + 7x^2 - 14x + 7 + 4x^3 - 12x^2 + 8x + x^4 - 4x^3 + 5x^2 - 2x$$

$y_{x-1} = x^4 - 1$.

Homework

Find $f(20.3), f(24.8), f(23.2), f^{-1}(19), f^{-1}(13)$ of the following table use the forward difference and find To find the function .

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Homework

Find $f(1.1), f(0.4), f(-0.1), f(3.8), f(1.5), f(4.1), f(5.3), f(-1.2), f(1.7), f^{-1}(10), f^{-1}(20)$ of the following table use the forward difference and find To find the function .

x	-1	0	1	2	3	4
y	0	-1	0	15	80	255