3.2.1 Canonical forms الصيغ القانونية

In this subsection we study the canonical forms of equ. (3.8). Suppose the functions A, B & C not all zero and let ξ and η be the new variables s.t $A^* = C^* = 0$

i.e

$$A^* = A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 = 0$$

and $$C^* = A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2 = 0$$

The above two equations are of the same form so we rewrite them in the form

$$A\gamma_x^2 + 2B\gamma_x\gamma_y + C\gamma_y^2 = 0 \quad \cdots (3.10)$$

where γ represents ξ or η.

Divide equation (3.10) by γ_y^2 to get

$$A\left(\frac{\gamma_x}{\gamma_y}\right)^2 + 2B\left(\frac{\gamma_x}{\gamma_y}\right) + C = 0 \quad \cdots (3.11)$$

$$d\gamma = \gamma_x dx + \gamma_y dy$$

and $d\gamma = 0$ when γ is constant

therefore

$$\frac{dy}{dx} = -\frac{\gamma_x}{\gamma_y} \quad \text{---} (3.12)$$

So equ. (10) becomes

$$A\left(\frac{dy}{dx}\right)^2 - 2B\frac{dy}{dx} + C = 0 \quad \cdots (3.13)$$

It's roots are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - AC}}{A}$$

or $$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - AC}}{A} \quad \cdots (3.14)$$

and $$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - AC}}{A} \quad \text{--} (3.15)$$

These two roots (3.14, 3.15) are called the characteristic equations.
Notice that each of these two roots is a 1st order diff. equ. and

each solution has arbitrary constant say ξ & η

3.2.2 Hyperbolic Type pdes

If $B^2 - AC > 0$ then the integration of (3.14) & (3.15) gives distinct real functions (دوال حقيقية مختلفة) which are

$$\xi(x,y) = \int \left(dy - \frac{B + \sqrt{B^2 - AC}}{A} dx \right)$$

$$= y - \int \frac{B + \sqrt{B^2 - AC}}{A} dx \quad \dots (3.16)$$

and $$\eta(x,y) = y - \int \frac{B - \sqrt{B^2 - AC}}{A} dx \quad \dots (3.17)$$

ξ & η are called the characteristic curves (المنحنيات المميزة) and they are lines when A, B & C are constants.

Substitute ξ & η in equ. (3.9) to get

$$u_{\xi\eta} = H_1 \quad \text{--- (3.18)}$$

where $H_1 = \dfrac{H^*}{2B^*}$, $B^* \neq 0$

Equ. (3.18) is called the

<u>1st canonical form</u> of the hyperbolic

type pde.

<u>Now</u> we can get a socalled

the 2nd canonical form by letting

$$\alpha = \xi + \eta \quad \text{--- (3.19)}$$
$$\beta = \xi - \eta$$

to reduce equ. (3.8) to

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2(\alpha, \beta, u, u_\alpha, u_\beta) \quad \text{--- (3.20)}$$

eg 2 Find the two canonical forms of

$$(x-1)u_{xx}+2xu_{xy}+(x+1)u_{yy}-x^2u_x$$
$$+yu_y-u=\sin(x-y+1)$$

Sol $A=x-1,\ B=x\ \&\ C=+(x+1)$

$$B^2-AC=x^2-(x-1)(x+1)$$
$$=x^2-x^2+1=1>0$$

∴ The given pde. is of hyperbolic type.

(3.14) & (3.15) are of the form

$$\frac{dy}{dx}-\frac{x+1}{x-1}=0 \quad \& \quad \frac{dy}{dx}-\frac{x-1}{x-1}=0$$

$$\xi(x,y)=\frac{1}{2}(y-x)-\ln|x-1|$$

$$\eta(x,y)=y-x$$

Now we evaluate x & y in terms of ξ & η

$$x = 1 + e^{\frac{1}{2}\eta - \xi}$$

$$y = \eta + 1 + e^{\frac{1}{2}\eta - \xi}$$

We evaluate the following

$$\xi_x = -\frac{1}{2} - \frac{1}{x-1} = -\frac{2 + e^{\frac{1}{2}\eta - \xi}}{2e^{\frac{1}{2}\eta - \xi}}$$

$$\eta_x = -1, \ \xi_y = \frac{1}{2}, \ \eta_y = 1$$

$$\xi_{xx} = e^{\eta - 2\xi}$$

$$\xi_{yy} = \xi_{xy} = \eta_{yy} = \eta_{xy} = \eta_{xx} = 0$$

$$\xi_x^2 = \frac{\left(2 + e^{\frac{1}{2}\eta - \xi}\right)^2}{4e^{\eta - 2\xi}}, \ \eta_x^2 = \eta_y^2 = 1$$

So the given pde becomes

$$u_{\xi\eta} = \frac{-2e^{\frac{1}{2}\eta - \xi}}{2e^{\eta - 2\xi} + 7e^{\frac{1}{2}\eta - \xi} + 5}\left[\frac{-2e^{-4\xi + 2\eta} + e^{-3\xi + \frac{3}{2}\eta} + 5e^{-2\xi + \eta} + (\eta + 6)e^{-\xi + \frac{1}{2}\eta} + 2}{2e^{-\xi + \frac{1}{2}\eta}} u_\xi\right.$$

$$+\left(e^{-2\xi+\eta}+3e^{-\xi+\frac{1}{2}\eta}+\eta+2\right)u_{\eta}-u-\sin(1-\eta)\Big]$$

which is the 1^{st} canonical form.

To find the 2^{nd} canonical form. Let

$$\alpha=\xi+\eta$$

and $\beta=\xi-\eta$

and the rest of calculations are left as an exercise to the student