Prof. Dr. Saad Naji Al-Azzawi
Assit. lec. Huda Emad AL-Deen

الفصل الدراسي الثاني

الفصل الثالث : المعادلات التفاضلية الجزئية من الرتبة الثانية

2^{nd} order P.D.Es

3.1 Introduction

The general form of a 2^{nd} order p.d.e in n independent variables $x_1, x_2, \dots, x_n$ and one dependent variable u is:

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G \quad \dots (3.1)$$

where $A_{ij} = A_{ji}$ and A_{ij}, B_i, F and G are functions defined on $D \subseteq \mathbb{R}^n$

We shall study the 2^{nd} order p.d.e's in two independent variables x & y and one dependent variable u where general form is:-

$$A(x,y) u_{xx} + 2B(x,y) u_{xy} + C(x,y) u_{yy} + D(x,y) u_x + E(x,y) u_y + F(x,y) u = G(x,y) \quad \text{----- } (3.2)$$

where A, B & C are not zero at the same point.

<u>Note</u> equ. (3-2) is called linear pde.

<u>Def</u> If the coefficients are constants then equ. (3-2) is called 2nd order pde with constant coefficients

<u>Def</u> If $G(x,y) \equiv 0$ then (3-2) is called homogeneous

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u = 0 \quad \text{--(3.3)}$$

otherwise ($G \neq 0$) nonhomogeneous

3.2 Classification of 2nd order pdes

The algebraic equation

$$A x^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$$

is called of

1) hyperbolic type if $B^2 - AC > 0$
2) Parabolic type if $B^2 - AC = 0$
3) Elleptic type if $B^2 - AC < 0$

Similarly for pde's

Def The pde (3.2) is of

1) Hyperbolic type if $B^2(x,y) - A(x,y)\,C(x,y) > 0$
2) Parabolic type if $B^2(x,y) - A(x,y)\,C(x,y) = 0$
3) Elleptic type if $B^2(x,y) - A(x,y)\,C(x,y) < 0$

for a point (x_0, y_0). If it is true for all $(x,y) \in D$ then they are called of $\left(\begin{array}{c}\text{hyperbolic type}\\ \text{parabolic type}\\ \text{elleptic type}\end{array}\right)$ when $\left(\begin{array}{c}B^2 - AC > 0\\ B^2 - AC = 0\\ B^2 - AC < 0\end{array}\right)$

The question is how to find the transformation(s) to get there canonical forms. The answer is yes we can find such transformation for equ. (3.2) and this transformation is

$$\xi = \xi(x,y) \qquad\qquad \text{------} \; (3.4)$$
$$\eta = \eta(x,y)$$

where ξ & η are C^1 functions and the Jacobian J!

$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$ at the point (x, y) under consideration.

Now

$$\left.\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} (\xi_x \eta_y + \xi_y \eta_x) + u_{\eta\eta} \eta_x \eta_y \\
&\quad + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 \\
&\quad + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned}\right\} \quad (3.5)$$

Substituting there quantitier in equ.(3.2) to get

$$A^* u_{\xi\xi} + 2B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + Fu = G \quad \dots (3.6)$$

where

$$\left.\begin{aligned}
A^* &= A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 \\
B^* &= A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y \\
C^* &= A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + 2B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + 2B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y
\end{aligned}\right\} \quad (3.7)$$

equ. (3.2) is equivalent to equ. (3.6) when $J \neq 0$, As we notice the classification depends on A, B & C therefore we rewrite equ. (3.2) in the form

$$A(x,y)u_{xx}+2B(x,y)u_{xy}+C(x,y)u_{yy}=H(x,y,u,u_x,u_y) \quad \dots (3.8)$$

and rewrite equ. (3.6) in the form

$$A^* u_{\xi\xi}+2B^* u_{\xi\eta}+C^* u_{\eta\eta}=H^*(\xi,\eta,u,u_\xi,u_\eta) \quad \dots (3.9)$$

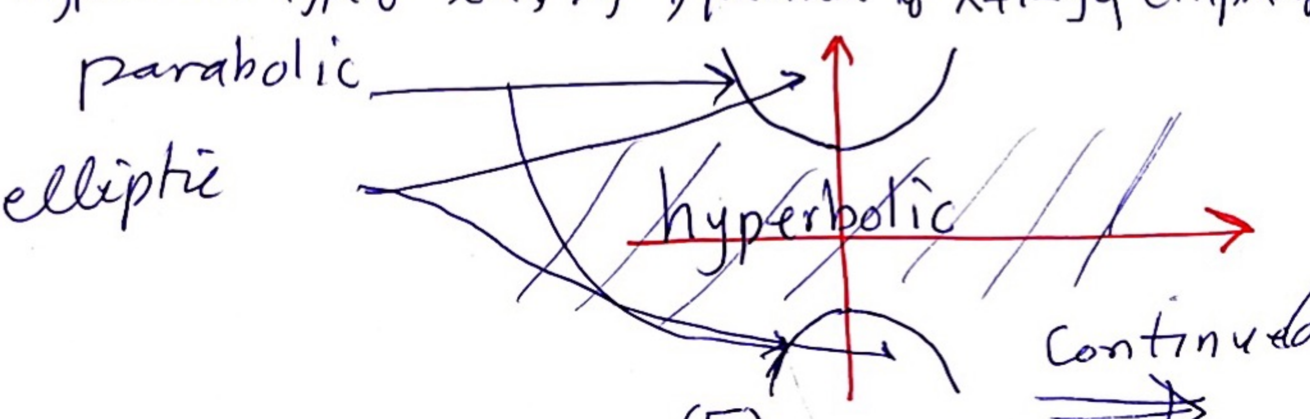
Example 1: Find the region $D \subseteq \mathbb{R}^2$ s.t

$$(x^2+1)u_{xx}-2xyu_{xy}+(y^2-1)u_{yy}+e^x u_x$$
$$+3x^2 u_y+5u=\sin(x+y)$$

is of hyperbolic, parabolic & elliptic type.

Sol. $B^2-AC=(-xy)^2-(x^2+1)(y^2-1)$
$$=\cancel{x^2y^2}-\cancel{x^2y^2}+x^2-y^2+1$$
$$=x^2-y^2+1$$

hyperbolic type of $x^2+1>y^2$, parabolic of $x^2+1=y^2$ & elliptic of $x^2+1<y^2$

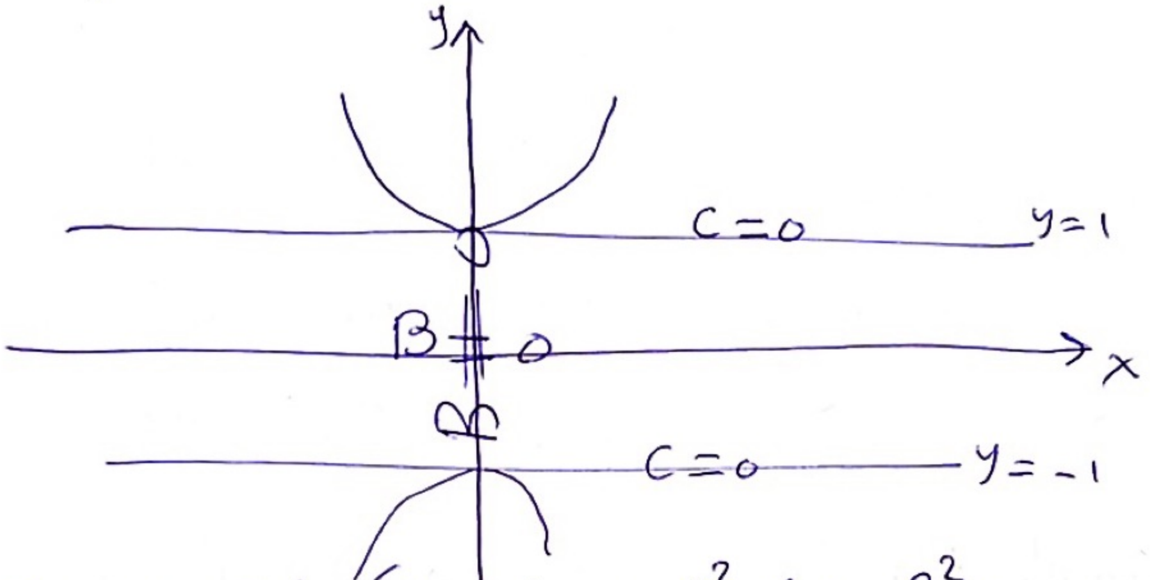


continued

Again $A = x^2+1 \neq 0 \quad \forall (x,y) \in$ plane

$B = -xy = 0$ on x-axis & y-axis

$C = y^2-1 = 0$ on $y = \mp 1$



On $y = \mp 1$ $(C=0) \Rightarrow B^2 - AC = B^2 > 0$

$\therefore$ The pde. is of hyperbolic type

On x-axis & y-axis $(y=0$ & $x=0)$

$B^2 - AC = 0 - (x^2+1)(y^2-1) = 1 > 0$

$\therefore$ It is of hyperbolic type