Lecture 5

Modeling & Simulation

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Auto-correlation test

The tests for autocorrelation are concerned with the dependence between numbers in a sequence. The list of the 30 numbers below (as an example) appears to have the effect that every 5th number has a very large value. If this is a regular pattern, we can't really say the sequence is random. The test to be described below requires the computation of the autocorrelation between every l numbers (l is also known as the lag), starting with the ith number. Thus, the autocorrelation ρ_{il} between the following numbers would be of interest:

$R_i\,,\; R_{i+l}\,,\; R_{i+2l}\,, \dots, R_{i+(M+1)l}.$

Then compute the value M (the largest integer to be found) such that:

$i + (M+1)l \leq N$, where N is the total number of values in the sequence, i is the beginning value. Since a nonzero autocorrelation implies a lack of independence, the following two tailed test is appropriate:

$$H_0: \quad \rho_{il} = 0$$

$$H_1: \quad \rho_{il} \neq 0$$

For large values of M, the distribution of the estimator of ρ_{il}, denoted $\widetilde{\rho_{il}}$, is approximately normal if the values $R_i\,,\; R_{i+l}\,,\; R_{i+2l}\,, \dots, R_{i+(M+1)l}$ are uncorrelated. Then the test statistic can be formed as follows:

$$Z_0 = \frac{\widetilde{\rho_{il}}}{\sigma_{\widetilde{\rho_{il}}}}$$

which is distributed normally with a mean of zero and a variance of 1, under the assumption of independence, for large M.

The formula for $\widetilde{\rho_{il}}$ in a slightly different form, and the standard deviation of the estimator, $\sigma_{\widetilde{\rho_{il}}}$ are given as follows:

$$\widetilde{\rho_{il}} = \frac{1}{M+1}\left[\sum_{k=0}^{M} R_{i+kl}\, R_{i+(k+1)l}\right] - 0.25$$

And

$$\sigma_{\widetilde{\rho_{il}}} = \frac{\sqrt{13M + 7}}{12(M + 1)}$$

After computing Z_0, do not reject the null hypothesis of independence if $\frac{-Z_\alpha}{2} \leq Z_0 \leq \frac{Z_\alpha}{2}$ where $\boldsymbol{\alpha}$ is the level of significance and $\frac{Z_\alpha}{2}$ is obtained from a given table.

Example

0.12	0.01	0.23	0.28	0.89	0.31	0.64	0.28	0.83	0.93
0.99	0.15	0.33	0.35	0.91	0.41	0.60	0.27	0.75	0.88
0.68	0.49	0.05	0.43	0.95	0.58	0.19	0.36	0.69	0.87

Test whether the 3rd, 8^{th}, 13^{th} and so on, numbers in the sequence at the table above are auto correlated using $\boldsymbol{\alpha} = 0.05$. Here, $i = 3$ (beginning with the third number), $l = 5$ (every five numbers), $N = 30$ (30 numbers in the sequence), thus compute $M = 4$ (largest integer such that $3 + (M + 1)5 < 30$). Then,

$$\widetilde{\rho_{il}} = \frac{1}{M + 1}\left[\sum_{k=0}^{M} R_{i+kl}\, R_{i+(k+1)l}\right] - 0.25$$

$$\widetilde{\rho_{35}} = \frac{1}{4 + 1}[(0.23)(0.28) + (0.28)(0.33) + (0.33)(0.27) + (0.27)(0.05) + (0.05)(0.36)] - 0.25 = -0.1945$$

And

$$\sigma_{\widetilde{\rho_{35}}} = \frac{\sqrt{13(4) + 7}}{12(4 + 1)} = 0.1280$$

Then, the test statistic assumes the value

$$Z_0 = \frac{\widetilde{\rho_{il}}}{\sigma_{\widetilde{\rho_{il}}}} = \frac{-0.1945}{0.1280} = -1.516$$

Now, the critical value from given table is

$$Z_{0.025} = 1.96$$

Therefore, the hypothesis of independence cannot be rejected on the basis of this test.

Queuing Theory

Queuing theory is the mathematical study of waiting lines, or queues.

Simulation of a Single-Server Queuing System

Here we introduce a single-server queuing model, and how to simulate it. A good example to think about for intuition is an ATM machine. We view the machine as a “server” that serves customers one at a time. The customers arrive randomly over time and wait in a queue (line), and upon beginning service, each customer spends a random amount of time in service before departing.

1.1 FIFO single-server model

There is one server (clerk, machine), behind which forms a queue (line) for arriving customers to wait in. The n^{th} customer is denoted by C_n and arrives at time t_n, where

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots,$$

with $\lim_{n\to\infty} t_n = \infty$.

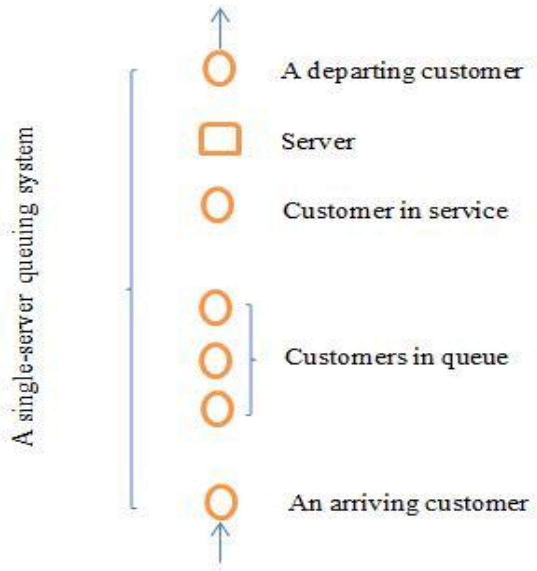
$T_n \stackrel{\text{def}}{=} t_{n+1} - t_n$ denotes the n^{th} interarrival time, the length of time between arrival of the successive customers C_n and C_{n+1}.

C_n requires a service time of length S_n, which is the length of time C_n spends in service with the server. We assume that the server processes service times at rate 1, meaning that, for example, if C_n enters service now with $S_n = 6$, then 4 units of time later there are 2 units of service time remaining to process. D_n, called the *delay* of C_n, denotes the length of time that C_n waits in the queue (line) *before entering service*; if C_n arrives finding the system empty, then C_n enters service immediately and so $D_n = 0$. Summarizing: C_n arrives at time t_n, waits in the queue for D_n units of time, then spends S_n units of time with the server before departing at time $t_n^d = t_n + D_n + S_n$, the n^{th} departure time. We are inherently assuming here that customers join the end of the queue upon arrival an enter service one at a time, and this is known as *first-in-first-out* (FIFO). But other service disciplines are useful in other applications, such as in computer processing, where *processor sharing* (PS) might be employed: If there are $k \geq 1$ “jobs” in the system, they all are in service together, but each is served at rate $1/k$. We will discuss disciplines later on, so for now we assume FIFO.

FIFO delay has a nice recursive property:

$$D_{n+1} = (D_n + S_n - T_n)^+, \ n \geq 0,$$

Consider a single-server queuing system



In which,

1. The inter arrival times $A_1, A_2, A_3, \ldots$, are independent identically distributed (iid) random variables.
2. The service times $S_1, S_2, S_3, \ldots$, of the successive customers are (iid) random variables that are independent of the interarrival times.

Note

A customer who arrives and finds the server:

1. Idle enters service immediately.
2. Busy joins the end of a single-queue.

- The server chooses a customer from the queue in a first-in, first-out (FIFO) manner.
- The simulation will begin in the empty and idle state.
- The arrival of the first customer occurs after the first inter arrival time A_1, rather than at time 0.

Suppose that we wish to simulate this system until a fixed number n of customers have completed their delays in queue (i.e. the nth customer enters service).

To measure the performance of this system, we will estimates (evaluate) three quantities:

1. The expected average delay in queue of the n customers completing their delays during the simulation.

$$average\ delay = \hat{d}(n) = \frac{(\sum_{i=1}^{n} D_i)}{n}$$

Where D_i is the delay in service for the ith customers, n is the number of customers enters service. Obviously, $D_1 = 0$ according to above note. Also, system providing very good service if many delays were zero.

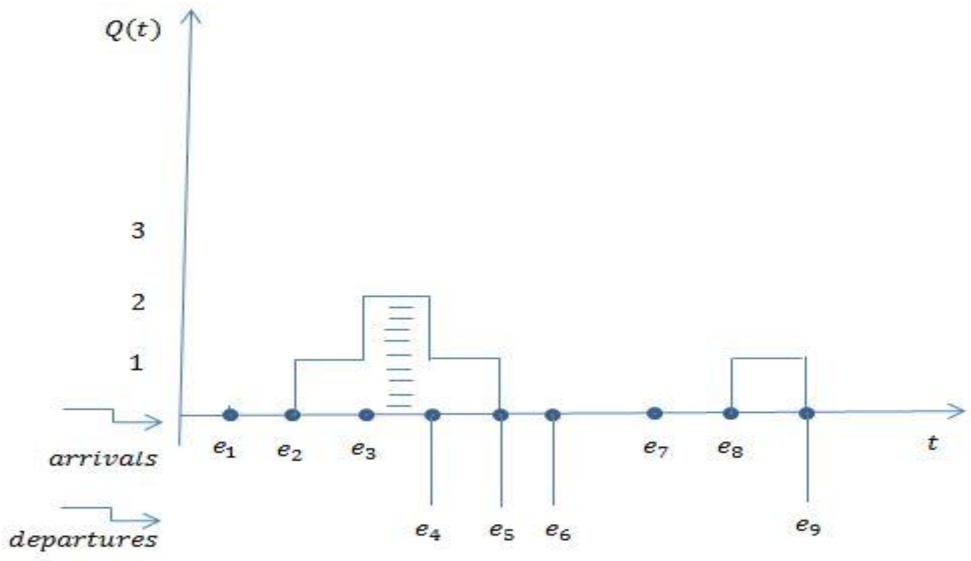
2. The expected average number of customers in queue (not being served). Let $Q(t)$ represents the number of customers in queue at time t. $T(n)$ be the time required to observe n delays in queue. Therefore,

$$average\ number\ of\ customers\ in\ queue = \hat{q}(n) = \frac{\int_0^{T(n)} Q(t)\, dt}{T(n)}$$

$$= \frac{area\ under\ Q(t)\ curve}{Time}$$

<u>Note</u>

Average number of customers in queue also known as average length of queue.



3. The expected proportion of time during the simulation (from 0 to $T(n)$) that the server is busy or the expected utilization of the server.

First we define the busy function by

$$B(t) = \begin{cases} 1 & if\ server\ busy\ at\ time\ t \\ 0 & if\ server\ idle\ at\ time\ t \end{cases}$$

Thus

The busy time for server = $\hat{u}(n) = \frac{\int_0^{T(n)} B(t)dt}{T(n)} = \frac{area\ under\ B(t)}{Time}$

Departure time of i^{th} customer $= d_i = \begin{cases} t_i + s_i & server\ idle \\ d_{i-1} + s_i & sever\ busy \end{cases}$

where t_i is the arrival time of i^{th} customer

s_i is the service time of i^{th} customer

$$s_i = \begin{cases} d_i - t_i & if\ server\ idle \\ d_i - d_{i-1} & if\ sever\ busy \end{cases}$$

Also, The delay of i^{th}customer= $D_i = \begin{cases} d_{i-1} - t_i & server\ busy \\ 0 & sever\ idle \end{cases}$

Thus the average delay $\hat{d}(n) = \frac{(\sum_{i=1}^{n} D_i)}{n}$