Dr. Auras Khalid

(c) Product of Matrices:

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B. Then the product matrix AB has the same number of rows as A and the same number of columns as B.

Thus the product of the matrices A_{mxp} and B_{pxn} is the matrix $(AB)_{mxn}$. The elements of AB are determined as follows:

The element C_{ij} in the ith row and jth column of $(AB)_{mxn}$ is found by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots\dots\dots + a_{in}b_{nj}$

for example, consider the matrices

$$A_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B_{2x2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Since the number of columns of A is equal to the number of rows of B ,the product AB is defined and is given as

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Thus c_{11} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the first column of B i.e., b_{11} , b_{21} and adding the product.

Similarly , c_{12} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the second column of B i.e., b_{12} , b_{22} and adding the product. Similarly for c_{21} , c_{22} .

Note :

1 . Multiplication of matrices is not commutative i.e., $AB \neq BA$ in general.
2 . For matrices A and B if $AB = BA$ then A and B commute to each other
3 . A matrix A can be multiplied by itself if and only if it is a square matrix.The product A.A in such cases is written as A^2.
Similarly we may define higher powers of a square matrix i.e., $A \,.\, A^2 = A^3$, $A^2 .\, A^2 = A^4$
4. In the product AB, A is said to be pre multiple of B and B is said to be post multiple of A.

Example 1: **If** $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **Find AB and BA.**

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 1+2 \\ -2+3 & -1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 4+3 \\ 1-1 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 0 & 5 \end{bmatrix}$$

This example shows very clearly that multiplication of matrices in general, is not commutative i.e., $AB \neq BA$.

Example 2: If

Example 2: **If** $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$**, find AB**

Solution:

Since A is a (2 x 3) matrix and B is a (3 x 2) matrix, they are conformable for multiplication. We have

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}$$

Remark:

If A, B and C are the matrices of order (m x p), (p x q) and (q x n) respectively, then

i. (AB)C = A(BC) i.e., Associative law holds.

ii. $\left.\begin{array}{l} C(A+B) = CA + CB \\ \text{and } (A+B)C = AC + BC \end{array}\right\}$ i.e distributive laws holds.

Note: that if a matrix A and identity matrix I are conformable for multiplication, then I has the property that

AI = IA = A i.e, I is the identity matrix for multiplication.

Exercise 9.1

Q.No. 1 Write the following matrices in tabular form:

i. $A = [a_{ij}]$, where i = 1, 2, 3 and j = 1, 2, 3, 4
ii. $B = [b_{ij}]$, where i = 1 and j = 1, 2, 3, 4
iii. $C = [c_{jk}]$, where j = 1, 2, 3 and k = 1

Q.No.2 Write each sum as a single matrix:

i. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

ii. $[1 \quad 3 \quad 5 \quad 6] + [0 \quad -2 \quad 1 \quad 3]$

iii. $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$

iv. $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

v. $2\begin{bmatrix} 6 & 1 \\ 0 & -3 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -5 & -1 \end{bmatrix}$

Q.3 Show that $\begin{bmatrix} b_{11} - a_{11} & b_{12} - a_{12} \\ b_{21} - a_{21} & b_{22} - a_{22} \end{bmatrix}$ is a solution of the matrix equation $X + A = B$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Q.4 Solve each of the following matrix equations:

i. $X + \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

ii. $X + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -2 & 0 \end{bmatrix}$

iii. $3X + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

iv. $X + 2I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

Q.5 Write each product as a single matrix:

i. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

ii. $\begin{bmatrix} 3 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

iii. $\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}$

iv. $\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

Q.6 If $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, find $A^2 + BC$.

Q.7 Show that if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then

(a) $(A + B)(A + B) \neq A^2 + 2AB + B^2$

(b) $(A + B)(A - B) \neq A^2 - B^2$

Q.8 Show that:

(i) $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 5 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a + 2b + 3c \\ 2a + b \\ 3a + 5b - c \end{bmatrix}$

The Determinant of a Matrix:

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. The determinants are defined only for square matrices. It is denoted by det A or |A| for a square matrix A.

The determinant of the (2 x 2) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11}\, a_{22} - a_{12}\, a_{21}$$

Example 3: If $A = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ find |A|

Solution:

$$|A| = \begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} = 9 - (-2) = 9 + 2 = 11$$

The determinant of the (3 x 3) matrix

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, denoted by $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

is given as, det A = |A|

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note: Each determinant in the sum (In the R.H.S) is the determinant of a submatrix of A obtained by deleting a particular row and column of A.

Example 4: If $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

find det A by expansion about (a) the first row

$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= 3(4 + 6) - 2(0 + 2) + 1\,(0 - 1)$$

$$= 30 - 4 - 1$$

$$|A| = 25$$

Solution of Linear Equations by Determinants:
(Cramer's Rule)

Consider a system of linear equations in two variables x and y,

$a_1x + b_1y = c_1$ (1)

$a_2x + b_2y = c_2$ (2)

The solutions for x and y of the system of equations (1) and (2) can be written directly in terms of determinants without any algebraic operations, as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This result is called Cramer's Rule.

Here $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = |A|$ is the determinant of the coefficient of x and y

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Then $x = \dfrac{|A_x|}{|A|}$ and $y = \dfrac{|A_y|}{|A|}$

Solution for a system of Linear Equations in Three Variables:

Consider the linear equations:

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Hence the determinant of coefficients is

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ if } |A| \neq 0$$

Then by Cramer's Rule the value of variables is:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_x|}{|A|}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_y|}{|A|}$$

and $$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{|A|} = \frac{|A_z|}{|A|}$$

Example

Use Cramer's rule to solve the system

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$x - 3y + 2z = 8$$

Here the determinant of the coefficients is:

$$|A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$
$$= -4(8+3) - 2(6-1) - 9(-9-4)$$
$$= -44 - 10 + 117$$
$$|A| = 63$$

for $|A_x|$, replacing the first column of $|A|$ with the corresponding constants 2, 5 and 8, we have

$$|A_x| = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}$$
$$= 2(11) - 2(2) - 9(-47) = 22 - 4 + 423$$

$$\boxed{|A_x| = 441}$$

Similarly,

$$|A_y| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}$$
$$= -4(2) - 2(5) - 9(19)$$
$$= -8 - 10 - 171$$

$$\boxed{|A_y| = -189}$$

and

$$|A_z| = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}$$
$$= -4(47) - 2(19) + 2(-13)$$
$$= -188 - 38 - 26$$

$$\boxed{|A_z| = -252}$$

Hence $x = \dfrac{|A_x|}{|A|} = \dfrac{441}{63} = 7$

$$y = \frac{|A_y|}{|A|} = \frac{-189}{63} = -3$$

$$z = \frac{|A_z|}{|A|} = \frac{-252}{63} = -4$$

So the solution set of the system is $\{(7, -3, -4)\}$

Use Cramer's rule to solve the following system of equations.

(i) $x - y = 2$, $x + 4y = 5$

(ii) $3x - 4y = -2$, $x + y = 6$

(iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$

(iv) $2x + 2y + z = 1$, $x - y + 6z = 21$, $3x + 2y - z = -4$

(v) $x + y + z = 0$, $2x - y - 4z = 15$, $x - 2y - z = 7$

(vi) $x - 2y - 2z = 3$, $2x - 4y + 4z = 1$, $3x - 3y - 3z = 4$

9.9 Special Matrices:

1. Transpose of a Matrix

If $A = [a_{ij}]$ is mxn matrix, then the matrix of order n x m obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted A^t or A'.

Example if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

2. Symmetric Matrix:

A square matrix A is called symmetric if $A = A^t$

for example if

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \quad \text{then} \quad A^t = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = A$$

Thus A is symmetric

3. Skew Symmetric:

A square matrix A is called skew symmetric if $A = -A^t$

for example if $B = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$, then

$$B^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = (-1) \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$B^t = -B$$

Thus matrix B is skew symmetric.

4. Singular and Non-singular Matrices:

A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$, for example if

$A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$, then $|A| = 0$, Hence A is singular and if $A = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}$, then $|A| \neq 0$,

Hence A is non-singular.

Example: Find k If $A = \begin{bmatrix} k-2 & 1 \\ 5 & k+2 \end{bmatrix}$ is singular

Solution: Since A is singular so $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$

$(k-2)(k+2) - 5 = 0$

$k^2 - 4 - 5 = 0$

$k^2 - 9 = 0 \Rightarrow \quad K = \pm 3$

5. Adjoint of a Matrix:

Let $A = (a_{ij})$ be a square matrix of order n x n and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A.

For example , if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Cofactor of A are:

$A_{11} = 5$, $A_{12} = -2$, $A_{13} = +1$

$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$

$A_{31} = 3$, $A_{32} = -2$, $A_{33} = 3$

Matrix of cofactors is

$$C = \begin{bmatrix} 5 & -2 & +1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix} \qquad C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$$

Hence adj A = C^t $= \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$

Note: Adjoint of a 2×2 Matrix:

The adjoint of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adjA is defined as

$$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

6. Inverse of a Matrix:

If A is a non-singular square matrix , then $A^{-1} = \dfrac{\text{adj } A}{|A|}$

For example if matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Then adj A $= \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Example 10: Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$|A| = 0 + 2(-2+3) - 3(-2+3) = 2 - 3$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0$, $A_{12} = 1$, $A_{13} = 1$

$A_{21} = 2$, $A_{22} = -3$, $A_{23} = 2$

$A_{31} = 3$, $A_{32} = -3$, $A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

9.11 Solution of Linear Equations by Matrices:

Consider the linear system:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \text{------} + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{------} + a_{2n}x_n = b_2 \\ \quad | \qquad\quad | \qquad\qquad\qquad | \\ a_{n1}x_1 + a_{n2}x_2 + \text{------} + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (1)$$

It can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$

Then latter equation can be written as,

$$AX = B$$

If $B \neq 0$, then (1) is called non-homogenous system of linear equations and if $B = 0$, it is called a system of homogenous linear equations.

If now $B \neq 0$ and A is non-singular then A^{-1} exists.

Multiply both sides of $AX = B$ on the left by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$1X = A^{-1}B$$

$$\text{Or} \quad X = A^{-1}B$$

Where $A^{-1}B$ is an n x 1 column matrix. Since X and $A^{-1}B$ are equal, each element in X is equal to the corresponding element in $A^{-1}B$. These elements of X constitute the solution of the given linear equations.

If A is a singular matrix, then of course it has no inverse, and either the system has no solution or the solution is not unique.

Example 11: Use matrices to find the solution set of

$$x + y - 2z = 3$$
$$3x - y + z = 5$$
$$3x + 3y - 6z = 9$$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Since $|A| = 3 + 21 - 24 = 0$

Hence the solution of the given linear equations does not exists.

Example 12: Use matrices to find the solution set of

$$4x + 8y + z = -6$$
$$2x - 3y + 2z = 0$$
$$x + 7y - 3z = -8$$

Solution:

Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}$

Since $|A| = -32 + 48 + 17 = 61$

So A^{-1} exists.

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

Now since,

$X = A^{-1} B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}$$

Q.5 Find the solution set of the following system by means of matrices:

(i) $2x - 3y = -1$
$x + 4y = 5$

(ii) $x + y = 2$
$2x - z = 1$
$2y - 3z = -1$

(iii) $x - 2y + z = -1$
$3x + y - 2z = 4$
$y - z = 1$

(iv) $-4x + 2y - 9z = 2$
$3x + 4y + z = 5$
$x - 3y + 2z = 8$

(v) $x + y - 2z = 3$
$3x - y + z = 0$
$3x + 3y - 6z = 8$

$$= \frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Hence Solution set: $\{(x, y, z)\} = \{(-2, 0, 2)\}$

Exercise 9.3

Q.1 Which of the following matrices are singular or non-singular.

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Q.2 Which of the following matrices are symmetric and skew-symmetric

(i) $\begin{bmatrix} 2 & 6 & 7 \\ 6 & -2 & 3 \\ 7 & 3 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Q.3 Find K such that the following matrices are singular

(i) $\begin{vmatrix} K & 6 \\ 4 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & K \\ -4 & 2 & 6 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ k & 3 & -6 \end{vmatrix}$

Q.4 Find the inverse if it exists, of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$