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Lecture 3
Inequalities & Limits

Rules for Inequalities

If a, b, and c are real numbers, then:

1. $a < b \Rightarrow a + c < b + c$
2. $a < b \Rightarrow a - c < b - c$
3. $a < b$ and $c > 0 \Rightarrow ac < bc$
4. $a < b$ and $c < 0 \Rightarrow bc < ac$
   Special case: $a < b \Rightarrow -b < -a$
5. $a > 0 \Rightarrow \frac{1}{a} > 0$
6. If a and b are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements.

Solving Inequalities

The process of finding the interval or intervals of numbers that satisfy an inequality in x is called **solving** the inequality.

EXAMPLE 1 Solve the following inequalities and show their solution sets on the real line.

(a) $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x-1} \geq 5$

Solution

(a)

$$2x - 1 < x + 3$$

$$2x < x + 4 \quad \text{Add 1 to both sides.}$$

$$x < 4 \quad \text{Subtract } x \text{ from both sides.}$$

The solution set is the open interval $(-\infty, 4)$ (Figure 1.1a).

(b)

$$-\frac{x}{3} < 2x + 1$$

$$-x < 6x + 3 \quad \text{Multiply both sides by 3.}$$

$$0 < 7x + 3 \quad \text{Add } x \text{ to both sides.}$$

$$-3 < 7x \quad \text{Subtract 3 from both sides.}$$

$$-\frac{3}{7} < x \quad \text{Divide by 7.}$$

(c) The inequality $6/(x - 1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x - 1)$ is undefined or negative. Therefore, $(x - 1)$ is positive and the inequality will be preserved if we multiply both sides by $(x - 1)$, and we have

$$\begin{aligned} \frac{6}{x-1} &\geq 5 & & \\ 6 &\geq 5x - 5 & & \text{Multiply both sides by } (x-1). \\ 11 &\geq 5x & & \text{Add 5 to both sides.} \\ \frac{11}{5} &\geq x. & & \text{Or } x \leq \frac{11}{5}. \end{aligned}$$

The solution set is the half-open interval $(1, 11/5]$ (Figure 1.1c). ■

Absolute Value Properties

1.	$\lvert -a \rvert = \lvert a \rvert$	A number and its additive inverse or negative have the same absolute value.
2.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$	The absolute value of a product is the product of the absolute values.
3.	$\left\lvert \frac{a}{b} \right\rvert = \frac{\lvert a \rvert}{\lvert b \rvert}$	The absolute value of a quotient is the quotient of the absolute values.
4.	$\lvert a + b \rvert \leq \lvert a \rvert + \lvert b \rvert$	The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values.

EXAMPLE 3 Illustrating the Triangle Inequality

$$\begin{aligned} \lvert -3 + 5 \rvert &= \lvert 2 \rvert = 2 < \lvert -3 \rvert + \lvert 5 \rvert = 8 \\ \lvert 3 + 5 \rvert &= \lvert 8 \rvert = \lvert 3 \rvert + \lvert 5 \rvert \\ \lvert -3 - 5 \rvert &= \lvert -8 \rvert = 8 = \lvert -3 \rvert + \lvert -5 \rvert \end{aligned}$$

■

Absolute Values and Intervals

If a is any positive number, then

5.	$\lvert x \rvert = a$	if and only if	$x = \pm a$
6.	$\lvert x \rvert < a$	if and only if	$-a < x < a$
7.	$\lvert x \rvert > a$	if and only if	$x > a$ or $x < -a$
8.	$\lvert x \rvert \leq a$	if and only if	$-a \leq x \leq a$
9.	$\lvert x \rvert \geq a$	if and only if	$x \geq a$ or $x \leq -a$

EXAMPLE 4 Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

Solution By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$\begin{aligned} 2x - 3 &= 7 & 2x - 3 &= -7 && \text{Equivalent equations without absolute values} \\ 2x &= 10 & 2x &= -4 && \text{Solve as usual.} \\ x &= 5 & x &= -2 \end{aligned}$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ■

EXAMPLE 5 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

Solution We have

$$\begin{aligned} \left|5 - \frac{2}{x}\right| < 1 &\Leftrightarrow -1 < 5 - \frac{2}{x} < 1 && \text{Property 6} \\ &\Leftrightarrow -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\ &\Leftrightarrow 3 > \frac{1}{x} > 2 && \text{Multiply by } -\tfrac{1}{2}. \\ &\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.} \end{aligned}$$

EXAMPLE 6 Solve the inequality and show the solution set on the real line:

(a) $|2x - 3| \le 1$ **(b)** $|2x - 3| \ge 1$

Solution

(a)

$$\begin{aligned} |2x - 3| &\le 1 \\ -1 \le 2x - 3 &\le 1 && \text{Property 8} \\ 2 \le 2x &\le 4 && \text{Add 3.} \\ 1 \le x &\le 2 && \text{Divide by 2.} \end{aligned}$$

The solution set is the closed interval $[1, 2]$ (Figure 1.4a).

(b)

$$|2x - 3| \ge 1$$

$$\begin{aligned} 2x - 3 &\ge 1 \quad \text{or} \quad 2x - 3 \le -1 && \text{Property 9} \\ x - \frac{3}{2} &\ge \frac{1}{2} \quad \text{or} \quad x - \frac{3}{2} \le -\frac{1}{2} && \text{Divide by 2.} \\ x &\ge 2 \quad \text{or} \quad x \le 1 && \text{Add } \tfrac{3}{2}. \end{aligned}$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Figure 1.4b). ■

Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well (such as the hyperbolic functions studied in Chapter 7).

Radian Measure

The **radian measure** of the angle ACB at the center of the unit circle (Figure 1.63) equals the length of the arc that ACB cuts from the unit circle. Figure 1.63 shows that $s = r\theta$ is the **length of arc** cut from a circle of radius r when the subtending angle θ producing the arc is measured in radians.

Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by

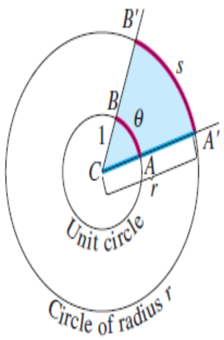
$$\pi \text{ radians} = 180°.$$

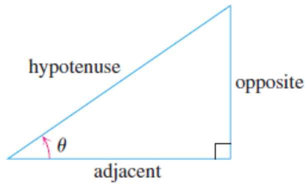
For example, $45°$ in radian measure is

$$45 \cdot \frac{\pi}{180} = \frac{\pi}{4} \text{ rad},$$

and $\pi/6$ radians is

$$\frac{\pi}{6} \cdot \frac{180}{\pi} = 30°.$$





$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.67 Trigonometric ratios of an acute angle.

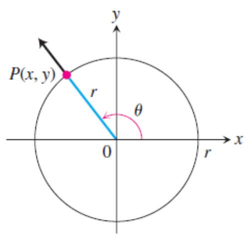


FIGURE 1.68 The trigonometric functions of a general angle θ are defined in terms of x, y, and r.

> **Angle Convention: Use Radians**
> From now on in this book it is assumed that all angles are measured in radians unless degrees or some other unit is stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is $60°$), not $\pi/3$ degrees. When you do calculus, keep your calculator in radian mode.

The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 1.67). We extend this definition to obtuse and negative angles by first placing the angle in standard position in a circle of radius r. We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle (Figure 1.68).

sine:	$\sin\theta = \dfrac{y}{r}$	**cosecant:**	$\csc\theta = \dfrac{r}{y}$
cosine:	$\cos\theta = \dfrac{x}{r}$	**secant:**	$\sec\theta = \dfrac{r}{x}$
tangent:	$\tan\theta = \dfrac{y}{x}$	**cotangent:**	$\cot\theta = \dfrac{x}{y}$

Notice also the following definitions, whenever the quotients are defined.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, $\tan\theta$ and $\sec\theta$ are not defined if $x = 0$. This means they are not defined if θ is $\pm\pi/2, \pm 3\pi/2, \dots$. Similarly, $\cot\theta$ and $\csc\theta$ are not defined for values of θ for which $y = 0$, namely $\theta = 0, \pm\pi, \pm 2\pi, \dots$.

The exact values of these trigonometric ratios for some angles can be read from the triangles in Figure 1.64. For instance,

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

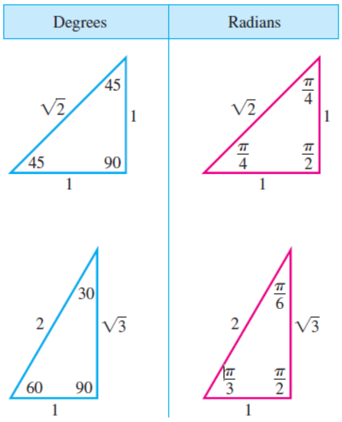


FIGURE 1.64 The angles of two common triangles, in degrees and radians.

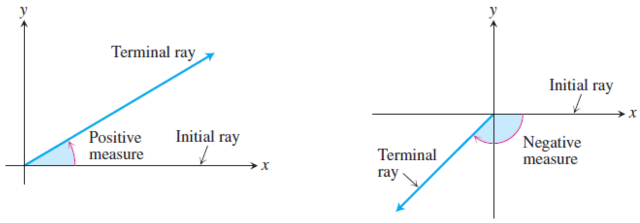
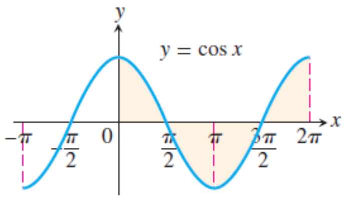


FIGURE 1.65 Angles in standard position in the xy-plane.

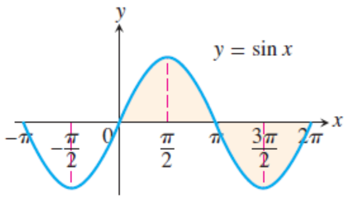
TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0



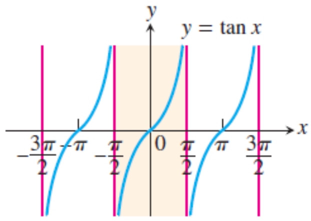
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



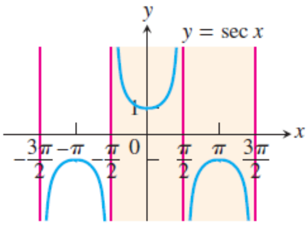
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



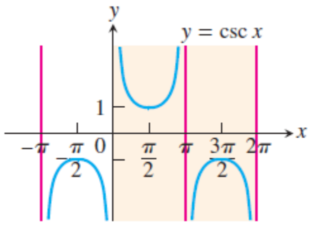
Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)



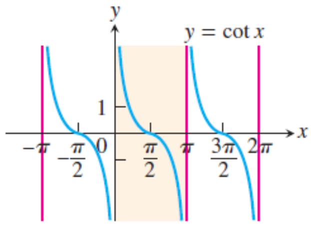
Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

(d)



Domain: $x \ne 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

(e)



Domain: $x \ne 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

Periods of Trigonometric Functions

Period π:
$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π:
$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Identities of Trigonometric Functions

$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \tag{2}$$

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \tag{3}$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad (4)$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (5)$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC and if θ is the angle opposite c, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \qquad (6)$$

Double Angle Formulas	Half Angle Formulas
$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= 2\cos^2\theta - 1$ $= 1 - 2\sin^2\theta$ $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$	$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$ $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$ $\tan\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

Limits and Continuity

OVERVIEW The concept of a limit is a central idea that distinguishes calculus from algebra and trigonometry. It is fundamental to finding the tangent to a curve or the velocity of an object.

In this chapter we develop the limit, first intuitively and then formally. We use limits to describe the way a function f varies. Some functions vary continuously; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump or vary erratically. The notion of limit gives a precise way to distinguish between these behaviors. The geometric application of using limits to define the tangent to a curve leads at once to the important concept of the derivative of a function. The derivative, which we investigate thoroughly in Chapter 3, quantifies the way a function's values change.

EXAMPLE 7 Finding Limits by Calculating $f(x_0)$

(a) $\lim_{x\to 2}(4) = 4$

(b) $\lim_{x\to -13}(4) = 4$

(c) $\lim_{x\to 3} x = 3$

(d) $\lim_{x\to 2}(5x - 3) = 10 - 3 = 7$

(e) $\lim_{x\to -2} \dfrac{3x+4}{x+5} = \dfrac{-6+4}{-2+5} = -\dfrac{2}{3}$

The Limit Laws

The next theorem tells how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 **Limit Laws**

If L, M, c and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $\lim_{x\to c}(f(x) + g(x)) = L + M$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $\lim_{x\to c}(f(x) - g(x)) = L - M$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x\to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 8 in Section 2.1) and the properties of limits to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim_{x\to -2}\sqrt{4x^2 - 3}$

Solution

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$ Sum and Difference Rules

$= c^3 + 4c^2 - 3$ Product and Multiple Rules

(b) $\lim_{x\to c}\frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)}$ Quotient Rule

$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5}$ Sum and Difference Rules

$= \frac{c^4 + c^2 - 1}{c^2 + 5}$ Power or Product Rule

(c) $\lim_{x\to -2}\sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2}(4x^2 - 3)}$ Power Rule with $r/s = 1/2$

$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3}$ Difference Rule

$= \sqrt{4(-2)^2 - 3}$ Product and Multiple Rules

$= \sqrt{16 - 3}$

$= \sqrt{13}$ ■

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

EXAMPLE 2 Limit of a Rational Function

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}=\frac{0}{6}=0$$

This result is similar to the second limit in Example 1 with $c = -1$, now done in one step. ■

EXAMPLE 3 Canceling a Common Factor

Evaluate

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x},\qquad \text{if } x\neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3.$$

EXAMPLE 4 Creating and Canceling a Common Factor

Evaluate

$$\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}.$$

Solution This is the limit we considered in Example 10 of the preceding section. We cannot substitute $x = 0$, and the numerator and denominator have no obvious common factors. We can create a common factor by multiplying both numerator and denominator by the expression $\sqrt{x^2+100}+10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

$$\begin{aligned}\frac{\sqrt{x^2+100}-10}{x^2}&=\frac{\sqrt{x^2+100}-10}{x^2}\cdot\frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10}\\&=\frac{x^2+100-100}{x^2\left(\sqrt{x^2+100}+10\right)}\\&=\frac{x^2}{x^2\left(\sqrt{x^2+100}+10\right)}&&\text{Common factor } x^2\\&=\frac{1}{\sqrt{x^2+100}+10}.&&\text{Cancel } x^2 \text{ for } x\neq 0\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}&=\lim_{x\to 0}\frac{1}{\sqrt{x^2+100}+10}\\&=\frac{1}{\sqrt{0^2+100}+10}&&\text{Denominator not 0 at } x=0\text{; substitute}\\&=\frac{1}{20}=0.05.\end{aligned}$$

Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.29 and confirm it algebraically using the Sandwich Theorem.

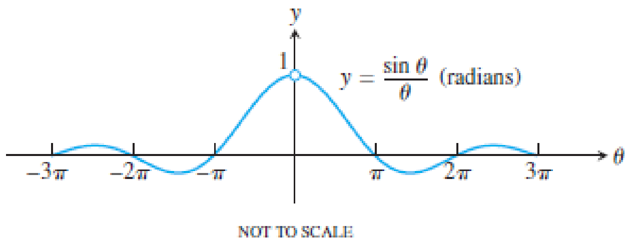


FIGURE 2.29 The graph of $f(\theta) = (\sin\theta)/\theta$.

THEOREM 7

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

EXAMPLE 5 Using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

Show that **(a)** $\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h - 1}{h} &= \lim_{h\to 0} -\frac{2\sin^2(h/2)}{h}\\ &= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta && \text{Let } \theta = h/2.\\ &= -(1)(0) = 0.\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} && \text{Now, Eq. (1) applies with } \theta = 2x.\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

■

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 7 Using Theorem 8

(a) $\lim_{x\to\infty} \left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x}$ Sum Rule

$= 5 + 0 = 5$ Known limits

(b) $\lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x}$

$= \lim_{x\to-\infty} \pi\sqrt{3} \cdot \lim_{x\to-\infty} \frac{1}{x} \cdot \lim_{x\to-\infty} \frac{1}{x}$ Product rule

$= \pi\sqrt{3} \cdot 0 \cdot 0 = 0$ Known limits ■

EXAMPLE 8 Numerator and Denominator of Same Degree

You Try It

$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$ Divide numerator and denominator by x^2.

$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$ See Fig. 2.33. ■

EXAMPLE 9 Degree of Numerator Less Than Degree of Denominator

$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$ Divide numerator and denominator by x^3.

$= \frac{0 + 0}{2 - 0} = 0$ See Fig. 2.34. ■