Remark (1.38):-

(1) Every linear space $L \neq \{0\}$ has a basis

(2) If $L = \{0\} \Longrightarrow dim\,(L) = 0$

(3) If L finite dimension linear space and S be a subspace of L, then $dim\, S \leq dim\, L$.

(4) If $dim\, S = \dim L \Longrightarrow S = L$

(5) If S $= \{\, x_1, \ldots, x_n\}$ be a linearly independent in L then there exists $c > 0$ such that $\|x\| = \| \sum_{i=1}^{n} \alpha_i x_1 \| \geq c \, \sum_{i=1}^{n} |\alpha_i|$.

(6) The dimension of quotient space (or factor space) is called codimension of M and denoted by $codm(M) = dim\,(\frac{L}{M})$.

The Convexity

Definition (1.39):- Let A be a subset of linear space L then , we say that A is **convex set** if satisfy the following condition :

$\forall x, y \in A\,, \lambda \in [0,1]$ then $\lambda x + (1-\lambda) y \in A$.

Examples About Convex Sets

Example(1.40):- If $A = (a, b) \subset \Longrightarrow A$ is convex set

Solution :

Let $x, y \in A\,, \lambda \in [0,1]$

$$x \in (a,b) \Rightarrow a < x < b \Rightarrow \lambda a < \lambda x < \lambda b \quad ...(1)$$

$y \in (a,b) \Rightarrow a < y < b \Rightarrow (1-\lambda)a < (1-\lambda)y < (1-\lambda)b \quad (2)$

By summing up (1) and (2)

$$\lambda a + (1-\lambda)a < \lambda x + (1-\lambda)y < \lambda b + (1-\lambda)b$$

$$a < \lambda x + (1-\lambda)y < b.$$

Example(1.41):- Every linear subspace is convex, but the converse is not true in general

Solution :

Let M be a subspace of linear space L $\Longrightarrow \alpha x + \beta y \in M$, $\forall x, y \in M$, $\alpha, \beta \in F$. Put $\alpha = \lambda, \beta = 1 - \lambda$, $0 \le \lambda \le 1$

$\Longrightarrow \lambda x + (1-\lambda)y \in M$, $\forall x, y \in M$, $0 \le \lambda \le 1$ $\Longrightarrow M$ is convex set

For the converse , consider the following example

Let $L = \mathbb{R}^2, M = \{(x,y) \in \mathbb{R}^2, x \ge 0 , y \ge 0\}$, then M is convex set but not subspace.

- ❖ To prove M is convex set.

Let $(x_1, y_1), (x_2, y_2) \in M$ and $0 \le \lambda \le 1$. To prove $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in M$.

$(x_1, y_1) \in M \Rightarrow x_1 \ge 0 , y_1 \ge 0$ and $(x_2, y_2) \in M \Rightarrow x_2 \ge 0 , y_2 \ge 0$.

Thus, $\lambda x_1 + (1-\lambda)y_1 \ge 0$ and $\lambda x_2 + (1-\lambda)y_2 \ge 0$.

Then , $\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \in M.$

- Show that M is not a subspace **(H.W.)**

Theorem(1.42): (Some Properties About Convex Sets)

1. The singleton set is convex set
2. The intersection of convex set is convex.
3. The empty set and the whole space are convex. **(H.W.)**
4. If A is convex set $\Longrightarrow$ αA also convex $\alpha \in F$. **(H.W.)**
5. If A and B are convex set $\Longrightarrow A + B$ also convex set.

Proof (1) Let $A = \{x\}$ to prove A is convex set.

Take $x \in A$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)x = x \in M$.

Proof (2) Let A and B are convex sets To prove $A \cap B$ is convex set.

Let $x, y \in A \cap B$ and $0 \leq \lambda \leq 1$ to prove $\lambda x + (1-\lambda)y \in A \cap B$.

$x, y \in A$ and A is convex $\Rightarrow$ $x + (1-\lambda)y \in A$ …. (1)

$x, y \in B$ and B is convex $\Rightarrow$ $x + (1-\lambda)y \in B$ …. (2)

From (1)&(2) we get $x + (1-\lambda)y \in A \cap B$. Then, $A \cap B$ is convex set.

Proof (5) Let $a_1 + b_1, a_2 + b_2 \in A + B$, then $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To prove $\lambda\,(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$, $\forall \lambda \in [0,1]$

Since A is convex set and $a_1, a_2 \in A \Rightarrow \lambda\, a_1 + (1-\lambda)a_2 \in A \;\; \forall \lambda \in [0,1]$…(1)

Since B is convex set and $b_1, b_2 \in B \Rightarrow \lambda\, b_1 + (1-\lambda)b_2 \in B \;\; \forall \lambda \in [0,1]$…(2)

By summing up (1) and (2) we get

$\lambda\, a_1 + (1-\lambda)a_2 + \lambda\, b_1 + (1-\lambda)b_2 \in A + B$

i.e., $\lambda\,(a_1 + b_1) + (1-\lambda)(a_2 + b_2) \in A + B$. Thus, $A + B$ is a convex set.

Linear operator and linear functional

Definition(1.43):- Let L and L' are linear spaces over the same field F . A mapping $T: L \to L'$ is called **Linear operator** or (**Linear transformation**) if

$$T(\alpha . x + \beta . y) = \alpha T(x) + \beta T(y)\,, \forall x, y \in L \text{ and } \alpha, \beta \in F$$

Note : The linear operator $T: L \to F$ is said to be **linear functional**.

Examples of Linear Functional

Example (1.44):- Let $T: \mathbb{R} \to \mathbb{R}$ such that $T(x) = mx\,, m \in \mathbb{R}$ the T is linear functional

Solution : Let $x_, y \in \mathbb{R}\,, \alpha, \beta \in \mathbb{R}$

$$T(\alpha x+\beta y)=m(\alpha x+\beta y)=m(\alpha x)+m(\beta y)$$
$$=\alpha(mx)+\beta(my)$$
$$=\ \alpha T(x)+\beta T(y)$$

Example(1.45):- Let $T:\mathbb{R}\to\mathbb{R}$ s.t. $T(x)=mx+b\ ,\ \ m,b\neq 0\in\mathbb{R}$ then T is not linear functional ?

Solution: Let $x\ _,\ y\in\mathbb{R}$

$$T(\alpha x+\beta y)=m(\alpha x+\beta y)+b=(m\alpha)x+(m\beta)y+b$$
$$=\alpha(mx)+\beta(my)+b\ \ldots..\qquad (1)$$

Now, $\alpha T(x)+\beta T(y)=\ \alpha(mx+b)+\beta(my+b)$

$$=\alpha mx+\beta my+(\alpha+\beta)b\ \ldots\ldots\quad (2)$$

Let $\alpha=1,\beta=2\ ,\ we\ get\ 1\neq 2$

Example(1.46):- Let $T: C[a,b]\to\mathbb{R}\ such\ that\ T\ (f)=\int_a^b f(x)dx$

Show that T is linear functional.

Solution : Let $f,g\in C[a,b]\ ,\alpha,\beta\in\mathbb{R}$. To prove that

$$T(\alpha f+\beta g)=\alpha T(f)+\beta T(g)$$

$$T(\alpha f+\beta g)=\int_a^b(\alpha f+\beta g)(x)dx$$
$$=\ \int_a^b\alpha f(x)\,dx+\ \int_a^b\beta g(x)dx$$
$$=\ \alpha\int_a^b f(x)dx+\ \beta\int_a^b g(x)dx=\ \alpha T(f)+\beta T(g).$$

Example(1.47):- Show that $T:\mathbb{R}^2\to\mathbb{R}^2$ such that

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2 + 1)$ is not linear operator

Solution : $T(0) = T((0,0)) = (0,1) \neq (0,0) \Longrightarrow T$ not linear functional.

Exercise

(1) If $T: \mathbb{R}^2 \to \mathbb{R}$ s.t $T(x_1, x_2) = {x^2}_1 + {x^2}_2$. Show that T is not linear functional.

(2) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x_1, x_2) = (x_1 + x_2, 0)$. Is T linear transformation ?

(3) If $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(x_1, x_2) = (\alpha x_1, x_2), \alpha \in \mathbb{R}$. Is T linear transformation ?

(4) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t $T\,(x_1, x_2, x_3) = (x_2, x_1 - x_2, 0)$. Is T linear transformation ?

Some Theorems About Linear operators

Theorem(1.48):- Let L, L', L'' are linear spaces over the same field F such that $T: L \to L'$ and g: $L' \to L''$ linear operators then

(1) $T(0_L) = 0_{L'}$ and $g(0_{L'}) = 0_{L''}$

(2) $T(x - y) = T(x) - T(y)$

(3) goT is linear operator .

Theorem(1.49):- Let L, L' are linear spaces over the same field F and $T_1, T_2: L \rightarrow L'$ are two linear operators. Define $+$ and $.$ as follows:

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall x \in L$$

$$(\lambda T_1)(x) = \lambda . T_1(x), \quad \forall x \in L, \ \lambda \in F$$

Then, $T_1 + T_2$ and αT_1 are linear operators

Proof :- Let $x, y \in L \quad and \ \alpha, \beta \in F$ then

(1) $(T_1 + T_2)(\alpha . x + \beta . y) = T_1(\alpha . x + \beta . y) + T_2(\alpha . x + \beta . y)$

$= \alpha . T_1(x) + \beta . T_1(y) + \alpha . T_2(x) + \beta . T_2(y)$

$= \alpha . (T_1(x) + T_2(x)) + \beta . (T_1(y) + T_2(y))$

$= \alpha . (T_1 + T_2)(x) + \beta . (T_2 + T_2)(y)$

(2) $\lambda T_1(\alpha . x + \beta . y) = \lambda . [\alpha . T_1(x) + \beta . T_1(y)]$

$= (\lambda\alpha) . T_1(x) + (\lambda\beta) . T_1(y)$

$= \alpha . \big(\lambda . T_1(x)\big) + \beta\big(\lambda . T_1(y)\big)$

$= \alpha . (\lambda T_1)(x) + \beta . (\lambda T_1)(y).$