$$\big(\alpha(f+g)\big)(x)=\alpha(f+g)(x)=\alpha(f(x)+g(x))$$

$=\alpha f(x)+\ \alpha g(x)=(\alpha f+\alpha g)(x)$

(d) $\forall f\in C^b[a,b],\alpha,\beta\in\mathbb{R}$, to prove $\quad(\alpha+\beta)f=\alpha f+\beta f$

$$\big((\alpha+\beta)f\big)(x)=(\alpha+\beta)f(x)=\alpha f(x)+\beta\ f(x)$$

$$=(\alpha f)(x)+(\beta f)(x)=(\alpha f+\beta f)(x)$$

(e) Let $f\in C^b[a,b]$ *and* 1 is the unity of $\mathbb{R}$, then $(1f)(x)=1\ f(x)=f(x)$.

General Properties of Linear Space (without prove)

Theorem(1.12):- Let $(L,+,.)$ be a linear space over F .Then

(1) $0.x=\ 0_L\ ,\quad\forall x\in L$

(2) $\lambda.0_L=\ 0_L\ ,\quad\lambda\in F$

(3) $(-\alpha.x)=(-\alpha).x=\alpha.(-x)\quad,\forall x\in L,\ \alpha\in F$

(4) If $x,y\in L\Longrightarrow\exists!\,z\in L$ such that $x+z=y$

(5) $\alpha.(x-y)=\ \alpha.x-\ \alpha.y\ ,\ \forall x,y\in L\ ,\ \alpha\in F$

(6) If $\alpha.x=0_L\Longrightarrow\ \alpha=0$ or $x=0_L$

(7) If $x\neq0_L$ *and* $\alpha_1x=\ \alpha_2x\ \Longrightarrow\ \alpha_1=\ \alpha_2$

(8) If $x\neq0_L\ ,\ \alpha\neq0\ ,y\neq0_L$ *and* $\alpha.x=\ \alpha.y\ \Longrightarrow x=y$

Linear subspace

Definition(1.13):- Let L be a linear space over F and $\emptyset\neq S\subseteq L$, then we say that S is **linear subspace** of L if S itself is a linear spase over F .

Theorem (1.14):- If L be a linear space over F and $\emptyset \neq S \subseteq L$, then S is linear subspace if satisfy the following conditions

(1) $x + y \in S\ , \forall x, y \in S$

(2) $\alpha . x \in S\ , \forall x \in S\ and\ \alpha \in F$

Or satisfy the equivalent condition of two conditions above ,

$\alpha . x + \beta . y \in S, \forall x, y \in S\ and\ \alpha, \beta \in F$

Remark(1.15):-

(1) A special subspace of L is improper subspace $S = L$

(2) Every other subspace of $L(\neq \{0\})$is called proper

(3) Another special subspace of any linear space L is S={0}

Example (1.16):- show that $S = \{(x, x_2) \in \mathbb{R}^2\ ;\ x_2 = 3x_1\}$ is subspace of $\mathbb{R}^2$?

Solution :- It is clear that $S \subseteq \mathbb{R}^2$, and $S \neq \emptyset$ because $(0,0) = 0 \in S$

To prove that $\alpha . x + \beta . y \in S\ , \forall \alpha, \beta \in F = \mathbb{R}\ , x\ , y \in \mathbb{R}^2$

$x = (x_1, x_2), y = (y_1, y_2)$

$\alpha . x + \beta . y = (\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)$

$\quad = (\alpha x_1 + \beta y_1\ , \alpha x_2 + \beta y_2)$

Now, $\alpha x_2 + \beta y_2 = \alpha(3x_1) + \beta(3y_1) = 3(\alpha x_1 + \beta y_1)$

$\Longrightarrow \alpha . x + \beta . y \in S$.

Example (1.17):- show that $S = \{\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} ; a, b \in \mathbb{R}\}$ is subspace of $M_{2\times2}(\mathbb{R})$?

Solution:- It is clear that $S \subseteq M_{2\times2}(\mathbb{R})$, and $S \neq \emptyset$ because $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$

$\forall x, y \in S \; and \; \alpha, \beta \in F.$

To prove that $\alpha . x + \beta . y = \alpha . \begin{pmatrix} 0 & a_1 \\ b_1 & 0 \end{pmatrix} + \beta . \begin{pmatrix} 0 & a_2 \\ b_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \alpha a_1 \\ \alpha b_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \beta a_2 \\ \beta b_2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & \alpha a_1 + \beta a_2 \\ \alpha b_1 + \beta b_2 & 0 \end{pmatrix} \in S$

Example (1.18):- show that $S = \{(x_1, x_2) \in \mathbb{R}^2 ; ax_1 + bx_2 = 0\}$ is subspace of $\mathbb{R}^2$? **(H.W.)**

Example (1.19):-The set $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, ; \, x_1 = 1 + x_2\}$ is not subspace of $\mathbb{R}^3$?

Solution:

Consider $\alpha = 2 \; and \; x = (2,1,0) \in S$, because $(2 = 1 + 1)$

$\alpha . x = 2.(2,1,0) = (4,2,0) \notin S$, because$(4 \neq 1 + 2)$

Hence, S is not subspace of $\mathbb{R}^3$.

Theorem(1.20):-Let S_1 and S_2 be two subspaces of linear space L . then

(1) $S_1 \cap S_2$ is subspace of linear space L .

(2) $S_1 + S_2$ is subspace of linear space L

(3) $S_1 \subseteq S_1 + S_2$, $S_2 \subseteq S_1 + S_2$. **(H.W.)**

Exercise :

(1) Which of the following subsets of $\mathbb{R}^3$ be a subspace of $\mathbb{R}^3$

a) $S_1 = \{x = (x_1, x_2, x_3);\ x_1 = x_2 \text{ and } x_3 = 0\}$

b) $S_2 = \{(x_1, x_2, x_3); x_3 = x_2 + 1\}$

c) $S_3 = \{(x_1, x_2, x_3);\ \ x_1, x_2, x_3\ \geq 0\}$

d) $S_4 = \{(x_1, x_2, x_3);\ x_1 - x_2 + x_3 = k\}$

(2) If S_1 and S_2 are subspaces of linear space L , then $S_1 \cup S_2$ not necessary subspace of L (Give example)

(3) If $S \neq \emptyset$ is any subset of L show that span S is subspace of L .

(4) Show that the Cartesian product $L = L_1 \times L_2$ of two linear spaces over the same field becomes a vector space , we define the two algebraic operations by

$$\left.\frac{(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)}{\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)}\right\} \ \frac{\left.\frac{\forall\ \ x = (x_1, x_2)}{y = (y_1, y_2)}\right\} \ \in L}{\alpha \in F}$$

(5) Let M be a subspace of a linear space L . The coset of an element $x \in L$ with respect to M is denoted by $x + M$ where

$x + M = \{z, z = x + m, m \in M\}$. Show that $(\frac{L}{M}, +, .)$ is linear space over under algebraic operations defined as