$= \ p \to q$ [def. of →]= L.H.S

<u>Theorem1.38</u>: (Properties of ↔)

Let p and q are two propositions. Prove the following properties without using truth tables:

1. $p \leftrightarrow p = T, \ p \leftrightarrow T = p, \ p \leftrightarrow F = \sim p$

2. $p \leftrightarrow \sim p = F$

3. $p \leftrightarrow q = q \leftrightarrow p$

4. $p \leftrightarrow q = \sim p \leftrightarrow \sim q$

5. $\sim p \leftrightarrow q = p \leftrightarrow \sim q$

6. $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

7. $\sim (p \leftrightarrow q) = p \leftrightarrow \sim q$

<u>Proof 1</u>: To prove $p \leftrightarrow T = p$

$p \leftrightarrow T = (p \to T) \wedge (T \to p)$ [def. of ↔]

$= (\sim p \vee T) \wedge (\sim T \vee p)$ [def. of →]

$= (\sim p \vee T) \wedge (F \vee p)$ $[\sim T = F]$

$= T \wedge p$ $[\sim p \vee T = T]$

$= p$

<u>Proof 6</u>: $\sim (p \leftrightarrow q) = \sim p \leftrightarrow q$

Take **L. H. S** : $\sim (p \leftrightarrow q) = \sim [(p \to q) \wedge (q \to p)]$ [def. of ↔]

$= \sim (p \to q) \vee \sim (q \to p)$ [De Morgan]

$= \sim (\sim p \vee q) \vee \sim (\sim q \vee p)$ [def. of →]

$= (p \land \sim q) \lor (q \land \sim p)$ [De Morgan]

$= [(p \land \sim q) \lor q] \land [(p \land \sim q) \lor \sim p)]$ [distributive (∨ on ∧)]

$= [(p \lor q) \land (\sim q \lor q)] \land [(p \lor \sim p) \land (\sim q \lor \sim p)]$ [dist. (∨ on ∧)]

$= [(p \lor q) \land T] \land [T \land (\sim q \lor \sim p)]$

$= (p \lor q) \land (\sim q \lor \sim p)$

$= (\sim p \to q) \land (q \to \sim p)$ [def. of →] $= \sim p \leftrightarrow q$ [def. of ↔]

<u>Mathematical Proof</u> البرهان الرياضي

A mathematical proof is a valid argument that establishes the truth of a mathematical statement.

البرهان الرياضي هو إثبات صحة عبارة رياضية من خلال حجة أو تعليل منطقي.

<u>Methods of Proving Mathematical Statements (or Theorems)</u>

1. <u>Direct Proof of a conditional statement $p \to q$</u>

Direct proofs lead from the hypothesis of a theorem to the conclusion.

<u>Definition1.39</u>: The integer number x is called **even** if there exist $k \in Z$ such that $x = 2k$.

<u>Definition1.40</u>: The integer number x is called **odd** if there exist $k \in Z$ such that $x = 2k + 1$.

<u>Theorem1.41</u>: If x is an odd natural number ($x \in$ O) then x^2 is odd

<u>Proof</u>: Assume that x is an odd natural number. We must prove x^2 is odd

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$x^2 = x.x = (2k+1)(2k+1) = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $s = 2k^2 + 2k \in N$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

<u>Theorem1.42</u>: (**H. W.**) If x is an even natural number ($x \in$ E) then x^2 is even.

<u>Theorem1.43</u>: The sum of two even natural numbers is even

<u>The theorem can be written as follows</u>: If $x, y \in E^+$ then $x + y \in E^+$ where $E^+ =$ set of positive even numbers.

<u>Proof</u>: Let p: $x\ and\ y$ are even positive numbers,

q: $x + y$ is an even positive number

Let $x = 2r$ and $y = 2s$ ($r, s \in N$). Then $x + y = 2r + 2s = 2(r + s)$ such that $r + s \in N$

$x + y = 2k$ where $k = r + s$. Therefore $x + y$ is a positive even number.

<u>Theorem1.44</u>: (H. W.)

i) If $x \in E$ and $y \in O$ then $x + y \in O$

ii) If $x \in E$ and $y \in O$ then $xy \in E$

iii) If $x, y \in E$ then $x + y \in E$

2. <u>Direct Proof of a biconditional statement $p \leftrightarrow q$</u>

To prove a proposition in the form $p \leftrightarrow q$, we prove its equivalence. i.e.,

$$\boldsymbol{p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)}$$

<u>Theorem1.45</u>: x is odd number $\leftrightarrow x + 1$ is an even number

<u>Proof</u>: Let p: x is odd number

q: $x + 1$ is an even number

من التعريف أعلاه يجب أن نبرهن بان **p → q** و **q → p**

1. **Prove $p \to q$:** Let $x \in O, x = 2k + 1\ ; k \in Z$

$x + 1 = 2k + 2 = 2(k + 1); \quad (k + 1) \in Z$

$x + 1 = 2r \ ; r = k + 1 \in Z$

$x + 1 \in E$

2. **Prove $q \to p$:** Let $x + 1 \in E$ To prove $x \in O$

$x + 1 = 2k \ ; k \in Z$

$x = 2k - 1; k \in Z$ ……(1)

Since $k \in Z,$ then $r = k - 1 \in Z$

$k = r + 1$……(2)

Substitute (2) in (1), $x = 2(r + 1) - 1 = 2r + 1; \quad r \in Z$

$x = 2r + 1 \in O$

Theorem1.46: x is even $\leftrightarrow$ x^2 is even

Proof: Let p: x is even number

q: x^2 is even number

من التعريف أعلاه يجب أن نبرهن بان $q \to p$ و $p \to q$

1. **Prove $p \to q$:** Let $x \in E, x = 2k\ ; k \in Z$

Prove $x^2 \in E$ (Theorem (1.42) مشابه لبرهان)

2. **Prove $q \to p$:** Let $x^2 \in E$ To prove $x \in E$

Take $x^2 + x = x(x + 1) \in E$ [from Theorem 1.44(ii)]

$\Longrightarrow x = x(x + 1) - x^2 \in E$ [Theorem 1.44(iii)]

$\Longrightarrow x \in E$

Theorem1.47: **(H. W.)** x is odd number if and only if x^2 is odd number.

2. **Proof by Contradiction**

البرهان بالتناقض هو أن نفرض عكس المطلوب إثباته ثم نحصل على تناقض مع الفرض

Theorem1.48: Prove that: If $x^2 \in O$ then $x \in O$

Proof: Assume that $x^2 \in O$. To prove $x \in O$

By contradiction, assume that $x \in E$

$x = 2k \,; k \in Z$

$x^2 = 4k^2 \in E$

تناقض مع الفرض لأنه في الفرض $x^2 \in O$

$\therefore x \in O.$

Theorem1.49: If x^2 is even then x is even

Proof: Assume that $x^2 \in E$. To prove $x \in E$

By contradiction, assume that $x \in O$

$x = 2k + 1 \,; k \in Z$

$x^2 = 4k^2 + 4k + 1 \in O$ تناقض مع الفرض

$\Longrightarrow x^2 \in E$. Hence, $x \in E$.

Theorem1.50: Prove that: If $n = ab$ where a and b are positive integers, then $a \leq \sqrt{n} \vee b \leq \sqrt{n}$.

Proof: Let p: $n = ab$ where a and b are positive integer **hypothesis**

q: $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ **conclusion**

The first step is to assume that the **conclusion** is false as follows:

Assume that $a \le \sqrt{n} \vee b \le \sqrt{n}$ is false (F). Hence, $\sim(a \le \sqrt{n} \vee b \le \sqrt{n})$ is true (T).

$\sim(a \le \sqrt{n} \vee b \le \sqrt{n}) = \sim(a \le \sqrt{n})$ and $\sim(b \le \sqrt{n})$ [De Morgan's law]

$= a > \sqrt{n}$ and $b > \sqrt{n}$

Multiply the two inequalities together, $ab > n$ هذه المتراجحة تناقض الفرض

This shows that $ab \ne n$ **contradiction with the hypothesis**

Thus, $a \le \sqrt{n} \vee b \le \sqrt{n}$ is true.

<u>Definition1.51</u>: Variable المتغير

An alphabetic letter $x,\ y, z, \ldots$ which represents a number that is either arbitrary or unknown.

<u>Example1.52</u>: ''$4x - 7 = 5$'': x is a variable

'' $\sqrt[3]{z} = 3$'': z is a variable

<u>Definition1.53</u>: Open Sentence الجملة المفتوحة

A sentence is called **open sentence** (or propositional function), if it contains <u>one or more variables</u>. Open sentence is denoted by $p(x)$, $q(x)$, $g(x)$…etc.

<u>Example1.54</u>: The following are open sentences:

$p(x)$: x is an odd number

$q(x, y)$: $x + y = 5$ such that $x, y \in N$

$r(z)$: $\sqrt[3]{z} = 3$ such that $z \in \mathbb{R}$

$s(y)$: computer y is working properly

<u>Example1.55</u>: Let the open sentence `` $\boldsymbol{p(x)}$: $x > 3$".

What are the truth value of $p(5)$ and $p(-1)$? Which values $x \in N$ that make $p(x)$ true?

<u>Solution</u>: $p(5)$: $5 > 3$ is a true proposition

$p(-1): -1 > 3$ is a false proposition

$p(x)$ is a true statement for $x \in \{4, 5, 6, \dots\}$.

Example1.56: Let the propositional function $\boldsymbol{q(x, y)}$: $x = y + 3$. What are the truth values of $q(1,2)$ and $q(3,0)$?

Solution: $q(1,2)$: $1 = 2 + 3$. This means $1 = 5$ which is false. Thus, $q(1,2)$ is false statement

$q(3,0)$: $3 = 0 + 3 = 3$. Hence $q(3,0)$ is true proposition

Example1.57: (H. W) Let the open sentence "$\boldsymbol{r(x, y, z)}$: $x + y = z$".

What are the truth values of $r(1,2,3)$ and $r(0,0,1)$?

Definition1.58: Solution Set (Truth set)

Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. In other words

$T_p = \{x \in A : p(x) \text{ is true}\}$

مجموعة الحل أو مجموعة الصدق: هي مجموعة العناصر التي تجعل التعبير المفتوح $p(x)$ عبارة صادقة.

Example1.59: Find the solution set for each of the following open sentences:

1) Let $p(x)$ be "$x + 2 > 7$" and $A = N$. Then

   $T_p = \{x \in N: x + 2 > 7\} = \{x \in N: x > 5\} = \{6,7, \dots\}$

   ماهي الأعداد الطبيعية الأكبر من ٥

2) Let $q(x)$ be "$x + 5 < 3$" and $A = N$. Then

   $T_q = \{x \in N: x + 5 < 3\} = \{x \in N: x < -2\} = \emptyset$

   ماهي الأعداد الطبيعية الاقل من2-.

3) Let $p(x)$ be ‘‘$x+5>1$’’ and $A=N$. Then

$T_p=\{x\in N: x+5>1\}=\{x\in N: x>-4\}=N$

ماهي الأعداد الطبيعية الأكبر من 4-

<u>Example1.60</u>: (H. W.) Find the following solution sets. Also determine $p(x)$ and A for each solution set

1) $T_P=\{x\in N: -2<x<2\}=\{1\}$

$p(x): \; -2<x<2$ $\qquad$ $A=N$

2) $T_P=\{x\in Z: -2<x<2\}$ **(H. W.)**

3) $T_P=\{x\in Z: -1<x<1\}$ **(H.W.)**

<u>Example1.61</u>: Assume we have the following statement:

‘‘$x > 2$ and $x < 5$’’

Which values of $x\in N$ that make the statement true? Which values of x that make the statement false? Discuss all the possible cases.

ماهي قيم x التي تجعل العبارة أعلاه صحيحة؟ وماهي القيم التي تجعلها خاطئة؟

<u>Solution</u>: For $A=\{3,4\}$, we have ‘‘$x > 2$ and $x < 5$’’ is **true** because the values in A are greater than 2 and less than 5.

For $A^c=N-A$ ={1,2,5, 6, 7, 8, …}, then ‘‘x > 2 and x < 5’’ is **false**

<u>Example1.62</u>: Assume we have the following statement:

‘‘$x \le -3 \text{ or } x \ge 6$’’