$(x+m)+(y+m)=(x+y)+m, \forall\, x+m, y+m \in \frac{L}{M}$

$\alpha.(x+m)=\alpha.x+m, \forall\, x+m \in \frac{L}{M}\ ,\ \alpha \in F$

Note : The space $(\frac{L}{M},+,.)$ is called quotient space (or factor space) .

Dimensional Linear Spaces

Definition (1.21):- A **linear combination** of vectors $x_1, x_2, \ldots, x_n$ of a linear space L is an expression of the from $\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are any scalars

i.e., x is linear combination of $x_1, x_2, \ldots, x_n$ if $\exists \alpha_1, \alpha_2, \ldots, \alpha_n$ s.t.

$x=\alpha_1 x_1+\alpha_2 x_2+\ \ldots+\alpha_n x_n$.

Example (1. 22):- Let $S=\{(1,2,3),(1,0,2)\}$, Express $x=(-1,2,-1)$, as a linear combination of x_1 and x_2 .

Solution: We must find scalars $\alpha_1, \alpha_2 \in F$ such that $x=\alpha_1.x_1+\alpha_2.x_2$

$(-1,2,-1)=\ \alpha_1.(1,2,3)+\alpha_2.(1,0,2)$

$=(\alpha_1, 2\alpha_1, 3\alpha_1)+(\alpha_2, 0, 2\alpha_2)$

So, $\alpha_1+\alpha_2=-1 \implies \alpha_2=-\alpha_1-1$

$$2\alpha_1 + 0 = 2 \Longrightarrow 2\alpha_1 = 2 \Longrightarrow \alpha_1 = 1$$

and, $3\alpha_1 + 2\alpha_2 = -1$

$\therefore\ \alpha_2 = -1 - 1 = -2$.

Example (1. 23):-If $S = \{(1,2,3), (1,0,2)\}$. Show that $x = (-1,2,0)$, is not linear combination of x_1, x_2 .

Solution:

Let $\alpha_1, \alpha_2 \in F$ and $x_1, x_2 \in S$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2$, we have

$$\left.\frac{\alpha_1 \quad + \alpha_2 = -1\ 2\alpha_1 + \ 0 = \ 2}{3\alpha_1 + 2\alpha_2 = 0}\right\}$$

$$\left(\frac{1 \qquad 1\ :-1}{2 \qquad 0\ :\ 2\,3 \qquad 2\ :\ 0}\right) \Longrightarrow \left(\frac{1 \quad 1\ :-1\,0 \quad -2\ :4}{0 \quad -1\ :3}\right)$$

The system has no solution

$\therefore x$ not linear combination of x_1, x_2.

Example (1.24):- Let $S = \{x_1,\ x_2,\ x_3\}$ where $x_1 = (1,2),\ x_2 = (0,1)\ and\ x_3 = (1,1)$. Express (1,0) as a linear combination of x_1, x_2 and x_3.

Solution:

We must find scalars $\alpha_1, \alpha_2, \alpha_3 \in F$ such that $x = \alpha_1 . x_1 + \alpha_2 . x_2 + \alpha_3 . x_3$

$$(1,0) = \alpha_1 . (1,2) + \alpha_2 . (0,1) + \alpha_3 . (1,1)$$

$$(1,0) = (\alpha_1, 2\alpha_1) + (0, \alpha_2) + (\alpha_3, \alpha_3)$$

$$\alpha_1 + \alpha_3 = 1 \Longrightarrow \alpha_1 = 1 - \alpha_3$$

$$2\alpha_1 + \alpha_2 + \alpha_3 = 0 \Longrightarrow 2(1-\alpha_3) + \alpha_2 + \alpha_3 = 0 \Longrightarrow -\alpha_3 + \alpha_2 = -2$$

$$\alpha_2 = -2 - \alpha_3$$

This system has multiple solutions in this case there are multiple possibilities for the α_i.

Definition (1.25):- Let $\emptyset \neq M \subseteq L$ the smallest subspace of L contains M is called **subspace generated** by M and denoted by [M] or span M.

Remark(1.26):-

1. Let $\emptyset \neq M \subseteq L$, the set of all linear combinations of vectors of M is called span of M .

2. $M \subset$span (M).

3. Span (M) = the intersection of all subspace of L containing M.

Example (1.27):- Find span $\{x_1, x_2\}$ where $x_1 = (1,2,3)$ and $x_2 = (1,0,2)$?

Solution :- The span $\{x_1, x_2\}$ is the set of all vectors $(x, y, z) \in \mathbb{R}^3$ such that

$$(x, y, z) = \alpha_1.(1,2,3) + \alpha_2.(1,0,2)$$

We wish to know for what values of (x, y, z) does this system of equations have solutions for α_1, α_2

$\alpha_1.(1,2,3) + \alpha_2.(1,0,2) = (x, y, z)$

$(\alpha_1, 2\alpha_1, 3\alpha_1) + (\alpha_2, 0{,}2\alpha_2) = (x, y, z)$

$\alpha_1 + \alpha_2 = x \Longrightarrow \alpha_2 = x - \alpha_1$

$2\alpha_1 = y \Longrightarrow \alpha_1 = \frac{1}{2}y$

$3\alpha_1 + 2\alpha_2 = z \Longrightarrow 6\alpha_1 + 4\alpha_2 - 2z = 0$

$6\left(\frac{1}{2}y\right) + 4\left(x - \frac{1}{2}y\right) - 2z = 0$

$3y + 4x - 2y - 2z = 0$

$4x + y - 2z = 0$

So, solutions when $4x + y - 2z = 0$

Thus span $\{x_1, x_2\}$ is the plane $4x + y - 2z = 0$

Example (1.28):-Show that $\{x_1, x_2\}$ span $\mathbb{R}^2$, when $x_1 = (1,1), x_2 = (2,1)$.

Solution : we being asked to show that any vectors in $\mathbb{R}^2$ can written as a linear combination of x_1, x_2. Let $(a, b) \in \mathbb{R}^2 and\ (a, b) = \alpha_1.(1,1) + \alpha_2.(2,1)$

$(\alpha_1, \alpha_1) + (2\alpha_2, \alpha_2) = (a, b)$

$\alpha_1 + 2\alpha_2 = a \Longrightarrow \alpha_1 = a - 2\alpha_2$

$\alpha_1 + \alpha_2 = b \Longrightarrow \alpha_2 = b - (a - 2\alpha_2)$

$-\alpha_2 = b - a \Longrightarrow \alpha_2 = a - b$

$\alpha_1 = a - 2(a-b) = 2b - a$ .Note that these two vectors span $\mathbb{R}^2$, that is every vector $\mathbb{R}^2$ can be expressed as a linear combination of them .

Example (1.29):-Show that $S = \{x_1, x_2, x_3\}\ span\ \mathbb{R}^2$, where $x_1 = (1,1), x_2 = (2,1), x_3 = (3,2)$. **(H.W.)**

Definition (1.30):- Let S = { $x_1, \dots, x_n$} be a subset of L , then S is called **linearly independent** if there exist $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

if $\alpha_1.x_1 + \alpha_2.x_2 + \cdots + \alpha_n.x_n = 0$ then $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

Definition (1.31):- Let S = { $x_1, x_2, \dots, x_n$} be a subset of L , then S is said to be **linearly dependent** if it is not linearly independent that is if

$\alpha_1.x_1 + \alpha_2.x_2 + \cdots + \alpha_n.x_n = 0$ but the $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zero .

Example (1.32):- Determine $S = \{x_1, x_2\}$ is linearly dependent or independent where $x_1 = (1,2,3)$, $x_2 = (1,0,2)$.

Solution : Let $\alpha_1, \alpha_2 \in F$

$\alpha_1(1,2,3) + \alpha_2(1,0,2) = (0,0,0)$, only solution is trivial solution $\alpha_1 = \alpha_2 = 0$. Thus, S is linearly independent.

Example (1.33):-Determine $S = \{x_1, x_2\}$ is linearly dependent or independent where $x_1 = (1,1,1), x_2 = (2,2,2)$?

Solution: Let $\alpha_1, \alpha_2 \in F$

$\alpha_1(1,1,1) + \alpha_2(2,2,2) = (0,0,0)$

$\alpha_1 + 2\alpha_2 = 0 \Longrightarrow \alpha_1 = 2\alpha_2$

So, S is linearly dependent

Theorem (1.34):- **(without prove)**

(1) Every m vectors set in $\mathbb{R}^n$, if $m > n$ then , the set is linearly dependent

(2) A linearly independent set in $\mathbb{R}^n$ has at most n vectors .

Remark (1.35):- Let L linear space over F , $S \subseteq L$ and $x_0 \in L$, then

(1) If $0_L \in S \Longrightarrow S$ is linear dependent . i.e., every subspace is linear dependent set

(2) If $x_0 \neq 0_2 \Longrightarrow \{x_0\}$ is linearly independent

Definition (1.36):- Let L be a linear space over F .A subset B of L is a **basis** if it is linearly independent and spans L i.e,

(1) B is linearly independent

(2) Span $(B) = L$ $\qquad$ B تولد الـ L

The number of elements in a basis for L is called the **dimension** of L and is denoted by dim (L)

Example(1.37):- Consider the linear space $(\mathbb{R}^3, +, ..)$

The dimension of L is 3. i.e., $dim\,(\mathbb{R}^3) = 3$

Since $B = \{(1,0,0), (0,1,0), (0,0,1)\}$ is basis for $\mathbb{R}^3$