Which values of x $\in N$ that make the statement true? Which values of x that make the statement false?

For $A = \{x \in N: x = 6, 7, \ldots\ldots\}$, the statement above is true
The statement is **false** for $A^c = N - A = \{1,2, \ldots ,5\}$

Quantifiers المسورات

Quantifiers are open sentences written in a special way.

المسورات هي جمل مفتوحة مكتوبة بطريقة معينة

There are two types of quantifiers:

1. **Universal quantifiers العبارة المسورة كلياً**
2. **Existential quantifiers العبارة المسورة جزئيا**

Universal quantifiers:

Let $p(x)$ be an open sentence on a set A. The notation

‘‘$\forall\ x \in A, p(x)$’’

Denote the **universal quantification تسوير كلي** of $p(x)$ and it reads as: ‘‘for all $x, p(x)$’’ or ‘‘for every $x, p(x)$’’ or ‘‘for each $x, p(x)$’’.

The symbol $\forall$ is called **universal quantifier** مسوراً كلياً.

The set A is called **domain المجال**

Example1.63: $\forall\ x \in N, x > 0$

All seasons in Iraq have rain

Remark1.64: 1. The universal quantifier $p(x)$ on a domain A is **true** if and only if $T_p = A$.

2. universal quantifier $p(x)$ on a domain A is **false** if and only if there exist $x \in A$ such that $p(x)$ is false.

<u>Example</u>1.65: Find the truth value of the following open sentences:

1. $\forall\, x \in \mathbb{R},\ x + 1 > x$

Let $A = \mathbb{R}\ and\ p(x): x + 1 > x$

Because $p(x)$ is true for all $x \in \mathbb{R}$, the solution set $T_p = \mathbb{R}$

$\Longrightarrow$ the quantification $\forall\, x \in R,\ x + 1 > x$ is **true.**

2. $\forall\, x \in N, x < 2$

Let $A = N\ and\ p(x): x < 2$

$p(x)$ is not true for all $x \in N$. Take $x = 3, p(3)$ is false.

$\Longrightarrow T_p \neq N$

3. $\forall\, x \in N, (x > 0\ and\ x = 0)$

The statement is **false**, there exists $x = 4 \in N$ such that $4 > 0\ and\ 4 \neq 0$.

4. $\forall\, x \in Z, |x| > 0$ (**H. W.)**
5. For all $x \in \{1, -1\}$, $x^2 - 1 = 0$ (**H. W.**)

<u>Existential quantifiers</u>:

Let $p(x)$ be an open sentence on a set A. The notation

‘‘$\exists\, x \in A, p(x)$’’

Denote the **existential quantification تسوير جزئي** of $p(x)$ and it read as: ‘‘there exists $x, p(x)$’’ or ‘‘there is $x, p(x)$’’ or ‘‘some $x, p(x)$’’.

The symbol $\exists$ is called **existential quantifier مسوراً جزئياً**.

The set A is called **domain المجال**

Example1.66: $\exists\ x\ \in N, x < 0$

There exist seasons in Iraq do not have rain

Remark1.67:

The existential quantifier $p(x)$ on a domain A is **true** if and only if $T_p \neq \emptyset$.

العبارة المسورة جزئياً تكون صادقة إذا وجد على الأقل عنصر واحد يحقق العبارة $p(x)$

The existential quantifier $p(x)$ on a domain A is **false** if and only if $T_p = \emptyset$.

العبارة المسورة جزئياً تكون كاذبة إذا لم يكن هناك عنصر في المجموعة A يحقق العبارة $p(x)$.

Example1.68: Find the truth value of the following open sentences:

1. $\exists x \in \mathbb{R}, x^2 = x$

A= R and p(x): $x^2 = x$

$T_p = \{0, 1\}$

$\Longrightarrow$ the existential quantifier $\exists x \in \mathbb{R}, x^2 = x$ is true

2. $\exists x \in N,\ 3x + 5 = 1$

$x = \frac{-4}{3} \notin \mathrm{N}$

$\Longrightarrow T_p = \emptyset$

$\Longrightarrow \exists x \in N,\ 3x + 5 = 1$ is false

3. $\exists x \in Z,\ [(x + 1)^2 = 0\ and\ x^2 - 1 = 0]$

$(x + 1)^2 = 0 \Longrightarrow x = -1$

And $x^2 - 1 = 0 \Longrightarrow x = -1, 1$

$T_p = \{-1\} \subset Z$

$\exists x \in Z,\ [(x+1)^2 = 0 \ and \ x^2 - 1 = 0]$ is true

De Morgan's law for the existential quantifier

$\sim [\exists\, x \in A, \sim p(x)] = \forall x \in A,\ p(x)$

قانون دي موركان للعلاقة بين التسوير الكلي والجزئي

Example1.69:

1. $\sim [\exists\, x \in E, x + 2 \notin E] = \forall x \in E, x + 2 \in E$

2. $\forall x \in N,\ \sqrt{3x} = \sqrt{3}\sqrt{x} = \sim[\exists x \in N, \sqrt{3x} \neq \sqrt{3}\sqrt{x}]$

Theorem1.70: Let p(x) be an open sentence and A is the domain. Then

1. $\sim[\forall x \in A,\ p(x)] = \exists\, x \in A, \sim p(x)$

2. $\sim[\forall x \in A,\ \sim p(x)] = \exists\, x \in A,\ p(x)$ (**H. W.**)

3. $\sim[\ \exists x \in A,\ p(x)] = \forall x \in A, \sim p(x)$ (**H. W.**)

Proof1: $\sim[\forall x \in A,\ p(x)] = \sim[\sim [\exists\, x \in A, \sim p(x)]]$ {from De Morgan}

$= \sim\sim[\exists\, x \in A, \sim p(x)]$

$= \exists\, x \in A, \sim p(x) \quad [\sim\sim p = p]$

Definition1.71: Nested Quantifiers المسورات المتداخلة

Two quantifiers are nested if one is within the area of the other.

في حالة وجود أكثر من متغير واحد في الجملة المفتوحة فان ذلك يستلزم وجود أكثر من مسور.

وهناك ثمانية طرق للتعبير عن المسورات المتداخلة وهي كالأتي:

Let $p(x, y)$ be an open sentence defined on the domain sets A and B. Then, the quantifiers can be expressed as follows:

1. $\forall x \in A,\ \forall y \in B,\ p(x,y)$

2. $\forall y \in B, \forall x \in A, p(x,y)$

3. $\exists x \in A,\ \exists y \in B,\ p(x,y)$

4. $\exists y \in B,\ \exists x \in A,\ p(x,y)$

5. $\forall x \in A,\ \exists y \in B,\ p(x,y)$

6. $\exists\ y \in B, \forall x \in A,\ p(x,y)$

7. $\exists\ x \in A, \forall y \in B,\ p(x,y)$

8. $\forall y \in B, \exists\ x \in A,\ p(x,y)$

Remark1.72: In the above definition, the quantifiers (1) and (2) are logically equivalent. i.e.,

$$\forall x \in A, \quad \forall y \in B, \quad p(x,y) = \forall y \in B, \quad \forall x \in A, \quad p(x,y)$$

Similarly, the quantifiers (3) and (4) are logically equivalent. i.e.,

$$\exists x \in A, \quad \exists y \in B, \quad p(x,y) = \exists y \in B, \quad \exists x \in A, \quad p(x,y)$$

Example1.73:

1. $\forall x \in \mathbb{R},\ \forall y \in N, x^2 + y^2 \ \geq 0\ (\boldsymbol{True}) = \ \forall y \in N,\ \forall x \in \mathbb{R},\ x^2 + y^2 \ \geq 0\ (\boldsymbol{True})$

2. $\exists x \in N, \exists y \in N,\ \ x + 2y < 0$ **(F)** $\equiv \exists y \in N, \exists x \in N,\ \ x + 2y \ < 0$

(F)

Remark1.74: In the above definition, the quantifiers (5) and (6) are not logically equivalent. i.e.,

$$\forall x \in A, \quad \exists y \in B, \quad p(x,y) \neq \ \exists y \in B, \quad \forall x \in A, \quad p(x,y)$$

Similarly, the quantifiers (7) and (8) are not logically equivalent. i.e.,

$$\exists x \in A, \quad \forall y \in B, \quad p(x,y) \neq \forall y \in B, \quad \exists x \in A, \quad p(x,y)$$

<u>Example1.75</u>:

$\exists x \in \mathbb{R},\ \forall y \in N, x+y=0$ **(False)**

المسورة أعلاه تعني بأنه يوجد عدد حقيقي x بحيث أن حاصل جمع x و y يساوي صفر لكل عدد طبيعي y.

$\forall y \in N,\ \exists x \in \mathbb{R},\ x+y=0$ **(True)**

العبارة تؤكد بأنه لكل عدد طبيعي y يوجد عدد حقيقي x بحيث $x+y=0$.

$\Longrightarrow \exists x \in \mathbb{R},\ \forall y \in N, x+y=0 \ \neq \forall y \in N,\ \exists x \in \mathbb{R},\ x+y=0$.

<u>Example1.76</u>: Let $x =$ computer, $y =$ student,

$p(x,y) =$ student uses the computer

Show that $\exists x,\ \forall y, p(x,y)\ \neq \forall y,\ \exists x, p(x,y)$

<u>Solution</u>:

$\exists x,\ \forall y,\ p(x,y)$

العبارة تعني بأنه يوجد كومبيوتر كل الطلاب تستخدمه

$\forall y, \quad \exists x, \quad p(x,y)$

العبارة تعني بأنه لكل طالب يوجد كومبيوتر يستخدمه

نلاحظ أن المعنى مختلف للعبارتين المسورتين

<u>De Morgan's laws for nested quantifiers</u>

Let x *and* y are two variables defined on the sets A and B, respectively and $p(x,y)$ an open sentence. Then:

1. $\sim[\forall\, x \in A, \forall\, y \in B,\ p(x,y)] = \exists x, \exists y,\ \sim p(x,y)$ **(H. W.)**
2. $\sim[\, \exists\, x \in A, \exists\, y \in B,\ p(x,y)] = \ \forall x, \forall\, y,\ \sim p(x,y)$

3. $\sim[\forall x \in A, \exists y \in B,\ p(x,y)] = \exists x, \forall y,\ \sim p(x,y)$ **(H. W.)**
4. $\sim[\exists x \in A, \forall y \in B,\ p(x,y)] = \forall x, \exists y,\ \sim p(x,y)$ **(H. W.)**

<u>Proof 2</u>:

Take the L. H. S

$\sim[\exists x \in A, \exists y \in B,\ p(x,y)] = \forall x \in A \sim [\exists y \in B,\ p(x,y)]$

$= \forall x \in A, \forall y \in B,\ \sim p(x,y)$

= R. H. S

<u>Example1.77</u>: Find the truth values of the following statements and of their negations:

1. $\forall x \in \mathbb{R}\ (x \neq 0), \exists y \in \mathbb{R},\ xy = 1$

The statement is **true** because $\forall x \in \mathbb{R}\ (x \neq 0),\ \exists y = \frac{1}{x} \in \mathbb{R},\ x\frac{1}{x} = 1$

Negation:

$\sim[\forall x \in \mathbb{R}\ (x \neq 0), \exists y \in \mathbb{R},\quad xy = 1]$

$= \exists x \in \mathbb{R}\ (x \neq 0),\ \forall y \in \mathbb{R},\ xy \neq 1$

The statement is **false**

Let $x = 2$ and $y = \frac{1}{2}$ then $xy = 1$

2. $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 \geq 0$ is true

يوجد عددين حقيقيين حاصل جمع مربعيهما عدد غير سالب

Negation:

$\sim[\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 \geq 0]$

$= \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 < 0$ is **false**

3. $\forall x \in N, \forall y \in N, x + y \in N$ **(H. W.)**

4. $\forall x \in N, \exists y \in Z, x + y \in N$ **(H. W.)**

Exercise1.78:

1. Express the following using connective operators and/or quantifiers

عبر عما يلي باستخدام ادوات الربط او المسورات

i) there exists p, and there exist q such that $pq = 32$

ii) for each x, there exists y such that $x < y$

iii) each even number is not odd number

iv) for each x, if x is natural number then x is an integer number

v) for each natural number x, x is even number or x is odd number

2. Find the negation of the following sentences:

i) $\forall x, \forall y, \exists z, x + y + z = 18$

ii) there exists y such for each $x, xy \leq 2$

iii) $\exists x, [p(x) \rightarrow q(x)]$