Exercise 2.11.

(1) Let $L = C^2$ be a linear space over $F = C$. Define $\| \| : C^2 \to \mathbb{R}$ such that $\|x\| = a|x_1| + b|x_2|$, $\forall x = (x_1, x_2) \in C^2$ and $a, b > 0$. Show that $\| \|$ is a norm on C^2. (**H.W.**)

(2) Consider the linear space $\mathbb{R}^2$. Let $\|x\| = \min\{|x_1|, |x_2|\}$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$. Show that $\| \|$ is not a norm on $\mathbb{R}^2$.

solution: Let $x = (0, -3) \in \mathbb{R}^2$

$\|x\| = \min\{|0|, |-3|\} = \min\{0, 3\} = 0$

Since $X \neq \mathbf{0}_{\mathbb{R}^2}$, but $\|x\| = 0$. Condition (2) of the definition of the norm is not valid. Hence, $\| \|$ is not a norm on $\mathbb{R}^2$.

(3) Consider the linear space $\mathbb{R}^2$. Let $\|x\| = |x_1|^2 + |x_2|^2$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$. Show that $\| \|$ does not satisfies condition (4).

solution: Let $x = (1, 3), \alpha = 2$

$|\alpha| \|x\| = 2(|x_1|^2 + |x_2|^2) = 2(1^2 + 3^2) = 20$

$\|\alpha x\| = \|2(1, 3)\| = \|(2, 6)\| = 2^2 + 6^2 = 40$

Thus, $|\alpha| \|x\| = 20 \neq \|\alpha x\| = 40$.

Some Important Inequalities

To give more examples about normed space, it is important to present some inequalities.

If $l^p = \{x = (x_1, x_2, ...) : x_i \in \mathbb{R}(or\ C)$ and $\sum_{i=1}^{\infty} |x_i|^p < \infty\}$ be a set of sequence space (see Example 1.6). Let $x = (x_1, x_2, ...) \in l^p$, $y = (y_1, y_2, ...) \in l^q$. Then

(1) Holder's Inequality

$$\sum_{i=1}^{\infty}|x_iy_i| \leq \Big[\sum_{i=1}^{\infty}|x_i|^p\Big]^{\frac{1}{p}}\Big[\sum_{i=1}^{\infty}|y_i|^q\Big]^{\frac{1}{q}},$$

where $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(2) Cauchy Schwarz's Inequality

$$\sum_{i=1}^{\infty}|x_iy_i| \leq \Big[\sum_{i=1}^{\infty}|x_i|^2\Big]^{\frac{1}{2}}\Big[\sum_{i=1}^{\infty}|y_i|^2\Big]^{\frac{1}{2}},$$

Note that Cauchy Schwarz's inequality is a special case of Holder's inequality where $p = q = 2$.

(3) Minkowski's Inequality

If $p \geq 1$

$$\Big[\sum_{i=1}^{\infty}|x_i + y_i|^p\Big]^{\frac{1}{p}} \leq \Big[\sum_{i=1}^{\infty}|x_i|^p\Big]^{\frac{1}{p}} + \Big[\sum_{i=1}^{\infty}|y_i|^p\Big]^{\frac{1}{p}},$$

Remark 2.12.

The three inequalities above hold for the linear spaces $L = \mathbb{R}^n$ and $L = C^n$.

Example 2.13.

Let $L = \mathbb{R}^2$ be a linear space over $\mathbb{R}$. If $x = (-1, 2), y = (0, 5) \in \mathbb{R}^2$.

(1) Verify Cauchy Shwarz inequality ($p = q = 2$).

(2) Verify Minkowski's inequality ($p = 3$).

Now we can give the following examples

Example 2.14.

(1) Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}}$ $\forall x \in \mathbb{R}^n$ or $C^n, x = (x_1, ..., x_n)$.

(2) Show that the linear space $\mathbb{R}^n$ over $\mathbb{R}$ (or C^n over C) is a normed space with $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$ $\forall x \in \mathbb{R}^n$ or $C^n, x = (x_1, ..., x_n)$ and $1 \leq p < +\infty$. (**H.W.**)

(3) Show that $(l^p, \| \|_p)$ is a normed space where $\|x\|_p = \left[\sum_{i=1}^{+\infty} |x_i|^p\right]^{\frac{1}{p}}$ $\forall x = (x_1, x_2, ...) \in l^p$ and $1 \leq p < +\infty$.

Solution (1): Let $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$ (or C^n) and $\alpha \in \mathbb{R}$ (or C).

(1) Since $|x_i| \geq 0, \quad \forall i = 1, ..., n$. Then, $\left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \geq 0$; that is $\|x\|_2 \geq 0$.

(2) $\|x\|_2 = 0 \iff \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0$

$$\iff |x_i|^2 = 0, \;\; \forall i = 1, ..., n$$

$$\iff x_i = 0, \;\; \forall i = 1, ..., n$$

$$\iff x = (x_1, ..., x_n) = \mathbf{0}_{\mathbb{R}^n}$$

(3) $\|x + y\|_2 = \|(x_1 + y_1, ..., x_n + y_n)\|_2$

$= \left[\sum_{i=1}^{n} |x_i + y_i|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$ (Minkowski's Inequality)

$$= \|x\|_2 + \|y\|_2$$

(4) $\|\alpha x\|_2 = \|(\alpha x_1, ..., \alpha x_n)\|_2 = \left[\sum_{i=1}^{n} |\alpha x_i|^2\right]^{\frac{1}{2}}$

$$= \left[\sum_{i=1}^{n} |\alpha|^2 |x_i|^2\right]^{\frac{1}{2}}$$

$$= |\alpha| \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = |\alpha| \, \|x\|_2.$$

Solution (3): Let $x = (x_1, x_2, ...), y = (y_1, y_2, ...) \in l^p$ and $\alpha \in \mathbb{R}$ (or C).

(1) Since $|x_i| \geq 0, \ \forall i \in N$. Then, $\left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} \geq 0$; that is $\|x\|_p \geq 0$.

(2) $\|x\|_p = 0 \iff \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} = 0 \iff \sum_{i=1}^{\infty} |x_i|^p = 0$

$$\iff |x_i|^2 = 0, \ \forall i \in N$$

$$\iff x_i = 0, \ \forall i \in N$$

$$\iff x = (0, 0, ...)$$

(3) $\|x + y\|_p = \|(x_1 + y_1, ..., x_n + y_n)\|_p$

$= \left[\sum_{i=1}^{\infty} |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{\infty} |y_i|^p\right]^{\frac{1}{p}}$ (Minkowski's Inequality)

$$= \|x\|_p + \|y\|_p$$

(4) $\|\alpha x\|_p = \|(\alpha x_1, ...)\|_p = \left[\sum_{i=1}^{\infty} |\alpha x_i|^p\right]^{\frac{1}{p}}$

$$= \left[\sum_{i=1}^{\infty} |\alpha|^p |x_i|^p\right]^{\frac{1}{p}}$$

$$= |\alpha| \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{\frac{1}{p}} = |\alpha| \, \|x\|_p$$

As an application to Example 2.14(1):

(1) Let $(\mathbb{R}^3, \| \, \|_2)$ be a normed space and $x = (x_1, x_2, x_3) = (1, -2, 4)$. Then, find $\|x\|_2$.

(2) Let $(C^2, \| \, \|_2)$ be a normed space and $x = (x_1, x_2) = (1 + i, -2i)$. Then, find $\|x\|_2$.

2.2 Product of Normed Spaces

Definition 2.15.
Let $(L, \| \|_L), (L', \| \|_L{}')$ be normed linear spaces over a field F. Let

$L \times L' = \{(x, y) : x \in L, y \in L'\}$ be the Cartesian product of L and L'.

Define $+$ on $L \times L'$ by

$(x_1, y_1) + (x_2, y_2) = (\underbrace{x_1 + x_2}_{\text{sum on L}}, \underbrace{y_1 + y_2}_{\text{sum on } L'}), \ \ \forall (x_1, y_1) + (x_2, y_2) \in L \times L'$.

Define a scalar multiplication

$\alpha.(x, y) = (\alpha x, \alpha y), \ \ \forall (x, y) \in L \times L', \forall \alpha \in F$.

Proposition 2.16.

Show that $(L \times L', +, .)$ is a linear space over F. (**H. W.**)

Remark 2.17.

The product linear space defined above can be made a normed space by different ways as we show in the following example.

Example 2.18.

Define $\| \| : L \times L' \to \mathbb{R}$ such that

(1) $\|(x, y)\|_1 = \|x\|_L + \|y\|_{L'}$

(2) $\|(x, y)\|_2 = \max\{\|x\|_L, \|y\|_{L'}\}$

(3) $\|(x, y)\|_3 = \min\{\|x\|_L, \|y\|_{L'}\}$ (**H. W.**)

Show that $(L \times L', \| \|_1), (L \times L', \| \|_2)$ are normed spaces.

Is $(L \times L', \| \|_3)$ is normed space?

Solution (1): To show $(L \times L', \| \|_1)$ is a normed space,

(i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0 \ \ \forall x \in L, \forall y \in L'$, then

$$\|x\|_L + \|y\|_{L'} = \|(x,y)\|_1 \geq 0.$$

(ii) $\|(x,y)\|_1 = 0 \iff \|x\|_L + \|y\|_{L'} = 0$

$$\iff \|x\|_L = \|y\|_{L'} = 0$$

$$\iff x = y = 0 \quad ((L, \|\ \|_L), (L', \|\ \|_{L'}) \text{ are normed spaces})$$

$$\iff (x,y) = (0,0)$$

(iii) For each $(x_1,y_1),(x_2,y_2) \in L \times L'$

$$\begin{aligned}\|(x_1,y_1)+(x_2,y_2)\|_1 &= \|(x_1+x_2, y_1+y_2)\|_1 \\ &= \|x_1+x_2\|_L + \|y_1+y_2\|_{L'} \\ &\leq \|x_1\|_L + \|x_2\|_L + \|y_1\|_{L'} + \|y_2\|_{L'} \\ &= (\|x_1\|_L + \|y_1\|_{L'}) + (\|x_2\|_L + \|y_2\|_{L'}) \\ &= \|(x_1,y_1)\|_1 + \|(x_2,y_2)\|_1\end{aligned}$$

(iv) For each $(x,y) \in L \times L'$ and for each $\alpha \in F$

$$\|\alpha(x,y)\|_1 = \|(\alpha x, \alpha y)\|_1 = \|\alpha x\|_L + \|\alpha y\|_{L'}$$

$$= |\alpha|\,\|x\|_L + |\alpha|\,\|y\|_{L'} = |\alpha|\,(\|x\|_L + \|y\|_{L'}) = |\alpha|\,\|(x,y)\|_1$$

Solution (2): Now, we show that $\|(x,y)\|_2 = \max\{\|x\|_L, \|y\|_{L'}\}$ is a norm on $L \times L'$

(i) Since $\|x\|_L \geq 0$ and $\|y\|_{L'} \geq 0 \ \ \forall x \in L, \forall y \in L'$, then

$\max\{\|x\|_L, \|y\|_{L'}\} = \|(x,y)\|_2 \geq 0.$

(ii) $\|(x,y)\|_2 = 0 \iff \max\{\|x\|_L, \|y\|_{L'}\} = 0$

$$\iff \|x\|_L = \|y\|_{L'} = 0$$

$$\iff x = y = 0 \quad ((L, \|\ \|_L), (L', \|\ \|_{L'}) \text{ are normed spaces})$$

$$\iff (x, y) = (0, 0)$$

(iii) For each $(x_1, y_1), (x_2, y_2) \in L \times L'$

$$\begin{aligned}
\|(x_1, y_1) + (x_2, y_2)\|_2 &= \|(x_1 + x_2, y_1 + y_2)\|_2 \\
&= \max\{\|x_1 + x_2\|_L, \|y_1 + y_2\|_{L'}\} \\
&\le \max\{\|x_1\|_L + \|x_2\|_L, \|y_1\|_{L'} + \|y_2\|_{L'}\} \\
&\le \max\{\|x_1\|_L, \|y_1\|_{L'}\} + \max\{\|x_2\|_L, \|y_2\|_{L'}\} \\
&= \|(x_1, y_1)\|_2 + \|(x_2, y_2)\|_2
\end{aligned}$$

(iv) For each $(x, y) \in L \times L'$ and for each $\alpha \in F$

$$\begin{aligned}
\|\alpha(x, y)\|_2 = \|(\alpha x, \alpha y)\|_2 &= \max\{\|\alpha x\|_L, \|\alpha y\|_{L'}\} \\
&= \max\{|\alpha| \|x\|_L, |\alpha| \|y\|_{L'}\} \\
&= |\alpha| \max\{\|x\|_L, \|y\|_{L'}\} = |\alpha| \|(x, y)\|_2
\end{aligned}$$

As an application to Example 2.18: Let $L = (\mathbb{R}, |\ |)$ and $L' = (\mathbb{R}^2, \|\ \|_2)$ where $\|x\|_2 = \big[\sum_{i=1}^{2} |x_i|^2\big]^{\frac{1}{2}}$. If $x = 3 \in L = \mathbb{R}$ and $y = (1, -2) \in L' = \mathbb{R}^2$. Find $\|(x, y)\|_1$ and $\|(x, y)\|_2$

Solution: $\|(x, y)\|_1 = \|(3, (1, -2))\|_1 = \|3\|_{\mathbb{R}} + \|(1, -2)\|_{\mathbb{R}^2}$

$$= |3| + \big[\sum_{i=1}^{2} |y_i|^2\big]^{\frac{1}{2}}$$

$$= 3 + \big[|1|^2 + |-2|^2\big]^{\frac{1}{2}} = 3 + \sqrt{5}.$$

Find $\|(x, y)\|_2$ (**H.W.**)