

**Example 1.26:** Let  $p$ : you can take the flight (True)

$q$ : you can buy a ticket (True)

Then the bi-conditional statement  $p \leftrightarrow q$  is

“you can take the flight if and only if you can buy a ticket”

Discuss the truth values of the bi-conditional statement.

**Solution:** The statement  $p \leftrightarrow q$  is **true**

“you buy a ticket  $\leftrightarrow$  can take the flight”

or

“you do not buy a ticket  $\leftrightarrow$  cannot take the flight”.

The statement  $p \leftrightarrow q$  is **false** when  $p$  and  $q$  have opposite truth values.

“you do not buy a ticket  $\leftrightarrow$  you can take the flight”

or

“you buy a ticket  $\leftrightarrow$  cannot take the flight”.

### **Properties of the biconditional operator** خواص أداة الشرط المزدوج

Let  $p, q$  and  $r$  are three propositions. Using the **truth table** show that: (H. W)

1.  $p \leftrightarrow q = q \leftrightarrow p$
2.  $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
3. **Find** the truth value of:  $p \leftrightarrow T, p \leftrightarrow F, p \leftrightarrow \sim p, p \leftrightarrow p$ .

### **Exercise 1.27:**

1. Find the truth value of the following statements:

[(if  $2 + 3 = 4$  then  $x + 4 = 4 + x$ ) and 8 is an even number] iff ( $2 \leq -10$  or  $2 \geq -10$ ).

**Solution:**  $[(F \rightarrow T) \wedge T] \leftrightarrow (F \vee T) = [T \wedge T] \leftrightarrow T = T \leftrightarrow T = T$

2. Let  $p$ : horse can swim

$q$ : Conjunction operator is useful

$$r: \sqrt{x+y} = \sqrt{x} + \sqrt{y} \text{ for } x, y \in N$$

Find the truth value of the following statements:

1.  $p \leftrightarrow r$
2.  $(p \rightarrow r) \wedge q$
3.  $[(p \rightarrow r) \vee (q \rightarrow \sim p)]$

3. Write the truth table of the following statements:

- i)  $\sim p \wedge q$
- ii)  $(p \wedge q) \rightarrow (p \vee q)$
- iii)  $(p \rightarrow q) \vee \sim (q \leftrightarrow p)$

4. Write the following statements using the connections operators  $\rightarrow, \leftrightarrow, \wedge, \vee$

- i) If  $p$  and  $q$  integer numbers and  $q \neq 0$  then  $\frac{p}{q}$  is a rational number
- ii) If  $x^2$  is integer number then  $x$  is even or odd number
- iii)  $xy > 0$  if and only if ( $x > 0$  and  $y > 0$ ) or ( $x < 0$  and  $y < 0$ )

**Definition 1.28:** A compound proposition that is always true is called a *tautology* or *lemma* or *theorem*.

يقال للعبارة المركبة التي تكون صادقة دائما بأنها نظرية أو تحصيل حاصل

A compound proposition that is always false is called a *contradiction*.

يقال للعبارة المركبة والتي تكون خاطئة دائما بأنها تناقض.

**Example1.29:** Show that  $(p \vee \sim p)$  is tautology and  $(p \wedge \sim p)$  is contradiction.

**Solution:**

$p$	$\sim p$	$p \vee \sim p$ tautology	$p \wedge \sim p$ contradiction
T	F	T	F
F	T	T	F

**Example1.30: (H. W)** which of the following compound statements is theorem (tautology) and which one is contradiction

$$p \wedge F, \quad p \vee T, \quad p \leftrightarrow \sim p, \quad [(p \rightarrow q) \wedge p] \wedge \sim q$$

**Definition1.31: Logical Equivalence** التكافؤ المنطقي

Two statements (propositions) that have **same** truth values are called **logically equivalent**. The notation  $p \equiv q$  or  $p = q$  denotes that p and q are logically equivalent.

تكون عبارتين متكافئة منطقيا إذا كان لهما نفس قيمة الصدق ويرمز للتكافؤ المنطقي بالرمز  $\equiv$  أو  $=$

**Example1.32:** show that  $\sim(p \vee q) = \sim p \wedge \sim q$  (logically equivalent).

**Hint:** make truth table

**Solution:** The truth table for  $\sim(p \vee q)$  and  $\sim p \wedge \sim q$  is

$p$	$q$	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F

T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

**Definition 1.33:** Let  $p, q$  and  $r$  three propositions, then define the following logical equivalence:

1.  $p \wedge q = \sim(\sim p \vee \sim q)$
2.  $p \rightarrow q = \sim p \vee q$
3.  $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

ملاحظة: التعريف أعلاه مهم جدا ويجب أن يحفظ

**De Morgan's Theorem:** Let  $p$  and  $q$  are two propositions. Then

1.  $\sim(p \wedge q) = \sim p \vee \sim q$  (H. W)
2.  $\sim(p \vee q) = \sim p \wedge \sim q$



**Proof (2):** Take the **right-hand side (R. H. S)**

$$\begin{aligned} \sim p \wedge \sim q &= \sim(\sim\sim p \vee \sim\sim q) \quad [\text{definition of } \wedge] \\ &= \sim(p \vee q) \quad [\text{double negation law: } \sim\sim p = p] \\ &= \text{Left hand side (L. H. S).} \end{aligned}$$

**Exercise 1.34:** Simplify the following statements:

1.  $\sim(p \vee \sim q)$

$$2. \sim(\sim p \rightarrow q)$$

$$3. \sim(\sim p \leftrightarrow q)$$

**Solution (1):**  $\sim(p \vee \sim q) = \sim p \wedge \sim\sim q$  [De Morgan's law]

$$= \sim p \wedge q \quad [\sim\sim q = q]$$

**Solution (2):**  $\sim(\sim p \rightarrow q) = \sim(\sim\sim p \vee q)$

$$= \sim(p \vee q) \quad [\sim\sim p = p]$$

$$= \sim p \wedge \sim q \quad [\text{De Morgan's law}]$$

### **Laws of Logical Equivalence** قوانين التطابق المنطقي

Let  $p, q$  and  $r$  are propositions. The following are some of the common logical equivalence rules:

**1. Commutative Law** قانون الإبدال:  $p \wedge q = q \wedge p$

$$p \vee q = q \vee p$$

$$p \leftrightarrow q = q \leftrightarrow p$$

**2. Associative Law** قانون التجميع:  $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$$(p \vee q) \vee r = p \vee (q \vee r)$$

$$(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$$

**3. Distributive Law (from left)** قانون التوزيع من اليسار:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

$$p \vee (q \rightarrow r) = (p \vee q) \rightarrow (p \vee r)$$

$$p \vee (q \leftrightarrow r) = (p \vee q) \leftrightarrow (p \vee r)$$

**4. Distributive Law (from right)** قانون التوزيع من اليمين:

$$(q \vee r) \wedge p = (q \wedge p) \vee (r \wedge p)$$

$$(q \wedge r) \wedge p = (q \wedge p) \wedge (r \wedge p)$$

$$(q \wedge r) \vee p = (q \vee p) \wedge (r \vee p)$$

$$(q \vee r) \vee p = (q \vee p) \vee (r \vee p)$$

$$(q \rightarrow r) \vee p = (q \vee p) \rightarrow (r \vee p)$$

$$(q \leftrightarrow r) \vee p = (q \vee p) \leftrightarrow (r \vee p)$$

**5. Idempotent Law** قانون تساوي القوى :  $p \wedge p = p$ ;  $p \vee p = p$

**6. Identity Law:**  $p \wedge T = p$ ;  $p \vee F = p$

**7. Domination Law:**  $p \wedge F = F$ ;  $p \vee T = T$

**Exercise1.35:** Simplify the following statements using laws of logical equivalence:

بسبب العبارات التالية باستخدام قوانين التطابق المنطقي

1.  $(p \vee q) \wedge \sim p$

2.  $(p \vee q) \vee (\sim p \wedge q)$

**Exercise1.36:** Prove (without using the truth table) that

برهن بدون استخدام جداول الصدق

$$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim q$$

**Solution:** Take the **L. H. S**

$$\begin{aligned} \sim (p \vee (\sim p \wedge q)) &= \sim p \wedge \sim(\sim p \wedge q) \text{ [ De Morgan's law]} \\ &= \sim p \wedge (\sim\sim p \vee \sim q) \text{ [ De Morgan's law]} \\ &= \sim p \wedge (p \vee \sim q) \text{ [by double negation law]} \\ &= (\sim p \wedge p) \vee (\sim p \wedge \sim q) \text{ [by distributive law]} \\ &= F \vee (\sim p \wedge \sim q) \text{ [}\sim p \wedge p = F\text{]} \\ &= (\sim p \wedge \sim q) \vee F \text{ [by commutative law]} \\ &= \sim p \wedge \sim q \text{ **R. H. S**} \end{aligned}$$

**Theorem1.37:** (Properties of  $\rightarrow$ )

Let  $p, q$  and  $r$  are three propositions. Prove the following properties **without using truth tables**:

1.  $p \rightarrow p = T$
2.  $\sim p \rightarrow p = p$
3.  $p \rightarrow T = T$
4.  $T \rightarrow p = p$
5.  $p \rightarrow F = \sim p$
6.  $F \rightarrow p = T$
7.  $p \rightarrow q = \sim q \rightarrow \sim p$
8.  $p \rightarrow q = (p \wedge \sim q) \rightarrow \sim p$

$$9. p \rightarrow q = (p \wedge \sim q) \rightarrow (r \wedge \sim r)$$

$$10. \sim(p \rightarrow q) = p \wedge \sim q$$

**Proof 1:** To prove  $p \rightarrow p = T$

$$\begin{aligned} p \rightarrow p &= \sim p \vee p \quad [\text{def. of } \rightarrow] \\ &= T \end{aligned}$$

**Proof 4:** To prove  $T \rightarrow p = p$

$$\begin{aligned} T \rightarrow p &= \sim T \vee p \quad [\text{def. of } \rightarrow] \\ &= F \vee p \quad [\sim T = F] \\ &= p \end{aligned}$$

**Proof 7:** To prove  $p \rightarrow q = \sim q \rightarrow \sim p$

$$\begin{aligned} p \rightarrow q &= \sim p \vee q \quad [\text{def. of } \rightarrow] \\ &= q \vee \sim p \quad [\vee \text{ is commutative}] \\ &= \sim q \rightarrow \sim p \end{aligned}$$

**Proof 8:** To prove  $p \rightarrow q = (p \wedge \sim q) \rightarrow \sim p$

$$\begin{aligned} \text{Take the R. H. S: } &(p \wedge \sim q) \rightarrow \sim p \\ &= \sim(p \wedge \sim q) \vee \sim p \quad [\text{def. of } \rightarrow] \\ &= (\sim p \vee \sim \sim q) \vee \sim p \quad [\text{De Morgan}] \\ &= (\sim p \vee q) \vee \sim p \quad [\sim \sim q = q] \\ &= \sim p \vee (q \vee \sim p) \quad [\vee \text{ is associative}] \\ &= \sim p \vee (\sim p \vee q) \quad [\vee \text{ is comm.}] \\ &= (\sim p \vee \sim p) \vee q \quad [\vee \text{ is asso.}] \\ &= \sim p \vee q \quad [p \vee p = p] \end{aligned}$$