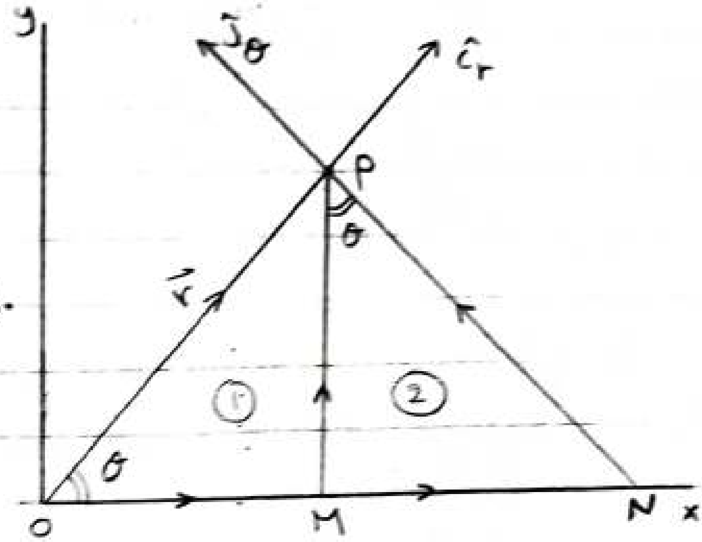
* Velocity and Acceleration in plane polar Coordinates

Vectorially, the position of the particle can be written as the product of the radial distance (r) by a unit radial vector ($\hat{i}_r$):.



$$\vec{r} = r\,\hat{i}_r$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\hat{i}_r)$$

$$\vec{V} = \dot{r}\,\hat{i}_r + r\frac{d\hat{i}_r}{dt} \text{ ---- } (a)$$

$$\vec{OM} = OM\,\hat{i} = (op\cos\theta)\,\hat{i} = r\cos\theta\,\hat{i}$$
$$\vec{Mp} = Mp\,\hat{j} = (op\sin\theta)\,\hat{j} = r\sin\theta\,\hat{j}$$

to Find $\frac{d\hat{i}_r}{dt}$ ⇒ in Δ(opM)

$$\vec{op} = \vec{oM} + \vec{Mp}$$
$$\cancel{r}\,\hat{i}_r = \cancel{r}\cos\theta\,\hat{i} + \cancel{r}\sin\theta\,\hat{j}$$
$$\therefore \hat{i}_r = \hat{i}\cos\theta + \hat{j}\sin\theta \text{ ---- } (1)$$

in Δ(MNP)

$$\vec{MP} = \vec{MN} + \vec{NP}$$
$$(Np\cos\theta)\,\hat{j} = (Np\sin\theta)\,\hat{i} + Np\,\hat{j}_\theta$$
$$\hat{j}_\theta = -\hat{i}\sin\theta + \hat{j}\cos\theta \text{ ---- } (2)$$

From eq (1)

$$\frac{d\hat{i}_r}{dt} = -\hat{i}\sin\theta\, d\theta + \hat{j}\cos\theta\, d\theta$$
$$= \dot{\theta}\,(-\hat{i}\sin\theta + \hat{j}\cos\theta)$$
$$= \dot{\theta}\,\hat{j}_\theta$$

equation (a)

$$\vec{V} = \dot{r}\,\hat{i}_r + r\frac{d\hat{i}_r}{dt}$$

$$\therefore \vec{V} = \dot{r}\,\hat{i}_r + r\dot{\theta}\,\hat{j}_\theta$$

* Acceleration in plan polar Coordinates

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\dot{r}\,\hat{i}_r + r\dot{\theta}\,\hat{j}_\theta)$$

$$\vec{a} = \ddot{r}\,\hat{i}_r + \dot{r}\frac{d\hat{i}_r}{dt} + (\dot{r}\dot{\theta}\,\hat{j}_\theta + \ddot{\theta}\,r\,\hat{j}_\theta + r\dot{\theta}\frac{d\hat{j}_\theta}{dt})$$

eq(2)

$$\hat{j}_\theta = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

$$\frac{d\hat{j}_\theta}{dt} = -\hat{i}\,\dot{\theta}\cos\theta - \hat{j}\,\dot{\theta}\sin\theta = -\dot{\theta}\,(\hat{i}\cos\theta + \hat{j}\sin\theta)$$

$$\frac{d\hat{j}_\theta}{dt} = -\dot{\theta}\,\hat{i}_r$$

$$\vec{a} = \ddot{r}\,\hat{i}_r + \dot{r}\dot{\theta}\,\hat{j}_\theta + \dot{r}\dot{\theta}\,\hat{j}_\theta + \ddot{\theta}\,r\,\hat{j}_\theta - r\dot{\theta}^2\,\hat{i}_r$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{i}_r + (2\dot{r}\dot{\theta} + \ddot{\theta}\,r)\,\hat{j}_\theta$$

Note:-

1. IF $r = \text{Constant}$ (Circular motion)
$\theta = \text{Vary}$

then:-

$$\vec{V} = r\dot{\theta}\,\hat{j}_\theta$$

$$\vec{a} = -r\dot{\theta}^2\,\hat{i}_r + \ddot{\theta}\,r\,\hat{j}_\theta$$

2. IF $r = \text{Vary}$ حركة عامة
$\theta = \text{Constant}$ (general motion)

$$\vec{V} = \dot{r}\,\hat{i}_r$$

$$\vec{a} = \ddot{r}\,\hat{i}_r$$

* Velocity and Acceleration in Cylindrical Coordinates :

- The cylindircal coordinates R, ϕ, Z

- The position vector is written as :

$$\vec{r} = R\hat{i}_R + Z\hat{K}$$

$$\vec{V} = \dot{R}\,\hat{i}_R + R\underbrace{\left(\frac{d\hat{i}_R}{dt}\right)}_{\dot{\phi} J_\phi} + \dot{Z}K$$

$$\vec{V} = \dot{R}\,\hat{i}_R + R\dot{\phi} J_\phi + \dot{Z}\hat{K}$$ حفظ

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\dot{R}\hat{i}_R + R\dot{\phi} J_\phi + \dot{Z}K\right)$$

$$\vec{a} = \ddot{R}\hat{i}_R + \dot{R}\left(\frac{d\hat{i}_R}{dt}\right) + \dot{R}\dot{\phi} J_\phi + R\ddot{\phi} J_\phi + R\dot{\phi}\underbrace{\left(\frac{dJ_\phi}{dt}\right)}_{-\dot{\phi}\hat{i}_R} + \ddot{Z}\hat{K}$$

$$\vec{a} = \ddot{R}\hat{i}_R + \dot{R}\dot{\phi} J_\phi + \dot{R}\dot{\phi} J_\phi + R\ddot{\phi} J_\phi - R\dot{\phi}^2\hat{i}_R + \ddot{Z}\hat{K}$$

$$\vec{a} = \left(\ddot{R} - R\dot{\phi}^2\right)\hat{i}_R + \left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right) J_\phi + \ddot{Z}\hat{K}$$

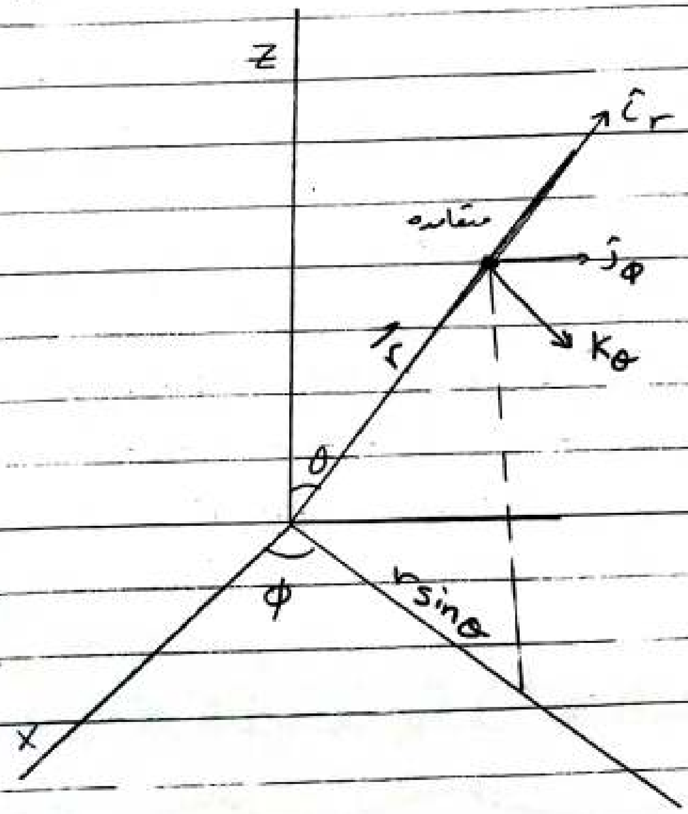
* Spherical Coordinates (r, θ, ϕ)

The position vector is

$$\vec{r} = r\,\hat{i}_r$$

$$\vec{V} = \dot{r}\,\hat{i}_r + r\dot{\phi}\sin\theta\,\hat{j}_\phi + r\dot{\theta}\,\hat{k}_\theta$$

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\hat{i} + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\hat{j}_\phi + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,k_\theta$$



eq.1) The position vector of a moving particle is given by the following expression. Find the velocity, speed, and acceleration as a function of time in each case.

1. $\vec{r} = \hat{i}ct - \hat{j}\frac{g}{2}t^2$
2. $\vec{r} = \hat{i}ct + \hat{j}\,b\cos\omega t + \hat{k}\,b\sin\omega t$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\hat{i}ct - \hat{j}\frac{gt^2}{2}\right)$$

$$= \hat{i}c - \hat{j}gt$$

$$v = |\vec{v}| = \sqrt{c^2 + g^2t^2}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -g\hat{j}$$

(2) $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\hat{i}ct + \hat{j}\,b\cos\omega t + \hat{k}\,b\sin\omega t\right)$

$$\vec{v} = c\hat{i} - b\omega\sin\omega t\,\hat{j} + b\omega\cos\omega t\,\hat{k}$$

$$v = |\vec{v}| = \left(c^2 + b^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t\right)^{1/2}$$

$$= \left(c^2 + b^2\omega^2(\sin^2\omega t + \cos^2\omega t)\right)^{1/2} = \left(c^2 + b^2\omega^2\right)^{1/2}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(c\hat{i} - b\omega\sin\omega t\,\hat{j} + b\omega\cos\omega t\,\hat{k}\right)$$

$$= -b\omega^2\cos\omega t\,\hat{j} - b\omega^2\sin\omega t\,\hat{k}$$

$$= -b\omega^2\left(\cos\omega t\,\hat{j} + \sin\omega t\,\hat{k}\right)$$

eq.2) The motion of a particle is given by

$$r = \hat{i}\cos\omega t + 2\hat{j}\sin\omega t$$

Find the angle between the acceleration vector and the velocity vector at time $(t = \frac{\pi}{4\omega})$

$$\vec{v}\cdot\vec{a} = va\cos\theta \Rightarrow \cos\theta = \frac{\vec{v}\cdot\vec{a}}{va}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(\hat{i}\cos\omega t + 2\hat{j}\sin\omega t)$$

$$\vec{v} = (-\omega\sin\omega t\,\hat{i} + 2\omega\cos\omega t\,\hat{j})$$

$$v = |\vec{v}| = (\omega^2\sin^2\omega t + 4\omega^2\cos^2\omega t)^{1/2}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(-\omega\sin\omega t\,\hat{i} + 2\omega\cos\omega t\,\hat{j})$$

$$= -\omega^2\cos\omega t\,\hat{i} - 2\omega^2\sin\omega t\,\hat{j}$$

$$a = |\vec{a}| = (\omega^4\cos^2\omega t + 4\omega^4\sin^2\omega t)^{1/2}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{a}}{va}$$

$$\cos\theta = \frac{(-\hat{i}\omega\sin\omega t + 2\omega\cos\omega t\,\hat{j})\cdot(-\hat{i}\omega^2\cos\omega t - 2\omega^2\sin\omega t\,\hat{j})}{(\omega^2\sin^2\omega t + 4\omega^2\cos^2\omega t)^{1/2}(\omega^4\cos^2\omega t + 4\omega^4\sin^2\omega t)^{1/2}}$$

الجواب النهائي $\cos\theta = \frac{-3}{5}$ $\therefore \theta = 127^\circ$

eq. 3) The position of two particle moving on a common circular path are give by:

$$r_1 = \hat{i}\, b \sin\omega t + \hat{j}\, b \cos\omega t$$

$$r_2 = \hat{i}\, b \cos\omega t - \hat{j}\, b \sin\omega t$$

Find the relative velocity, the magnitude of the relative velocity, and the time rate of change of the distance between the two particles, all as functions of the time t.

$$\vec{V}_{12} = \vec{V}_2 - \vec{V}_1$$

$$\vec{V}_1 = \frac{dr_1}{dt} = \frac{d}{dt}(\hat{i}\, b \sin\omega t + \hat{j}\, b \cos\omega t)$$

$$\vec{V}_1 = \hat{i}\, b\omega \cos\omega t - \hat{j}\, b\omega \sin\omega t$$

$$\vec{V}_2 = \frac{d\vec{r}_2}{dt} = \frac{d}{dt}(\hat{i}\, b \cos\omega t - \hat{j}\, b \sin\omega t)$$

$$\vec{V}_2 = -\hat{i}\, b\omega \sin\omega t - \hat{j}\, b\omega \cos\omega t$$

$$\vec{V}_{12} = (-\hat{i}\, b\omega \sin\omega t - \hat{j}\, b\omega \cos\omega t) - (\hat{i}\, b\omega \cos\omega t - \hat{j}\, b\omega \sin\omega t)$$

$$= \hat{i}(-b\omega \sin\omega t - b\omega \cos\omega t) + \hat{j}(b\omega \sin\omega t - b\omega \cos\omega t)$$

$$= \hat{i}(-b\omega)(\sin\omega t + \cos\omega t) + \hat{j}(b\omega)(\sin\omega t - \cos\omega t)$$

الكميات الإتجاهيه لانستطيع استخراج عامل مشترك منها (الا للكميات العددية)

$$V_{12} = |\vec{V}_{12}| = \sqrt{b^2\omega^2(\sin\omega t + \cos\omega t)^2 + b^2\omega^2(\sin\omega t - \cos\omega t)^2}$$

$$= b\omega \sqrt{\sin^2\omega t + 2\sin\omega t \cos\omega t + \cos^2\omega t + \sin^2\omega t - 2\sin\omega t \cos\omega t + \cos^2\omega t}$$

$$= b\omega \sqrt{2(\sin^2\omega t + \cos^2\omega t} \qquad = b\omega\sqrt{2}$$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$$= (\hat{i}\, b\cos\omega t - \hat{j}\, b\sin\omega t) - (\hat{i}\, b\sin\omega t + \hat{j}\, b\cos\omega t)$$

$$= \hat{i}\, b\cos\omega t - \hat{j}\, b\sin\omega t - \hat{i}\, b\sin\omega t - \hat{j}\, b\cos\omega t$$

$$= \hat{i}(b\cos\omega t - b\sin\omega t) - \hat{j}(b\cos\omega t + b\sin\omega t)$$

$$= \hat{i}\, b\,(\cos\omega t - \sin\omega t) - \hat{j}\, b\,(\cos\omega t + \sin\omega t)$$

$$r_{12} = |\vec{r}_{12}| = \sqrt{b^2(\cos\omega t - \sin\omega t)^2 + b^2(\sin\omega t + \cos\omega t)^2}$$

$$= b\sqrt{2}$$

$$\frac{dr_{12}}{dt} = 0$$

∴ التغير الزمني للمسافة بين الجسيمين = صفر

eq.4) The acceleration of a particle varies with time according to the equation

$$a = \hat{i}At + \hat{j}Bt^2 + \hat{k}Ct^3$$

At time $t=0$, the velocity is (v_0) and the position is (r_0). Find the position vector as a function of t.

$$\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow d\vec{v} = \vec{a}\,dt \Rightarrow \int_{v_0}^{v} dv = \int_0^t (\hat{i}At + \hat{j}Bt^2 + \hat{k}Ct^3)\,dt$$

$$\vec{v} - \vec{v}_0 = \hat{i}\frac{At^2}{2}\Big]_0^t + \hat{j}\frac{Bt^3}{3}\Big]_0^t + \hat{k}\frac{ct^4}{4}\Big]_0^t$$

$$\vec{v} = \hat{i}\frac{At^2}{2} + \hat{j}\frac{Bt^3}{3} + \hat{k}\frac{ct^4}{4} + v_0$$

$$\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow \int_{r_0}^{r} dr = \int_0^t \vec{v}\,dt$$

$$\vec{r} - \vec{r}_0 = \int_0^t \left(\hat{i}\frac{At^2}{2} + \hat{j}\frac{Bt^3}{3} + \hat{k}\frac{ct^4}{4} + v_0\right) dt$$

$$\vec{r} - \vec{r}_0 = \hat{i}\frac{At^3}{6}\Big]_0^t + \hat{j}\frac{Bt^4}{12}\Big]_0^t + \hat{k}\frac{ct^5}{20}\Big]_0^t + v_0 t$$

$$r = \hat{i}\frac{At^3}{6} + \hat{j}\frac{Bt^4}{12} + \hat{k}\frac{ct^5}{20} + 2r_0$$