Chapter 3...... Inverse Operator

Linear Equations with Constant Coefficients

We will use the inverse operator method to solve homogeneous and nonhomogeneous partial differential equations with constant coefficients. This method, although basically developed and frequently used for solving ordinary differential equations, becomes useful for finding general solutions of partial differential equations with constant coefficients. The problem of finding the general solutions of second order partial differential equations with constant coefficients and determining their particular solutions under auxiliary (initial) conditions is also discussed in a later section. Before we discuss the partial differential equations with constant coefficients we will first review in §3.1 the technique of inverse operators from the theory of ordinary differential equations. This review should prove useful in discussing the homogeneous and nonhomogeneous partial differential equations with constant coefficients.

3.1. Inverse Operators

If D represents $\frac{d}{dx}$, then $\frac{1}{D}$ is defined as the inverse operator of D, i.e,

$$\frac{1}{D}\varphi(x) = \int \varphi(x)\,dx.$$

If $f(D)$ represents a polynomial in D with constant coefficients, then $f(D)$ is a linear differential operator, and we define its inverse as $\frac{1}{f(D)}$. Thus,

$$f(D)\left[\frac{1}{f(D)}\varphi(x)\right] = \varphi(x). \qquad (3.1)$$

Note that

$\frac{1}{f(D)}[f(D)\varphi(x)]$ is not necessarily equal to $\varphi(x)$. However, if

$\frac{1}{f(D)}\varphi(x) = \psi(x)$, then $\psi(x)$ contains arbitrary constants, and $\psi(x) = \varphi(x)$, for some value of these arbitrary constants. In the sequel we will ignore arbitrary constants. We will list some formulas for the operator pair $f(D)$ and $\frac{1}{f(D)}$

Properties of inverse operator of ordinary differential equations

1. $f(D)\left[\frac{1}{f(D)}\varphi(x)\right]=\varphi(x),\ \frac{1}{f(D)}[f(D)\varphi(x)]=\varphi(x)+c$

2. $\frac{1}{f_1(D)f_2(D)}\phi(x)=\frac{1}{f_1(D)}\left(\frac{1}{f_2(D)}\phi(x)\right)=\frac{1}{f_2(D)}\left(\frac{1}{f_1(D)}\phi(x)\right).$

3. $\frac{1}{f(D)}[c_1\phi_1(x)+c_2\phi_2(x)]=c_1\frac{1}{f(D)}\phi_1(x)+c_2\frac{1}{f(D)}\phi_2(x).$

4. $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$, provided that $f(a)\neq 0$.

5. $f(D)\phi(x)e^{ax}=e^{ax}f(D+a)\phi(x)$

6. $\frac{1}{f(D)}\phi(x)e^{ax}=e^{ax}\frac{1}{f(D+a)}\phi(x).$

7. $\frac{1}{(D-a)^m}e^{ax}=\frac{x^m e^{ax}}{m!}$

8. $\frac{1}{(D-a)^m f(D)}e^{ax}=\frac{x^m}{m!f(a)}e^{ax},\ a\neq 0.$

9. $\frac{1}{D^2+a^2}\begin{cases}\cos\\ \sin\end{cases}bx=\frac{\begin{cases}\cos\\ \sin\end{cases}bx}{a^2-b^2},|a|\neq|b|.$

10. $\frac{1}{f(D^2)}\begin{cases}\cos\\ \sin\end{cases}ax=\frac{\begin{cases}\cos ax\\ \sin\end{cases}ax}{f(-a^2)}$, provided that $f(-a^2)\neq 0$.

11. $\frac{1}{(D^2+a^2)}\begin{cases}\cos ax\\ \sin ax\end{cases}=\frac{x}{2a}\begin{cases}\sin ax\\ -\cos ax\end{cases}.$

12. $\frac{1}{aD^2+bD+c}\begin{cases}\cos\\ \sin\end{cases}\omega x=\frac{(c-a\omega^2)\begin{cases}\cos\\ \sin\end{cases}\omega x\pm b\omega\begin{cases}\sin\\ \cos\end{cases}\omega x}{(c-a\omega^2)^2+b^2\omega^2}.$

13. $\frac{1}{f(D)}x^n=\frac{1}{a_n[1+g(D)]}x^n$

$$=\frac{1}{a_n}[1-g(D)+g^2(D)-g^3(D)+\cdots+g^n(D)+\cdots]x^n$$

where the terms of degree $n+1$ or higher are ignored and a_n depend on $f(D), g(D)$.

Proof of Formula 3. If $c=0$ then

$$\frac{1}{f(D)}[f(D)\varphi(x)]=f(D)\left[\frac{1}{f(D)}\varphi(x)\right]=\varphi(x).$$

$$\frac{1}{f(D)}[c_1\phi_1(x)+c_2\phi_2(x)]$$

$$=\frac{1}{f(D)}\left[c_1 f(D)\left[\frac{1}{f(D)}\varphi_1(x)\right]+c_2 f(D)\left[\frac{1}{f(D)}\varphi_2(x)\right]\right]$$

$$=\frac{1}{f(D)}f(D)\left(c_1\frac{1}{f(D)}\varphi_1(x)+c_2\frac{1}{f(D)}\varphi_2(x)\right)$$

$$=c_1\frac{1}{f(D)}\varphi_1(x)+c_2\frac{1}{f(D)}\varphi_2(x)$$

Proof of Formula 4. If $\phi(x)=e^{ax}$, we know that

$$De^{ax}=ae^{ax},\ D^2e^{ax}=a^2e^{ax},\ \ldots\ D^ne^{ax}=a^ne^{ax}$$

Let $f(D)=D^n \Longrightarrow f(a)=\ a^n$

$$\Longrightarrow f(D)e^{ax}=f(a)e^{ax}, \qquad (*)$$

If we take $\frac{1}{f(D)}$ as the inverse operator of $f(D)$, to $(*)$ then obviously

$$\frac{1}{f(D)}f(D)e^{ax}=\frac{1}{f(D)}f(a)e^{ax}=f(a)\frac{1}{f(D)}e^{ax},$$

$$e^{ax}=f(a)\frac{1}{f(D)}e^{ax},$$

$$\Longrightarrow \frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax} \text{ provided } f(a)\neq 0$$

Proof of Formula 5.

$$De^{ax}\phi(x)=e^{ax}D\phi(x)+ae^{ax}\phi(x)$$

$$De^{ax}\phi(x)=e^{ax}(D+a)\phi(x)$$

$$D^2e^{ax}\phi(x)=D(e^{ax}(D+a)\phi(x))$$

$$=e^{ax}D(D+a)\phi(x)+ae^{ax}(D+a)\phi(x)$$

$$=e^{ax}(D(D+a)+a(D+a))\phi(x)$$

$$D^2e^{ax}\phi(x)=e^{ax}(D+a)^2\phi(x)$$

$$D^ne^{ax}\phi(x)=e^{ax}(D+a)^n\phi(x) \quad (3.2)$$

Let $f(D)=D^n \Longrightarrow f(D+a)=(D+a)^n$

$$f(D)e^{ax}\phi(x)=e^{ax}f(D+a)\phi(x) \qquad (3.3)$$

Proof of Formula 6. From formula 5 $D^n e^{ax}\phi(x) = e^{ax}(D+a)^n\phi(x)$

Let $f(D) = \frac{1}{D^n} \Rightarrow D^n = \frac{1}{f(D)} \Rightarrow (D+a)^n = \frac{1}{f(D+a)}$, substitute in (3.2)

$$\frac{1}{f(D)}e^{ax}\phi(x) = e^{ax}\frac{1}{f(D+a)}\phi(x) \qquad (3.4)$$

Proof of Formula 7 & 8.

$\frac{1}{(D-a)^m}e^{ax} \overset{?}{=} \frac{x^m e^{ax}}{m!}$ & $\frac{1}{(D-a)^m f(D)}e^{ax} \overset{?}{=} \frac{x^m}{m!f(a)}e^{ax}$

Proof 7: first we have to show $\frac{1}{D^m}(1) \overset{?}{=} \frac{x^m}{m!}$

$$Dx^m = mx^{m-1}, D^2x^m = D(Dx^m) = m(m-1)x^{m-2}, \cdots,$$

$$D^m x^m = m! \qquad (**)$$

نأخذ للطرفين في$(**)$ $\frac{1}{D^m}$ لنحصل على

$$\frac{1}{D^m}D^m x^m = x^m = \frac{1}{D^m}(m!) = m!\frac{1}{D^m}(1)$$

$$\frac{1}{D^m}(1) = \frac{x^m}{m!} \qquad (3.5)$$

Using formula 6 with $f(D) = (D-a)^m, \phi(x) = 1 \Rightarrow f(D+a) = D^m$ and (3.5) we get

ملاحظة لانستطيع استخدام formula 4 وذلك لان $f(a) = 0$

$$\frac{1}{(D-a)^m}e^{ax} \overset{6}{=} e^{ax}\frac{1}{f(D+a)}(1) = e^{ax}\frac{1}{D^m}(1) \overset{(3.5)}{=} e^{ax}\frac{x^m}{m!}.$$

Proof 8: using formula 4, 3 and 7

$$\frac{1}{(D-a)^m}\frac{1}{f(D)}e^{ax} \overset{4}{=} \frac{1}{(D-a)^m}\frac{1}{f(\mathrm{a})}e^{ax} \overset{3}{=} \frac{1}{f(\mathrm{a})}\frac{1}{(D-a)^m}e^{ax} \overset{7}{=} \frac{e^{ax}}{f(\mathrm{a})}\frac{x^m}{m!}$$

Proof of Formula 9 & 10.

To show that $D^2 = -a^2$ in polar form

$$D\begin{cases}\cos ax\\ \sin ax\end{cases} = \begin{cases}-a\sin ax\\ a\cos ax\end{cases} = a\begin{cases}-\sin ax\\ \cos ax\end{cases}$$

$$D^2\begin{cases}\cos ax\\ \sin ax\end{cases} = \begin{cases}-a^2\sin ax\\ -a^2\cos ax\end{cases} = -a^2\begin{cases}\sin ax\\ \cos ax\end{cases}$$

$$D^3\begin{cases}\cos ax\\ \sin ax\end{cases} = a^3\begin{cases}\sin ax\\ -\cos ax\end{cases}, D^4\begin{cases}\cos ax\\ \sin ax\end{cases} = a^4\begin{cases}\sin ax\\ \cos ax\end{cases}$$

$$D^5\begin{cases}\cos ax\\ \sin ax\end{cases}=a^5\begin{cases}-\sin ax\\ \cos ax\end{cases},D^6\begin{cases}\cos ax\\ \sin ax\end{cases}=-a^6\begin{cases}\sin ax\\ \cos ax\end{cases},D^7\begin{cases}\cos ax\\ \sin ax\end{cases}$$

$$=a^7\begin{cases}\sin ax\\ -\cos ax\end{cases}$$

$$1. D^{2n}\begin{cases}\cos ax\\ \sin ax\end{cases}=(-1)^n a^{2n}\begin{cases}\cos ax\\ \sin ax\end{cases}, 2.\ D^{2n-1}\begin{cases}\cos ax\\ \sin ax\end{cases}=(-1)^n a^{2n-1}\begin{cases}\sin ax\\ -\cos ax\end{cases}$$

From 1 we get $D^{2n}=(-1)^n a^{2n}$ or $(D^2)^n=(-a^2)^n \Longrightarrow D^2=-a^2$,

So formula 9 $\frac{1}{D^2+b^2}\begin{cases}\cos ax\\ \sin ax\end{cases}=\frac{1}{b^2-a^2}\begin{cases}\cos ax\\ \sin ax\end{cases}, |a|\neq|b|$

And formula 10 $\frac{1}{f(D^2)}\begin{cases}\cos ax\\ \sin ax\end{cases}=\frac{1}{f(-a^2)}\begin{cases}\cos ax\\ \sin ax\end{cases}$

Proof of Formula 11 $\frac{1}{(D^2+a^2)}\begin{cases}\cos ax\\ \sin ax\end{cases}\stackrel{?}{=}\frac{x}{2a}\begin{cases}\sin ax\\ -\cos ax\end{cases}$

$$(D^2+a^2)(x\sin ax)=D^2x\sin ax+a^2x\sin ax$$

$$=D(\sin ax+ax\cos ax)+a^2x\sin ax$$

$$=a\cos ax+ax\cos ax-a^2x\sin ax+a^2x\sin ax=2a\cos ax$$

$$(D^2+a^2)(x\sin ax)=2a\cos ax$$

ناخذ $\frac{1}{(D^2+a^2)}$ للطرفين

$$\left\{\frac{1}{(D^2+a^2)}2a\cos ax=x\sin ax\right\}\div 2a$$

$$\frac{1}{(D^2+a^2)}\cos ax=\frac{x}{2a}\sin ax$$

Similarly $(D^2+a^2)(-x\cos ax)=-D^2x\cos ax-a^2x\cos ax$

$$=-D(\cos ax-ax\sin ax)-a^2x\cos ax$$

$$=a\sin ax+a\sin ax+a^2x\cos ax-a^2x\cos ax=2a\sin ax$$

$$(D^2+a^2)(-x\cos ax)=2a\sin ax$$

$$\frac{1}{(D^2+a^2)}2a\sin ax=-x\cos ax$$

$$\frac{1}{(D^2+a^2)}\sin ax=-\frac{x}{2a}\cos ax$$

Proof of Formula 13

If $\phi(x)=x^n$, and let $f(D)=a_0D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n$

$$f(D) = a_n\left[1 + \frac{a_{n-1}}{a_n}D + \frac{a_{n-2}}{a_n}D^2 + \cdots + \frac{a_0}{a_n}D^n\right]$$

$$= a_n\left(1 + \frac{1}{a_n}(a_{n-1}D + a_{n-2}D^2 + \cdots + a_0D^n)\right) = a_n(1 + g(D))$$

$$f(D) = a_n(1 + g(D))$$

Where $g(D) = \frac{1}{a_n}(a_{n-1}D + a_{n-2}D^2 + \cdots + a_0D^n)$

Which gives

$$\frac{1}{f(D)} = \frac{1}{a_n(1 + g(D))}$$

Taylor expansion $\frac{1}{1+x} = 1 - x + x^2 - \cdots (-1)^n x^n + \cdots$.

So

$$\frac{1}{f(D)}x^n = \frac{1}{a_n(1 + g(D))}x^n$$

$$= \frac{1}{a_n}\big(1 - g(D) + g^2(D) - \cdots (-1)^n g^n(D)\big)x^n$$

$$0 = g^{n+1}(D) = g^{n+2}(D) = \cdots , \qquad \text{for example } (x^3)'''' = 0$$

Examples:

Therefore, the particular integral y_p of the equation $f(D)y = Ae^{ax}$ using formula 4 is given by

$$y_p = \frac{A}{f(a)}e^{ax},\ f(a) \neq 0$$

Example 3.1. Consider $y'' + y' + y = e^{2x}$ or $(D^2 + D + 1)y = e^{2x}$. Then

$$y_p = \frac{1}{D^2 + D + 1}e^{3x} = \frac{1}{f(3)}e^{3x} = \frac{1}{3^2 + 3 + 1}e^{3x} = \frac{1}{13}e^{3x}$$

Example 3.2. Consider $(D^4 + 8)y = e^x$. Find y_p .

$$y_p = \frac{1}{D^4 + 8}e^x = \frac{1}{1^4 + 8}e^x = \frac{1}{9}e^x$$

Now, in order to find the particular integral y_p of $f(D)y = Ax^n$, we apply the inverse $\frac{1}{f(D)}$ of $f(D)$ to the ordinary differential equation and get

$$y = \frac{1}{f(D)}(Ax^n)$$

$$= A \cdot \frac{1}{a_n}(1 + g(D))^{-1}x^n$$

$$y_p = \frac{A}{a_n}[1 - g(D) + (g(D))^2 - (g(D))^3 + \cdots]x^n \quad (3.6)$$

where terms of degree $n + 1$ and higher in D are ignored in the above expansion on the right side.

Example 3.3. Consider $y'' + y' + 2y = x^4 \rightarrow (D^2 + D + 2)y = x^4$. Find y_p ? By formula 13

$$y_p = \frac{1}{D^2 + D + 2}x^4 = \frac{1}{2\left(1 + \frac{1}{2}D + \frac{1}{2}D^2\right)}x^4$$

$$= \frac{1}{2}\left[1 + \frac{1}{2}\mathrm{D} + \frac{1}{2}D^2\right]^{-1} x^4, \quad g(D) = \frac{1}{2}D + \frac{1}{2}D^2$$

$$= \frac{1}{2}\left[1 - \left(\frac{1}{2}D + \frac{1}{2}D^2\right) + \left(\frac{1}{2}D + \frac{1}{2}D^2\right)^2 - \left(\frac{1}{2}D + \frac{1}{2}D^2\right)^3 + \left(\frac{1}{2}D + \frac{1}{2}D^2\right)^4 - \cdots\right]x^4$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}D - \frac{1}{2}D^2 + \frac{1}{4}D^2 + \frac{1}{2}D^3 + \frac{1}{4}D^4 - \frac{1}{8}D^3 - \frac{3}{8}D^4 + \frac{1}{16}D^4 + O(D^5)\right]x^4,$$

where $O(D^5)$ means terms containing D^5 and higher powers in D, so $O(D^5) = 0$ Thus,

$$y_p = \frac{1}{2}\left[x^4 - 2x^3 - 6x^2 + 3x^2 + 12x + 6 - 3x - 9 + \frac{3}{2}\right]$$

$$= \frac{1}{2}\left[x^4 - 2x^3 - 3x^2 + 9x - \frac{3}{2}\right].$$