Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\, dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n \, du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}\, dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5 (3x^2 + 1)\, dx.$

Sol: let $u = x^3 + x.$ then $du = \frac{du}{dx}\, dx = (3x^2 + 1)\, dx,$

so that by substitution we have :

$$\begin{aligned} \int (x^3 + x)^5 (3x^2 + 1)\, dx &= \int u^5\, du && \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\, dx. \\ &= \frac{u^6}{6} + C && \text{Integrate with respect to } u. \\ &= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u. \end{aligned}$$

EXAMPLE:

Find the integral $\int \sqrt{2x + 1}\, dx.$

SOL: let u=2x+1 and n=1/2, $du = \frac{du}{dx}\, dx = 2\, dx$

because of the constant factor 2 is missing from the integral. So we write

$$\begin{aligned} \int \sqrt{2x + 1}\, dx &= \frac{1}{2} \int \underbrace{\sqrt{2x + 1}}_{u} \cdot \underbrace{2\, dx}_{du} \\ &= \frac{1}{2} \int u^{1/2}\, du && \text{Let } u = 2x + 1,\ du = 2\, dx. \\ &= \frac{1}{2} \frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \end{aligned}$$

EXAMPLE: $\int \cos(7\theta + 3)\, d\theta$.

SOL: Let $u = 7\theta + 3$ so that $du = 7\, d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7. Then,

$$\begin{aligned}
\int \cos(7\theta + 3)\, d\theta &= \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\, d\theta && \text{Place factor 1/7 in front of integral.}\\
&= \frac{1}{7}\int \cos u\, du && \text{Let } u = 7\theta + 3,\ du = 7\, d\theta.\\
&= \frac{1}{7}\sin u + C && \text{Integrate.}\\
&= \frac{1}{7}\sin(7\theta + 3) + C && \text{Substitute } 7\theta + 3 \text{ for } u.
\end{aligned}$$

EXAMPLE:

$$\begin{aligned}
\int x^2 \sin(x^3)\, dx &= \int \sin(x^3)\cdot x^2\, dx\\
&= \int \sin u \cdot \frac{1}{3}\, du && \text{Let } u = x^3,\ du = 3x^2\, dx,\ (1/3)\, du = x^2\, dx.\\
&= \frac{1}{3}\int \sin u\, du\\
&= \frac{1}{3}(-\cos u) + C && \text{Integrate.}\\
&= -\frac{1}{3}\cos(x^3) + C && \text{Replace } u \text{ by } x^3.
\end{aligned}$$

EXAMPLE: Evaluate $\int x\sqrt{2x + 1}\, dx$

SOL: $u = 2x + 1$ to obtain $x = (u - 1)/2$, and find that $x\sqrt{2x + 1}\, dx = \frac{1}{2}(u - 1)\cdot \frac{1}{2}\sqrt{u}\, du.$

The integration now becomes

$$\begin{aligned}
\int x\sqrt{2x + 1}\, dx &= \frac{1}{4}\int (u - 1)\sqrt{u}\, du = \frac{1}{4}\int (u - 1)u^{1/2}\, du && \text{Substitute.}\\
&= \frac{1}{4}\int (u^{3/2} - u^{1/2})\, du && \text{Multiply terms.}\\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2} \int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}}, C\right. && \text{Integrate.}\\
&= \frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} + C && \text{Replace } u \text{ by } 2x + 1. \ \blacksquare
\end{aligned}$$

$$\int \frac{2z\, dz}{\sqrt[3]{z^2 + 1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2 + 1,\ du = 2z\, dz.$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2 + 1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

Method 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2 + 1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2 + 1},\ u^3 = z^2 + 1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}(z^2 + 1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2 + 1)^{1/3}.$$

Example: The Integrals of $\boldsymbol{sin^2 x}$ and $\boldsymbol{cos^2 x}$

$$\textbf{(a)} \int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$
$$\textbf{(b)} \int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \blacksquare$$

DEFINITION: If u is a differentiable function that is never zero, $\int \frac{1}{u}\,du = \ln|u| + C.$

In general $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

EXAMPLE

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln|u|\Big]_{-5}^{-1} \qquad u = x^2-5,\ du = 2x\,dx,\ u(0) = -5,\ u(2) = -1$$
$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5$$

<u>The Integrals of tan x, cot x, sec x, and esc x</u>

$$\text{1-} \quad \int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u} \qquad u = \cos x > 0 \text{ on } (-\pi/2, \pi/2),\ du = -\sin x\,dx$$
$$= -\ln|u| + C = -\ln|\cos x| + C$$
$$= \ln\frac{1}{|\cos x|} + C = \ln|\sec x| + C. \qquad \text{Reciprocal Rule}$$

$$\text{2-} \quad \int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} \qquad u = \sin x,\ du = \cos x\,dx$$
$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\text{3-} \quad \int \sec x\,dx = \int \sec x\,\frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C \qquad \begin{array}{l} u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x)\,dx \end{array}$$

4- $$\int \csc x\,dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)}\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C \qquad \begin{array}{l} u = \csc x + \cot x \\ du = (-\csc x \cot x - \csc^2 x)\,dx \end{array}$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

EXAMPLE:

$$\int_0^{\pi/6} \tan 2x\,dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\,du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

$$\int e^u\,du = e^u + C$$

EXAMPLE :

(a) $$\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3}\,du \qquad u = 3x,\ \frac{1}{3}du = dx,\ u(0) = 0,\ u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$$

$$= \frac{1}{3}\int^{\ln 8} e^u\,du$$

$$= \frac{1}{3}\int_0 e^u \, du$$

$$= \frac{1}{3} e^u \Big]_0^{\ln 8}$$

(b) $$\int_0^{\pi/2} e^{\sin x} \cos x \, dx = e^{\sin x} \Big]_0^{\pi/2}$$ Antiderivative from Example 2c

$$= e^1 - e^0 = e - 1$$

The integral of a^u

$$\int a^u \, du = \frac{a^u}{\ln a} + C.$$

EXAMPLE :

(a) $\frac{d}{dx} 3^x = 3^x \ln 3$

(b) $\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3) \frac{d}{dx}(-x) = -3^{-x} \ln 3$

(c) $\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3) \frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3) \cos x$

(d) $\int 2^x\,dx = \dfrac{2^x}{\ln 2} + C$

(e) $\int 2^{\sin x}\cos x\,dx = \int 2^u\,du = \dfrac{2^u}{\ln 2} + C$

$= \dfrac{2^{\sin x}}{\ln 2} + C$

Example :

(a) $\dfrac{d}{dx}\log_{10}(3x+1) = \dfrac{1}{\ln 10}\cdot\dfrac{1}{3x+1}\dfrac{d}{dx}(3x+1) = \dfrac{3}{(\ln 10)(3x+1)}$

(b) $\int \dfrac{\log_2 x}{x}\,dx = \dfrac{1}{\ln 2}\int \dfrac{\ln x}{x}\,dx \qquad \log_2 x = \dfrac{\ln x}{\ln 2}$

$= \dfrac{1}{\ln 2}\int u\,du \qquad u = \ln x,\quad du = \dfrac{1}{x}dx$

$= \dfrac{1}{\ln 2}\dfrac{u^2}{2} + C = \dfrac{1}{\ln 2}\dfrac{(\ln x)^2}{2} + C = \dfrac{(\ln x)^2}{2\ln 2} + C$

Integration Formulas

1. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)

2. $\int \dfrac{du}{a^2+u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$ (Valid for all u)

3. $\int \dfrac{du}{u\sqrt{u^2-a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{3} - \dfrac{\pi}{4} = \dfrac{\pi}{12}$

(b) $\int \dfrac{dx}{\sqrt{3-4x^2}} = \dfrac{1}{2}\int \dfrac{du}{\sqrt{a^2-u^2}}$

$= \dfrac{1}{2}\sin^{-1}\left(\dfrac{u}{a}\right) + C$