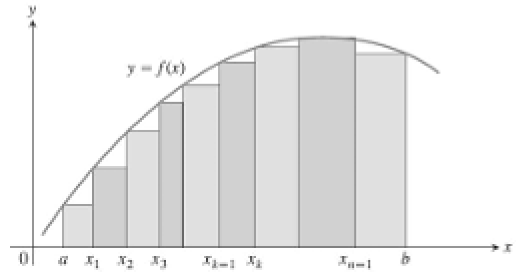
.2 INTEGRATION:



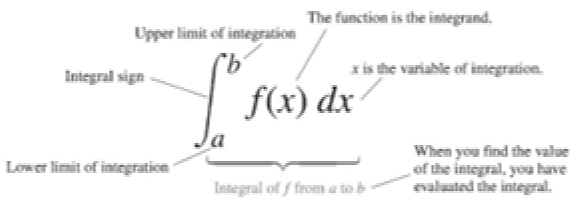
1) The Definite Integral

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$

$\Delta x_k = \Delta x = (b-a)/n$ for all k

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

$\Delta x = (b-a)/n$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5, \qquad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$

Then:

1. $\int_{4}^{1} f(x)\,dx = -\int_{1}^{4} f(x)\,dx = -(-2) = 2$

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$

$= 2(5) + 3(7) = 31$

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

2.1 Integration by Substitution

THEOREM Substitution in Definite Integrals: If g' is continuous on the interval *[a,* b] and *f* continuous on the range of ***g(x) = u,*** then

$$\int_{a}^{b} f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

SOL:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_{0}^{2} \sqrt{u}\,du$$

$$= \frac{2}{3} u^{3/2} \Big]_{0}^{2}$$

Evaluate the new definite integral.

EXAMPL

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

(a) $\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta\,d\theta = \int_{1}^{0} u \cdot (-du)$

Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$,
$-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) =$
When $\theta = \pi/2$, $u = \cot(\pi/2) =$

$$= -\int_{1}^{0} u\,du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $$\int_{-\pi/4}^{\pi/4} \tan x\,dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x}\,dx$$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$ Let $u = \cos x$, $du = -\sin x\,dx$. When $x = -\pi/4$, $u = \sqrt{2}/2$. When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= -\ln|u|\Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$ Integrate, zero width interval

THEOREM:
Let f be continuous on the symmetric interval [-a, a].

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE: Evaluate $\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx$.

SOL: Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval [- 2, 2] , so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx = 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

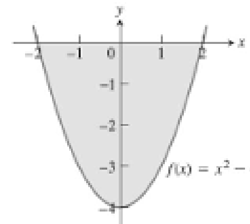
DEFINITION: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b.

$$A = \int_a^b f(x)dx$$

If f(x) is negative then $A = \int_a^b |f(x)|dx$

EXAMPLE

Let $f(x) = x^2 - 4$, compute (a) the definite integral over the interval [-2,2], and (b) the area between the graph and the x-axis over [-2,2].



Solution:

(a) $\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3}.$

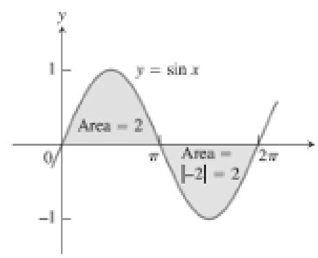
(b) The area between the graph and the x-axis is $\left|-\frac{32}{3}\right| = \frac{32}{3}$

EXAMPLE: Find the area between the graph $f(x) = x^3 - 2x^2 - x$

SOL: f(x)=0 then $(x^2 - 1)(x - 2) = 0$ that is x=1, -1 and x=2

$$A = A_1 + A_2 = \int_{-1}^{1} |f(x)|dx + \int_{1}^{2} |f(x)|dx$$

$$= \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]$$



EXAMPLE: Let the *function* $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

Solution

(a) The definite integral for $f(x) = sinx$ is given by

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

Total enclosed area $= \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$

EXAMPLE: Find $\int_{-1}^{2} |x-1| dx$

Since $|x-1| = \begin{cases} x-1 & x \geq 1 \\ -x+1 & x < 1 \end{cases}$ then $\int_{-1}^{2} |x-1| dx = \int_{-1}^{1} (-x+1) dx + \int_{1}^{2} (x-1) dx$

.3 Indefinite Integrals and the Substitution Method

Since any two antiderivatives of f differ by a constant, the indefinite integral notation means that for any antiderivative F of f,

$$\int f(x)\, dx = F(x) + C,$$

where C is any arbitrary constant.

THEOREM:
The Substitution Rule If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}\,dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Sol: let $u = x^3 + x$. then $du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$

so that by substitution we have :

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u.$$

EXAMPLE:

Find the integral $\int \sqrt{2x + 1}\,dx.$

SOL: let u=2x+1 and n=1/2, $du = \frac{du}{dx}\,dx = 2\,dx$

because of the constant factor 2 is missing from the integral. So we write

$$\int \sqrt{2x + 1}\,dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}\cdot\underbrace{2\,dx}$$

Sol: let $u = x^3 + x$. **then** $du = \frac{du}{dx}dx = (3x^2 + 1)\,dx$,

so that by substitution we have :

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

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because of the constant factor 2 is missing from the integral. So we write

$$\int \sqrt{2x + 1}\,dx = \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u}\cdot\underbrace{2\,dx}_{du}$$

$$= \frac{1}{2}\int u^{1/2}\,du \qquad \text{Let } u = 2x + 1,\ du = 2\,dx.$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}(2x + 1)^{3/2} + C \qquad \text{Substitute } 2x + 1 \text{ for } u.$$

EXAMPLE: Find $\int \sec^2(5t + 1)\cdot 5\,dt.$

SOL: Let $u = 5t + 1$ and $du = 5\,dx$. Then,

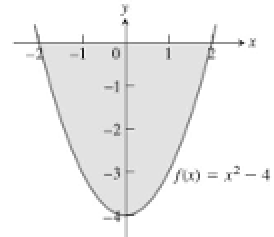
$$\int \sec^2(5t + 1)\cdot 5\,dt = \int \sec^2 u\,du \qquad \text{Let } u = 5t + 1,\ du = 5\,dt.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5t + 1) + C \qquad \text{Substitute } 5t + 1 \text{ for } u.$$

EXAMPLE

Let $f(x) = x^2 - 4$, compute (a) the definite integral over the interval [-2,2], and (b) the area between the graph and the x-axis over [-2,2].



Solution:

(a) $\int_{-2}^{2} f(x)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3}.$

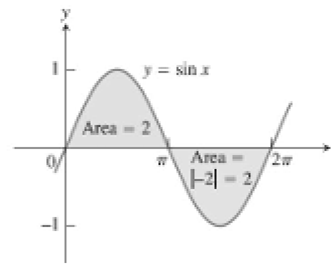
(b) The area between the graph and the x-axis is $\left|-\frac{32}{3}\right| = \frac{32}{3}$

EXAMPLE: Find the area between the graph $\boldsymbol{f(x) = x^3 - 2x^2 - x}$

SOL: f(x)=0 then $(\boldsymbol{x^2 - 1})(\boldsymbol{x - 2}) = \boldsymbol{0}$ that is x=1, -1 and x=2

A=$\boldsymbol{A_1 + A_2 = \int_{-1}^{1}|f(x)|dx + \int_{1}^{2}|f(x)|dx}$

$$=\left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]$$



EXAMPLE: Let the *function f(x)* = sin *x* between *x* = 0 and *x* = 2π. Compute

(a) the definite integral of *f(x)* over [0, 2π].

(b) the area between the graph of *fix)* and the x-axis over [0, 2π].

Solution

(a) The definite integral for $f(x) = sinx$ is given by

$$\int_{0}^{2\pi} \sin x\,dx = -\cos x\Big]_{0}^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

(b) To compute the area between the graph of *f(x)* and the x-axis over $[0, 2\pi]$ we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to $\sin x=0$ i.e. $x=0$, $x=\pi$ *or* $x=2\pi$

Now subdivide $[0, 2\pi]$ into two pieces: the interval $[0, \pi]$ and the interval $[\pi, 2\pi]$.

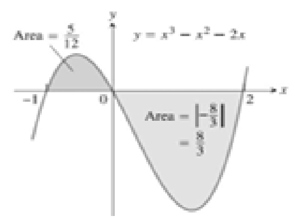
$$\int_0^{\pi} \sin x\, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2$$
$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2$$

Area $= |2| + |-2| = 4$.

EXAMPLE:

Find the area of the region between the x-axis and the graph of

$\boldsymbol{f(x) = x^3 - x^2 - 2x, \quad -1 \le x \le 2}$



Solution

First find the zeros of f. $f(x) = x^3 - x^2 - 2x = 0$

$$x(x^2 - x - 2) = 0$$
$$x(x+1)(x-2) = 0$$

x = 0, -1, and 2 . The zeros subdivide [-1,2] into two subintervals: [-I, 0], on which f≥ 0, and [0, 2], on which *f≤0*. We integrate *f* over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$
$$\int_{0}^{2} (x^3 - x^2 - 2x)\, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

Total enclosed area $= \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$