$$\alpha = \int_0^1 u(t)dt\,,\ \ \beta = \int_0^1 t^2 u(t)dt\,,\ \ \gamma = \int_0^1 tu(t)dt.$$

To determine the constants α, β and γ, we substitute into each equation of to obtain

$$\alpha = \int_0^1 (4-\alpha) + (45-30\beta)t + (26-12\gamma)t^2 dt = \frac{211}{6} - \alpha - 15\beta - 4\gamma,$$

$$\beta = \int_0^1 t^2(4-\alpha) + (45-30\beta)t + (26-12\gamma)t^2 dt$$

$$= \frac{1067}{60} - \frac{1}{3}\alpha - \frac{15}{2}\beta - \frac{12}{5}\gamma,$$

$\gamma = \int_0^1 t(4-\alpha) + (45-30\beta)t + (26-12\gamma)t^2 dt = \frac{47}{2} - \frac{1}{2}\alpha - 10\beta - 3\gamma.$

$$2\alpha + 15\beta + 4\gamma = \frac{211}{6}$$

$$\frac{1}{3}\alpha + \frac{17}{2}\beta - \frac{12}{5}\gamma = \frac{1067}{60},$$

$$\frac{1}{2}\alpha + 10\beta + 4\gamma = \frac{47}{2}$$

Unlike the previous examples, we obtain a system of three equations in three unknowns α, β, and γ. Solving this system of algebraic equations gives $\alpha = 3$, $\beta = \frac{43}{30}, \gamma = \frac{23}{12}$. Substituting, the exact solution is given by

$u(x) = 1 + 2x\ +\ 3x2.$

Exercises 3.2.5 Use the direct computation method to solve the following Fredholm integral equations:

1. $u(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)u(t)dt$

2. $u(x) = -8 + 11x - x^2 + x^3 - \int_0^1 (12x - 20t)u(t)dt$

3. $u(x) = 1 + 7x + 20x^2 + x^3 - \int_0^1 (10xt^2 + 20x^2t)u(t)dt$

4. $u(x) = 2\sqrt{3} - \frac{1}{x} + \sec x \tan x - \int_0^{\frac{\pi}{6}} xu(t)dt$

5. $u(x) = 2\pi^3 - \ln\left(2 + \sqrt{3}\right)x + \sec x \tan x - \int_0^{\frac{\pi}{3}} xtu(t)dt$

6. $u(x) = 1 + \ln x - \int_{0^+}^1 \ln(xt^2)u(t)dt\,, 0 < x \le 1$

7. $u(x) = 1 + \pi^2 \sec^2 x - \int_0^{\frac{\pi}{4}} \sec^2 x \, u(t)dt$

8. $u(x) = 1 + x + e^x - \frac{2}{3}\int_0^1 xtu(t)dt.$

3.2.6 The Successive Approximations Method

Given Fredholm integral equation of the second kind

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt, \quad (3.17)$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$u_0(x) = \text{any selective real valued function,}$$

$$u_{n+1}(x) = f(x) + \lambda \int_a^b K(x,t)u_n(t)dt, n \geq 0. \quad (3.18)$$

$u_0(x) =$ all terms not included inside the integral sign,

$$u_1(x) = \lambda \int_a^b K(x,t)u_0(t)dt,$$

The successive approximations method gives the exact solution, if it exists, by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x). \qquad (3.19)$$

However, the Adomian decomposition method gives the solution as infinite series of components by $u(x) = \sum_{n=0}^{\infty} u_n(x)$. (3.20)

This series solution converges rapidly to the exact solution if such a solution exists. The successive approximations method, or the iteration method will be illustrated by studying the following examples.

Example 3.23 Solve the Fredholm integral equation by using the successive approximations method

$$u(x) = x + e^x - \int_0^1 xtu(t)dt.$$

For the zeroth approximation $u_0(x)$, we can select $u_0(x) = 0$. The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = x + e^x - \int_0^1 xtu_n(t)dt, n \geq 0.$$

Substituting into we obtain

$$u_1(x) = x + e^x - \int_0^1 xtu_0(t)dt = e^x + x,$$

$$u_2(x) = x + e^x - \int_0^1 xtu_1(t)dt = e^x - \frac{x}{3},$$

$$u_3(x) = x + e^x - \int_0^1 xtu_2(t)dt = e^x + \frac{x}{9},$$

$$\cdots\cdots\cdots$$

$$u_{n+1}(x) = x + e^x - \int_0^1 xtu_n(t)dt = e^x + \frac{(-1)^n}{3^n}x,$$

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x$$

Example 3.24 Solve the Fredholm integral equation by using the successive approximations method $u(x) = x + \lambda \int_{-1}^{1} xtu(t)dt$.

For the zeroth approximation $u_0(x)$, we can select $u_0(x) = x$. The method of successive approximations admits the use of the iteration formula

$$u_{n+1}(x) = x + \lambda \int_{-1}^{1} xtu_n(t)dt\,, n \geq 0.$$

Substituting we obtain

$$u_1(x) = x + \frac{2}{3}\lambda x,$$

$$u_2(x) = x + \frac{2}{3}\lambda x + (\frac{2}{3})^2\lambda^2 x,$$

$$u_3(x) = x + \frac{2}{3}\lambda x + (\frac{2}{3})^2\lambda^2 x + (\frac{2}{3})^3\lambda^3 x,$$

$$\cdots\cdots\cdots$$

$$u_{n+1}(x) = x + \frac{2}{3}\lambda x + (\frac{2}{3})^2\lambda^2 x + (\frac{2}{3})^3\lambda^3 x + \cdots + (\frac{2}{3})^{n+1}\lambda^{n+1} x$$

$$= x\sum_{n=0}^{\infty} (\frac{2\lambda}{3})^n.$$

The solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \frac{3x}{3-2\lambda}, -\frac{3}{2} < \lambda < \frac{3}{2},$$

Example 3.25 Solve the Fredholm integral equation by using the successive approximations method

$$u(x)=\sin x+\sin x\int_0^{\frac{\pi}{2}}\cos t\,u(t)dt.$$

we select $u_0(x)=0$. We next use the iteration formula

$$u_{n+1}(x)=\sin x+\sin x\int_0^{\frac{\pi}{2}}\cos t\,u_n(t)dt\,,n\geq 0.$$

Substituting we obtain

$$u_1(x)=\sin x+\sin x\int_0^{\frac{\pi}{2}}0=\sin x.$$

$$u_2(x)=\sin x+\sin x\int_0^{\frac{\pi}{2}}\cos t\sin t\,dt=\frac{3}{2}\sin x.$$

$$u_3(x)=\sin x+\frac{3}{2}\sin x\int_0^{\frac{\pi}{2}}\cos t\sin t\,dt=\frac{7}{4}\sin x.$$

$$u_4(x)=\sin x+\frac{7}{4}\sin x\int_0^{\frac{\pi}{2}}\cos t\sin t\,dt=\frac{15}{8}\sin x.$$

$u_{n+1}(x)=\frac{2^{n+1}-1}{2^n}\sin x=\left(2-\frac{1}{2^n}\right)\sin x.$

The solution $u(x)$ is given by $u(x)=\lim\limits_{n\to\infty}u_{n+1}(x)=2\sin x.$

Example 3.26 Solve the Fredholm integral equation by using the successive approximations method

$$u(x)=x+\sec^2 x-\int_0^{\frac{\pi}{4}}xu(t)dt.$$

For the zeroth approximation u0(x), we may select $u_0(x)=0$. We next use the iteration formula

$u_{n+1}(x)=x+\sec^2 x-\int_0^{\frac{\pi}{4}}xu_n(t)dt\,,n\geq 0$. This in turn gives

$$u_1(x)=\sec^2 x+x,\quad u_2(x)=\sec^2 x+\frac{\pi^2}{32}x,$$

$$u_3(x)=\sec^2 x+\frac{\pi^4}{1024}x,\qquad u_4(x)=\sec^2 x+\frac{\pi^6}{32768}x\ \ldots$$

$u_{n+1}(x) = \sec^2 x + (-1)^n (\frac{\pi^2}{32})^n x$. Notice that $\lim_{n\to\infty} (\frac{\pi^2}{32})^n = 0$. Consequently, the solution $u(x)$ of is given by $u(x) = \lim_{n\to\infty} u_{n+1}(x) = \sec^2 x$.

Exercises 3.2.5 Use the successive approximations method to solve the following Volterra integral equations:

1. $u(x) = x + \int_0^x (x - t)u(t)dt.$
2. $u(x) = 1 + 2x + 4\int_0^x (x - t)u(t)dt.$
3. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x - t)^3 u(t)dt$
4. $u(x) = 1 - 2\sinh x + \int_0^x (x - t + 2)u(t)dt$
5. $u(x) = 1 - x\sin x + \int_0^x tu(t)dt.$
6. $u(x) = 2x\cosh x - 4\int_0^x tu(t)dt.$
7. $u(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x u(t)dt.$

3.2.7 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{k} \frac{u^{(n)}(b)}{n!}(x - b)^n, \qquad (3.20)$$

converges to $f(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \qquad (3.21)$$

Following the discussion presented before in Chapter 3, the series solution method that stems mainly from the Taylor series for analytic functions, will be used for solving Fredholm integral equations. We will assume that the solution $u(x)$ of the Fredholm integral equations

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

is analytic, and therefore possesses a Taylor series of the form given in (4.191), where the coefficients a_n will be determined recurrently. Substituting (4.191) into both sides of (4.192) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \lambda \int_a^b K(x,t)(a_0 + a_1 t + a_2 t^2 + \cdots)dt,$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (4.192) will be converted to a traditional integral in (4.193) or (4.194) where instead of integrating the unknown function $u(x)$, terms of the form $t^n, n \geqslant 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

Example 4.27

Solve the Fredholm integral equation by using the series solution method

$$u(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)u(t)dt$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4.195) leads to

$$\sum_{n=0}^{\infty} a_n x^n = (x+1)^2 + \int_{-1}^{1} \left((xt + x^2t^2)\sum_{n=0}^{\infty} (a_n t^n)\right) dt.$$

Evaluating the integral at the right side gives

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + \left(2 + \frac{2}{3}a_1 + \frac{2}{5}a_3 + \frac{2}{7}a_5 + \frac{2}{9}a_7\right)x + \left(1 + \frac{2}{3}a_0 + \frac{2}{5}a_2 + \frac{2}{7}a_4 + \frac{2}{9}a_6 + \frac{2}{11}a_8\right)x^2.$$

Equating the coefficients of like powers of x in both sides of (4.198) gives

$$a_0 = 1, a_1 = 6, a_2 = \frac{25}{9}, a_n = 0, n \geqslant 3.$$

The exact solution is given by

$$u(x) = 1 + 6x + \frac{25}{9}x^2,$$

obtained upon substituting (4.199) into (4.196).

Example 4.28

Solve the Fredholm integral equation by using the series solution method

$$u(x) = x^2 - x^3 + \int_0^1 (1 + xt)u(t)dt$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4.201) leads to

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - x^3 + \int_0^1 \left((1 + xt)\sum_{n=0}^{\infty}(a_n t^n)\right)dt$$

Evaluating the integral at the right side, and equating the coefficients of like powers of x in both sides of the resulting equation we find

$$a_0 = -\frac{29}{60}, a_1 = -\frac{1}{6}, a_2 = 1, a_3 = -1, a_n = 0, n \geqslant 4.$$

Consequently, the exact solution is given by

$$u(x) = -\frac{29}{60} - \frac{1}{6}x + x^2 - x^3.$$

Example 4.29

Solve the Fredholm integral equation by using the series solution method

$$u(x) = -x^4 + \int_{-1}^1 (xt^2 - x^2t)u(t)dt$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n,$$

into both sides of Eq. (4.206) leads to

$$\sum_{n=0}^{\infty} a_n x^n = -x^4 + \int_{-1}^1 \left((xt^2 - x^2t)\sum_{n=0}^{\infty}(a_n t^n)\right)dt.$$

Evaluating the integral at the right side, and equating the coefficients of like powers of x in both sides of the resulting equation we find

$$a_0 = 0, \quad a_1 = -\frac{30}{133}, a_2 = \frac{20}{133}, a_3 = 0, a_4 = -1$$
$$a_n = 0, \quad n \geqslant 5.$$

Consequently, the exact solution is given by

$$u(x) = -\frac{30}{133}x + \frac{20}{133}x^2 - x^4.$$

Example 4.30

Solve the Fredholm integral equation by using the series solution method

$$u(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} u(t)dt$$

Example 4.30

Solve the Fredholm integral equation by using the series solution method

$$u(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} u(t)dt$$

Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n,$$

into both sides of Eq. (4.211) gives

$$\sum_{n=0}^{\infty} a_n x^n = -1 + \cos x + \int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} (a_n t^n)\right) dt$$

Evaluating the integral at the right side, using the Taylor series of $\cos x$, and proceeding as before we find

$$a_0 = 1, a_{2j+1} = 0, a_{2j} = \frac{(-1)^j}{(2j)!}, j \geqslant 0.$$

Consequently, the exact solution is given by

$$u(x) = \cos x.$$

3.3 Fredholm Integral Equations of the First Kind

Given Fredholm integral equation of the first kind

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt, \ (3.24)$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter.

3.4.1 The Method of Regularization:

The method of regularization was established independently by Phillips [11] and Tikhonov [12]. The method of regularization consists of replacing ill posed problem by well-posed problem. The method of regularization trans forms the linear Fredholm integral equation of the first kind

$$f(x) = \int_a^b K(x,t)u(t)dt\,, \quad x \in D, \qquad (3.25)$$

to the approximation Fredholm integral equation b

$$\mu u_\mu(x) = f(x) - \int_a^b K(x,t)u_\mu(t)dt\,, \quad x \in D. (3.26)$$

where µ is a small positive parameter. It is clear that (4.259) is a Fredholm integral equation of the second kind that can be rewritten

$$u_\mu(x) = \frac{1}{\mu} f(x) - \frac{1}{\mu}\int_a^b K(x,t)u_\mu(t)dt\,, x \in D, \qquad (3.26)$$

Example 4.36

Combine the method of regularization and the direct computation method to solve the Fredholm integral equation of the first kind

$$\frac{1}{4}e^x = \int_0^{\frac{1}{4}} e^{x-t}\,u(t)dt. \qquad (4.27)$$

Using the method of regularization, Equation (4.262) can be transformed to

$$u_\mu(x) = \frac{1}{4\mu}e^x - \frac{1}{\mu}\int_0^{\frac{1}{4}} e^{x-t}\,u_\mu(t)dt. \qquad (4.28)$$

The resulting Fredholm integral equation of the second kind will be solved by the direct computation method. Equation (4.263) can be written as

$$u_\mu(x) = \left(\frac{1}{4\mu} - \frac{\alpha}{\mu}\right)e^x, \qquad (4.29)$$

where

$$\alpha = \int_0^{\frac{1}{4}} e^{-t} u_\mu(t) dt. \qquad (3.30)$$

To determine α, we substitute (4.264) into(4.265), integrate the resulting integralandsolvetofindthat

$$\alpha = \frac{1}{1+4\mu} . \qquad (3.31) .$$

This in turn gives

$$u_\mu(x) = \frac{e^x}{1+4\mu} . \qquad (3.32)$$

The exact solution u(x) of (4.262) can be obtained by

$$u(x) = \lim_{\mu \to 0} u_\mu(x) = e^x.$$

3.4.2 The Homotopy Perturbation Method

The homotopy perturbation method was introduced and developed by. The homotopy perturbation method couples. A homotopy with an embedding parameter p ∈ [0,1] is constructed, and the impeding parameter p is considered a small parameter. The method was derived and illustrated in [10], and several differential equations were examined.

Substituting(4.308)into(4.307),and proceeding as before we obtain the recurrence relation

$$u_0(x) = 0, u_1(x) = f(x),$$

$$u_{n+1}(x) = u_n(x) - \int_a^b K(x,t) un(t) dt , \quad n \geqslant 1. (4.310)$$

If the kernel is separable, i.e. $K(x,t) = g(x)h(t)$,the n the following condition

$$\left| 1 - \int_a^b K(t,t) dt \right| < 1$$

Must be justified for convergence. The proof of this conditionis left to the reader. We will concern ourselves only on the case where K(x,t)=g(x)h(t).The HPM will be used to solve the following Fredholm integral equations of the first kind.

Example 4.41 Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$\frac{1}{3}e^{x} = \int_{0}^{\frac{1}{3}} e^{x-t}\, u(t)dt. \,(4.312)$$

Notice that

$$\left|1 - \int_{0}^{\frac{1}{3}} K(t,t)dt\right| = \frac{2}{3} < 1$$

$$u_0(x) = 0, u_1(x) = \frac{1}{3}e^{x},$$

$$u_{n+1}(x) = u_n(x) - \int_{0}^{\frac{1}{3}} K(x,t)u_n(x)dt\,, n \geqslant 1. \,(4.314)$$

This inturn gives

$$u_0(x) = 0, u_1(x) = \frac{1}{3}e^{x},$$

$$u_2(x) = \frac{1}{3}e^{x} - \int_{0}^{\frac{1}{3}} e^{x-t}u_n(x)dt = \frac{2}{9}\, e^{x},$$

$$u_3(x) = \frac{2}{9}e^{x} - \int_{0}^{\frac{1}{3}} e^{x-t}\frac{2}{9}e^{t}dt = \frac{4}{27}\, e^{x},$$

$$u_4(x) = \frac{4}{27}e^{x} - \int_{0}^{\frac{1}{3}} e^{x-t}\frac{4}{27}e^{t}dt = \frac{8}{81}\, e^{x}, \cdots$$

and so on. Consequently, the approximate solution is given by

$$u(x) = e^{x}\left(\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots\right) = \frac{e^{x}}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^{2} + \left(\frac{2}{3}\right)^{3} + \cdots\right)$$

$$= \frac{e^{x}}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n} = \frac{e^{x}}{3}\frac{1}{1-\frac{2}{3}} = e^{x}$$

Example 4.2 Use the homotopy perturbation methodto solve the Fredholm integral equation of the first kind

$$\frac{1}{4}e^{-x} = \int_{0}^{\frac{1}{4}} e^{x-t}\, u(t)dt. \,(4.312)$$

Notice that

$$\left|1-\int_0^{\frac{1}{4}} K(t,t)dt\right| = \frac{3}{4} < 1$$

$$u_0(x) = 0, u_1(x) = \frac{1}{4}e^{-x},$$

This in turn gives

$$u_2(x) = \frac{1}{4}e^{-x} - \int_0^{\frac{1}{4}} e^{x-t}u_n(x)dt = \frac{3}{16}\ e^{-x},$$

$$u_3(x) = \frac{2}{9}e^{-x} - \int_0^{\frac{1}{4}} e^{x-t}\frac{2}{9}e^{t}dt = \frac{9}{64}\ e^{-x},$$

$$u_4(x) = \frac{3^3}{4^4}e^{-x}, u_5(x) = \frac{3^4}{4^5}e^{-x}, \cdots$$

and so on. Consequently, the approximate solution is given by

$$u(x) = e^{-x}\left(\frac{1}{4} + \frac{3}{4^2} + \frac{3^2}{4^3} + \frac{3^3}{4^4} + \cdots\right) = \frac{e^{-x}}{4}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right)$$

$$= \frac{e^{-x}}{4}\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n = \frac{e^{-x}}{4}\frac{1}{1-\frac{3}{4}} = e^{-x}$$

Example 4.3 Use the homotopy perturbation method to solve the Fredholm integral equation of the first kind

$$x = \int_0^1 xt\,u(t)dt. (4.312)$$

Notice that

$$\left|1-\int_0^1 t^2 dt\right| = \frac{2}{3} < 1$$

$$u_0(x) = 0, u_1(x) = x,$$

This in turn gives

$$u_2(x) = x - \int_0^1 xt^2 dt = \frac{2}{3}x,$$

$$u_3(x) = \frac{2}{3}x - \int_0^1 xt\frac{2}{3}tdt = \frac{4}{9}x = \left(\frac{2}{3}\right)^2 x,$$

$$u_4(x) = \left(\frac{2}{3}\right)^3 x, u_5(x) = \left(\frac{2}{3}\right)^4 x, \cdots$$

and so on. Consequently, the approximate solution is given by

$$u(x) = x\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right) = x\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = x\frac{1}{1-\frac{2}{3}} = 3x$$

Exercises 4.4.2 Use the homotopy perturbation method to solve the Fredholm integral equations of the first kind

1. $\frac{1}{2}(1 - e^{-2})e^{3x} = \int_0^1 e^{3x-4t}\, u(t)dt$
2. $\frac{1}{2}e^{3x} = \int_0^{\frac{1}{2}} e^{3x-3t}\, u(t)dt$
3. $\frac{3}{4}x = \int_0^1 xt^2 u(t)dt$
4. $\frac{6}{5}x^2 = \int_0^1 x^2t^2u(t)dt$
5. $\frac{2}{5}x^2 = \int_{-1}^1 x^2t^2u(t)dt$
7. $\frac{1}{6}x^2 = \int_0^1 x^2t^2u(t)dt$
9. $-\frac{1}{4}x = \int_0^1 xtu(t)dt$