Chapter 2

Volterra Integral Equations

2.0 Introduction

It was stated in Chapter 1 that Volterra integral equations for the first kind Volterra integral equations, the unknown function u(x) occurs only under the integral sign in the form:

$$f(x) = \int_0^x K(x,t)u(t)dt. \qquad (2.1)$$

However, Volterra integral equations of the second kind, the unknown function $u(x)$ occurs inside and outside the integral sign. The second kind is represented in the form:

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt. \qquad (2.2)$$

2.1. The Successive Approximations Method طريقة التقريبات المتتالية

The successive approximations method, also called the Picard iteration method provides a scheme that can be used for solving initial value problems or integral equations.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

Given the linear Volterra integral equation of the second kind

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt, \qquad (2.3)$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt, \qquad (2.4)$$

where the zeroth approximation $u_0(x)$ can be any selective real valued function. We always start with an initial guess for $u_0(x) = u(0)$, will be determined as

$$u_1(x) = f(x) + \lambda \int_0^x K(x,t)u_0(t)dt \ ,$$

$$u_2(x) = f(x) + \lambda \int_0^x K(x,t)u_1(t)dt \ ,$$

$$u_3(x) = f(x) + \int_0^x K(x,t)u_2(t)dt \ , \cdots$$

$$\cdots \quad \cdots$$

$$u_n(x) = f(x) + \lambda \int_0^x K(x,t)u_{n-1}(t)dt \ , (2.5)$$

The question of convergence of $u_n(x)$ is justified by noting the following theorem.

Theorem 3.1 *If $f(x)$ in (2.5) is continuous for the interval $0 \le x \le a$, and the kernel $K(x,t)$ is also continuous in the triangle $0 \le x \le a, 0 \le t \le x$, the sequence of successive approximations $u_n(x), n \ge 0$, converges to the solution $u(x)$ of the integral equation under discussion.*

The successive approximations method, or the Picard iteration method will be illustrated by the following examples.

Example 2.1 Solve the Volterra integral equation by using the successive approximations method

$$u(x) = 1 - \int_0^x (x - t)u(t)dt.$$

For the zeroth approximation $u_0(x)$, we can select $u_0(x) = 1$. The method of successive approximations admits the use of the iteration formula obtain

$$u_1(x) = 1 - \int_0^x (x-t)u_0(t)dt = 1 - \frac{x^2}{2} \ ,$$

$$u_2(x) = 1 - \int_0^x (x-t)u_1(t)dt = 1 - \frac{x^2}{2} + \frac{x^4}{4!} ,$$

$$u_3(x) = 1 - \int_0^x (x-t)u_2(t)dt = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!},$$

$$\cdots \ \cdots \ \cdots$$

$$u_n(x) = 1 - \int_0^x (x-t)u_{n-1}(t)dt = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^n}{n!} + \cdots ,$$

Consequently, the solution $u(x)$ is $u(x) = \lim_{n\to\infty} u_n(x) = \cos x$.

Example 2.2 Solve the Volterra integral equation by using the successive approximations method

$$u(x) = x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt.$$

For the zeroth approximation $u_0(x)$, we can select $u_0(x) = 0$. The method of successive approximations admits the use of the iteration formula

$$u_1(x) = x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_0(t)dt = x + \frac{1}{2}x^2,$$

$$u_2(x) = x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_1(t)dt = x + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^5}{5!},$$

$$u_3(x) = x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_2(t)dt = x + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!},$$

$$\cdots \ \cdots \ \cdots$$

$$u_n(x) = x + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^n}{n!} + \cdots ,$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots + \frac{x^n}{n!} + \cdots - 1 - \frac{x^3}{3!}$$

$$u(x) = \lim_{n\to\infty} u_n(x) = e^x - 1 - \frac{x^3}{3!}.$$

Example 2.3 Solve the Volterra integral equation by using the successive approximations method

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt.$$

For the zeroth approximation $u_0(x)$, we may select $u_0(x) = x$. Then by using the iteration formula

$$u_1(x) = 1 + \frac{x^3}{3} - \ x\sin x + x\cos x,$$

$$u_2(x) = -1 + \frac{x^2}{2} + \frac{x^5}{15} + (2 - 3x - x^2)\sin x + (2 + 3x - x^2)\cos x,$$

$$u_n(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \frac{x^7}{7!} + \ \cdots,$$

Notice that we used the Taylor expansion for sin x and cos x to determine the approximations $u_3(x), u_4(x)$. The solution $u(x)$ is given by

$$u(x) = \lim_{n\to\infty} u_n(x) = \sin x + \cos x.$$

Exercises 2.5

1. $u(x) = x + \int_0^x (x-t)u(t)dt$
2. $u(x) = \frac{1}{6}x^3 - \int_0^x (x-t)u(t)dt$
3. $u(x) = \frac{1}{6}x^3 + \int_0^x (x-t)u(t)dt$
4. $u(x) = 1 + x^2 - \int_0^x (x-t+1)^2 u(t)dt$
5. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt$

2.2 The Adomian Decomposition Method

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad (2.6)$$

or equivalently

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots,$$

where the components $u_n, n \geq 0$ are to be determined in a recursive manner. The decomposition method concerns itself with finding the components $u_0, u_1, u_2, \cdots$ individually. As will be seen through the text, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (3.4) into the Volterra integral equation (3.3) to obtain

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_0^x K(x,t) \left(\sum_{n=0}^{\infty} u_n(t)\right) dt, \quad (2.7)$$

or equivalently

$$u_0(x) + u_1(x) + \cdots = f(x) + \lambda \int_0^x K(x,t) \left(u_0(t) + u_1(t) + u_2(t) + \cdots\right) dt$$

The zeroth component $u_0(x)$ is identified by all terms that are not included

Example 2.4 Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t) dt.$$

We notice that $f(x) = 1, \lambda = -1, K(x,t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form given in (2.7). Substituting the decomposition series (2.7) into both sides of example gives

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \sum_{n=0}^{\infty} u_n(t)\, dt,$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 - \int_0^x [u_0(t) + u_1(t) + u_2(t) + \cdots]\, dt.$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1, u_{k+1}(x) = -\int_0^x u_k(t)dt,$$

so that

$u_0(x) = 1, u_1(x) = -\int_0^x u_0(t)dt = -\int_0^x dt = -x,$

$$u_2(x) = -\int_0^x u_1(t)dt = -\int_0^x -tdt = \frac{x^2}{2!},$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x \frac{t^2}{2!}dt = -\frac{x^3}{3!},$$

$$u_4(x) = -\int_0^x u_3(t)dt = -\int_0^x -\frac{t^3}{3!}dt = \frac{t^4}{4!},$$

and so on. Using (3.4) gives the series solution:

$$u(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

that converges to the closed form solution: $u(x) = e^{-x}$.

Example 2.5 Solve the following Volterra integral equation:

$$u(x) = 1 + \int_0^x (t - x)u(t)dt.$$

We notice that $f(x) = 1,\ \lambda = 1,\ K(x,t) = t - x$. Substituting the decomposition series into both sides to gives

$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x (t - x)\,[u_0(t) + u_1(t) + u_2(t) + \cdots]\,dt$

Proceeding as before we set the following recurrence relation:

$$u_0(x) = 1, u_1(x) = -\int_0^x (t - x)u_0(t)dt = -\int_0^x (t - x)dt = -\frac{x^2}{2!},$$

$$u_2(x) = -\int_0^x (t - x)u_1(t)dt = \frac{1}{2!}\int_0^x (t - x)t^2dt = \frac{x^4}{4!},$$

$$u_3(x) = -\int_0^x (t - x)u_2(t)dt = -\frac{1}{4!}\int_0^x (t - x)t^4dt = -\frac{x^6}{6!},$$

$$u_4(x) = -\int_0^x (t-x)u_3(t)dt = \frac{1}{6!}\int_0^x (t-x)t^6 dt = -\frac{x^8}{8!},$$

and so on. The solution in a series form is given by

$$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

and in a closed form by $u(x) = \cos x$, obtained upon using the Taylor expansion for cos x.

Example 2.7 Solve the following Volterra integral equation:

$$u(x) = 1 - x - \frac{x^2}{2!} - \int_0^x (t-x)u(t)dt.$$

Notice that $f(x) = 1 - x - \frac{x^2}{2!}, \lambda = -1, K(x,t) = t - x$. Substituting the decomposition series into both sides to gives or equivalently. This allows us to set the following recurrence relation that gives

$$u_0(x) = 1 - x - \frac{x^2}{2!},$$

$$u_1(x) = -\int_0^x (t-x)u_0(t)dt = -\int_0^x (t-x)\left(1 - t - \frac{t^2}{2!}\right)dt = \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!},$$

$$u_2(x) = -\int_0^x (t-x)u_1(t)dt = \int_0^x (t-x)(\frac{t^2}{2!} - \frac{t^3}{3!} - \frac{t^4}{4!})dt = \frac{x^4}{4!} - \frac{x^5}{5!} - \frac{x^6}{6!},$$

$$u_3(x) = -\int_0^x (t-x)u_2(t)dt = \int_0^x (t-x)(\frac{t^4}{4!} - \frac{t^5}{5!} - \frac{t^6}{6!})dt = \frac{x^6}{6!} - \frac{x^7}{7!} - \frac{x^8}{8!},$$

$$u_4(x) = -\int_0^x (t-x)u_3(t)dt = \int_0^x (t-x)(\frac{t^6}{6!} - \frac{t^7}{7!} - \frac{t^8}{8!})dt = \frac{x^8}{8!} - \frac{x^9}{9!} - \frac{x^{10}}{10!},$$

and so on. The solution in a series form is given by

$$u(x) = 1 - (x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots),$$

and in a closed form by $u(x) = 1 - \sinh x$, obtained upon using the Taylor expansion for sinh x.

Example 2.8 We now consider the Volterra integral equation:

$$u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt.$$

Identifying the zeroth component u0(x) by the first four terms that are not included under the integral sign, and using the ADM we set the recurrence relation as

$$u_0(x) = x + x^4 + \frac{x^2}{2} + \frac{x^5}{5},$$

$$u_1(x) = -\int_0^x u_0(t)dt = -\int_0^x \left(t + t^4 + \frac{1}{2}t^2 + \frac{1}{5}t^5\right)dt = -\frac{x^2}{2} - \frac{x^5}{5} - \frac{x^3}{6} - \frac{x^6}{30},$$

$$u_2(x) = -\int_0^x u_1(t)dt = -\int_0^x \left(-\frac{t^2}{2} - \frac{t^5}{5} - \frac{t^3}{6} - \frac{t^6}{30}\right)dt = \frac{x^3}{6} + \frac{x^6}{30} + \frac{x^4}{24} + \frac{x^7}{210}$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x (\frac{t^3}{6} + \frac{t^6}{30} + \frac{t^4}{24} + \frac{t^7}{210})dt$$

$$= -\frac{x^4}{24} - \frac{x^7}{210} - \frac{x^5}{120} - \frac{x^8}{1680},$$

We can easily notice the appearance of identical terms with opposite signs. This phenomenon of such terms is called noise terms phenomenon that will be presented later. Canceling the identical terms with opposite terms gives the exact solution $u(x) = x + x^4$.

Example 2.9 We finally solve the Volterra integral equation:

$$u(x) = 2 + \frac{1}{3}\int_0^x xt^3 u(t)dt.$$

Proceeding as before we set the recurrence relation

$$u_0(x) = 2,$$

$$u_1(x) = \frac{1}{3}\int_0^x xt^3 u_0(t)dt = \frac{2}{3}\int_0^x xt^3 dt = \frac{x^5}{6},$$

$$u_2(x) = \frac{1}{3}\int_0^x xt^3 u_1(t)dt = \frac{1}{18}\int_0^x xt^8 dt = \frac{x^{10}}{162},$$

$$u_3(x) = \frac{1}{3}\int_0^x xt^3 u_2(t)dt = \frac{1}{486}\int_0^x xt^{13} dt = \frac{x^{15}}{6804},$$

$$u_4(x) = \frac{1}{3}\int_0^x xt^3 u_3(t)dt = \frac{1}{20412}\int_0^x xt^{18}dt = \frac{x^{20}}{387828},$$

and so on. The solution in a series form is given by

$u(x) = 2 + \frac{x^5}{6} + \frac{x^{10}}{6.3^3} + \frac{x^{15}}{6.3^4.14} + \frac{x^{20}}{6.3^5.19} + \cdots$. It seems that an exact solution is not obtainable. The obtained series solution can be used for numerical purposes. The more components that we determine the higher accuracy level that we can achieve.

Exercises 2.2.1

1. $u(x) = 6x - x^3 + \int_0^x (x - t)u(t)dt$

2. $u(x) = x - \frac{2}{3}x^3 - 2\int_0^x u(t)dt$

3. $u(x) = 1 - x + \int_0^x (x - t)u(t)dt$

4. $u(x) = x - \int_0^x (x - t)u(t)dt$

2.3 The Modified Decomposition Method

we can set $f(x) = f_1(x) + f_2(x)$. we introduce a qualitative change in the formation of the recurrence relation. To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation:

$u_0(x) = f_1(x),\ u_1(x) = f_2(x) + \lambda \int_0^x K(x,t)u_0(t)dt,$

$u_{k+1}(x) = \lambda \int_0^x K(x,t)u_k(t)dt,\ k \geq 1.$

This shows that the difference between the standard recurrence relation and the modified recurrence relation rests only in the formation of the first two components $u_0(x)$ and $u_1(x)$ only. The other components

Example 2.7 Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sin x + (e - e^{\cos x}) - \int_0^x e^{\cos t}\, u(t)dt.$$

We first split $f(x)$ given by $f(x) = \sin x + (e - e^{\cos x})$, into two parts, namely

$f_1(x) = \sin x\,,\ f_2(x) = e - e^{\cos x}$.

We next use the modified recurrence formula to obtain

$u_0(x) = f_1(x) \ = \ sin\ x,$

$$u_1(x) = (e - e^{\cos x}) - \int_0^x e^{\cos t}\ u0(t)dt \ = \ 0,$$

$$u_2(x) = -\int_0^x e^{\cos t} u_1(t)dt \ = \ 0,\ \ k \geq 1.$$

$$0 = u_2(x) = u_3(x) = u_4(x)\cdots$$

$$u(x) = \sum_{n=0}^{\infty} u_n(x) = u_0(x) + u_1(x) + u_2(x) + \cdots$$

It is obvious that each component of u_j , $j \geq 1$ is zero. This in turn gives the exact solution by $u(x) = \sin x$.

Example 2.8 Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sec x \tan x + (e^{\sec x} - e) - \int_0^x e^{\sec t}\, u(t)dt, \qquad x < \frac{\pi}{2}.$$

Proceeding as before we split $f(x)$ into two parts

$$u_0(x) = f_1(x) = \sec x \tan x\,, \qquad f_2(x) = e^{\sec x} - e\,.$$

We next use the modified recurrence formula to obtain

$u_1(x) = 0 = u_k(t), k \geq 2$. This in turn gives the exact solution by

$u(x) = \sec x \tan x$.

Example 2.10 Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = 1 + x^2 + \cos x - x - \frac{1}{3}x^3 - \sin x + \int_0^x u(t)dt.$$

$f(x) = 1 + x^2 + \cos x - x - \frac{1}{3}x^3 - \sin x$, into two parts, the first three terms and the next three terms, hence we set $f_1(x) = 1 + x^2 + \cos x$, $f_2(x) = -(x + \frac{1}{3}x^3 + \sin x)$. Using the modified recurrence formula gives $u_0(x) = 1 + x^2 + \cos x$,

$u_1(x) = -(x + \frac{1}{3}x^3 + \sin x) + \int_0^x (1 + t^2 + \cos t)\, dt = 0, u_k(x) \ = 0,$

$$u(x) = 1 + x^2 + \cos x.$$

Exercises 2.2.2

1. $u(x) = 2x + 3x^2 + \left(e^{x^2} + x^3 - 1\right) - \int_0^x (e^{t^2+t^3})u(t)dt$
2. $u(x) = e^{-x^2} - \frac{x}{2}\left(1 - e^{-x^2}\right) - \int_0^x xtu(t)dt.$
3. $u(x) = \cosh x + x\sinh x - \int_0^x xtu(t)dt.$
4. $u(x) = 1 + \sin x + x + x^2 - x\cos x - \int_0^x xtu(t)dt.$
5. $u(x) = \sec x^2 - (1 - e^{\tan x})x - x\int_0^x e^{\tan t}\,u(t)dt$
6. $u(x) = \cosh x + \frac{x}{2}(1 - e^{\sinh x}) + \frac{x}{2}\; x\int_0^x e^{\sinh t}\,u(t)dt$
7. $u(x) = x^3 - x^5 + 5\int_0^x t\,u(t)dt.$

2.4 The Laplace Transform Method

The Laplace transform method is a powerful technique that can be used for solving initial value problems and integral equations as well. In the convolution theorem for the Laplace transform, it was stated that if the kernel K(x, t) of the integral equation:

$$u(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt,$$

depends on the difference $x - t$, then it is called a difference kernel. Examples of the difference kernel are e^{x-t}, $cos(x - t)$, and $x - t$. The integral equation can thus be expressed as

$$u(x) = f(x) + \lambda\int_0^x K(x-t)u(t)dt,$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by

$$\mathcal{L}\{f_1(x)\} = F_1(s), \mathcal{L}\{f_2(x)\} = F_2(s).$$

The Laplace convolution product of these two functions is defined by

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt = (f_2 * f_1)(x)$$
$$= \int_0^x f_2(x-\ t)f_1(t)dt \,. (2.12)$$

the convolution product $(f_1 * f_2)(x)$ is given by

$$\mathcal{L}\{(f_1 * f_2)(x)\} \ = \ \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s). (2.13)$$

we find

$$U(s) = F(s) + \lambda V(s)U(s), \qquad (*)$$

where

$$U(s) = \mathcal{L}\{u(x)\}, \qquad V(s) = \mathcal{L}\{K(x)\}, \qquad F(s) = \mathcal{L}\{f(x)\}. (2.14)$$

Solving (3.213) for $U(s)$ gives

$$U(s) = \frac{F(s)}{1-\lambda V(s)} \ , \lambda V(s) \neq 1. (2.15)$$

The solution u(x) is obtained by taking the inverse Laplace transform of both sides of (3.215) where we find

$$u(x) = \mathcal{L}^{-1}\left\{\frac{F(s)}{1-\lambda V(s)}\right\}. (2.16)$$

Recall that the right side of (3.216) can be evaluated by using Table 1.1 in Section 1.5. The Laplace transform method for solving Volterra integral equations will be illustrated by studying the following examples.

Example 2.23 Solve the VIE by using the Laplace transform method

$$u(x) = 1 + \int_0^x u(t)dt.$$

Notice that the kernel $K(x-t) = 1,\ \lambda = 1$. Taking Laplace transform of both sides gives

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} + \mathcal{L}\{1 * u(x)\},$$

so that $\mathcal{L}\{u(x)\} = \frac{1}{s} + \frac{1}{s}\mathcal{L}\{u(x)\}$, or equivalently $\mathcal{L}\{u(x)\} = \frac{1}{s} - 1$. By taking the inverse Laplace transform of both sides $u(x) = e^x$.

Example 2.24 Solve the Volterra integral equation by using the Laplace transform method $u(x) = 1 - \int_0^x (x-t)u(t)dt$.

Notice that the kernel $K(x-t) = (x-t), \lambda = -1$. Taking Laplace transform of both sides gives

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\{(x-t) * u(x)\},$$

so that $U(s) = \frac{1}{s} - \frac{1}{s^2}U(s)$, or equivalently $U(s) = \frac{s}{s^2+1}$. By taking the inverse Laplace transform of both sides, the exact solution $u(x) = \cos x$, is readily obtained.

Example 2.25 Solve the Volterra integral equation by using the Laplace transform method $u(x) = \frac{1}{3!}x^3 - \int_0^x (x-t)u(t)dt$. Taking Laplace transform of both sides gives $\mathcal{L}\{u(x)\} = \frac{1}{3!}\mathcal{L}\{x^3\} - \mathcal{L}\{(x) * u(x)\}$. This gives

$\mathcal{L}\{u(x)\} = \frac{1}{3!} \times \frac{3!}{s^4} - \frac{1}{s^2}\mathcal{L}\{u(x)\}$, so that

$\mathcal{L}\{u(x)\} = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$. Taking the inverse Laplace transform of both sides of gives the exact solution $u(x) = x - \sin x$.

Example 2.26 Solve the *VIE* by using the Laplace transform method

$$u(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)\, u(t)dt$$

Recall that we should use the linear property of the Laplace transforms. Taking Laplace transform of both sides gives

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{\sin x + \cos x\} + 2\mathcal{L}\{\sin(x-t) * u(x)\},$$

so that $U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + 1 + \frac{2}{s^2+1}U(s)$, (3.234) or equivalently $U(s) = \frac{1}{s-1}$. Taking the inverse Laplace transform of both sides gives the exact solution $u(x) = e^x$.

Exercises 2.2.6 Use the Laplace transform method to solve the *VIE*:

1. $u(x) = x + \int_0^x (x-t)u(t)dt$
2. $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$