Chapter 4
Laplace Equation

ظهرت هذه المعادله اول مره عام 1752 عن طريق اويلر في موضوع هيدروداينمك , وسميت المعادله باسم لابلاس لان بير سيمون دي لا بلاس اول من درس حلها بشكل واسع وبشكل خاص حول جذب الاجسام وهي من اهم المعادلات في المعادلات التفاضليه الجزئيه

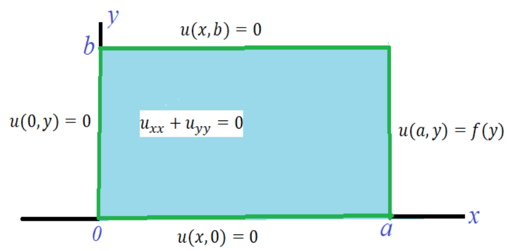
Chapter Four Laplace Equation.

$u_{xx} + u_{yy} = 0$ *or* $\nabla^2 u(x,y) = 0$ (1) in two dimensions

$u_{xx} + u_{yy} + u_{zz} = 0$ or $\nabla^2 u(x,y,z) = 0$ (2) in Three dimensions

The problem of finding a solution of Laplace equation on given Boundary values is known. Dirichlet problem and the values of the normal derivative are prescribed on the boundary

4.1. Dirichlet problem for a Rectangle:



$$u_{xx} + u_{yy} = 0 \qquad u(x,b) = 0,$$

$$u(x,0) = 0, u(x,b) = 0, \quad 0 \le y \le b \qquad (3)$$

$$u(0,y) = 0, u(a,y) = f(y), \quad 0 < x < a \qquad (4)$$

Let

$$u(x,y) = X(x)Y(y)\,, \qquad X(x) \ne 0, Y(y) \ne 0 \qquad (5)$$

$$X''Y + XY'' = 0$$

$$\frac{X'}{X} = -\frac{Y'}{Y} = \lambda$$

λ is separation Constant

$$X'' - \lambda X = 0 \qquad (6)$$

$$Y'' + \lambda Y = 0 \qquad (7)$$

$$\begin{aligned} u(x,0) = 0 &\Rightarrow X(x)Y(0) = 0 \Rightarrow X(x) \ne 0 \Rightarrow Y(0) = 0 \\ u(x,b) = 0 &\Rightarrow X(x)Y(b) = 0 \Rightarrow X(x) \ne 0 \Rightarrow Y(b) = 0 \end{aligned} \qquad (8)$$

$$u(0,y) = 0 \Rightarrow X(0)Y(y) = 0 \Rightarrow Y(y) \ne 0 \Rightarrow X(0) = 0 \qquad (9)$$

to solve (7) + (8)

$$m^2 + \lambda = 0 \Longrightarrow m^2 = -\lambda$$

- If $\lambda = 0 \Longrightarrow Y(y) = c_1 y + c_2$

$$Y(0) = 0, \;\; Y(b) = 0$$

$$Y(0) = c_2 = 0\,, Y(b) = c_1 b = 0 \Longrightarrow c_1 = 0$$

$Y(y) = 0$ Contradiction (no solution) (neglect)

- if $\lambda < 0$ then $\lambda = -\delta^2 \Longrightarrow m^2 + \lambda = 0 \Longrightarrow m^2 = \delta^2 \Longrightarrow \; m = \pm\delta$

$$Y(y) = c_1 e^{\delta y} + c_2 e^{-\delta y}, Y(0) = c_1 + c_2 = 0 \Longrightarrow c_1 = -c_2$$

$$Y(b) = c_1 e^{\delta b} + c_2 e^{-\delta b} = 0$$

$$c_2\left(-e^{\delta b} + e^{-\delta b}\right) = 0 \Longrightarrow c_2 = 0 = c_1$$

$Y(y) = 0$ Contradiction (no solution)

- if $\lambda > 0$, let $\lambda = \delta^2 \Longrightarrow \;\; m^2 = -\delta^2$

$$m = \pm\delta i, \;\; Y_1 = \sin\delta y, \;\; Y_2 = \cos\delta y$$

$$Y(y) = c_1 \cos\delta y + c_2 \sin\delta y$$

$$Y(0) = 0 \to c_1 = 0, \;\; Y(y) = c_2 \sin\delta y$$

$$Y(b) = 0 \to c_2 \sin\delta b = 0, c_2 \neq 0 \Longrightarrow \sin\delta b = 0$$

$$\delta b = n\pi, \Rightarrow \delta_n = \frac{n\pi}{b}\,, \;\; \lambda_n = \left(\frac{n\pi}{b}\right)^2, \;\; n = 1,2,\cdots$$

$$Y_n(y) = c_n \sin\frac{n\pi y}{b} \qquad (10)$$

to solve (6) + (9): $\quad m^2 - \lambda = 0 \Longrightarrow m^2 = \left(\frac{n\pi}{b}\right)^2 \Longrightarrow \; m = \pm\,\frac{n\pi}{b}$

$$X_n(0) = 0$$

$$X_n(x) = k_n e^{\frac{n\pi}{b}x} + j_n e^{-\frac{n\pi}{b}x}$$

$$X_n(0) = k_n + j_n = 0 \;\; \Longrightarrow j_n = -k_n$$

$$X_n(x) = k_n\left(e^{\frac{n\pi}{b}x} - e^{-\frac{n\pi t}{b}x}\right) = 2k_n \sinh\frac{n\pi}{b}x \qquad (11)$$

The fundamental solution

$$u_n(x,y) = 2k_n \sinh\frac{n\pi}{b}x \cdot c_n \sin\frac{n\pi}{b}y \Longrightarrow \; b_n = 2k_n c_n$$

$$u_n(x,y) = b_n \sinh\frac{n\pi}{b}x\, sin\frac{n\pi}{b}y \qquad (12)$$

the general solution is

$$u(x,y)=\sum_{n=1}^{\infty} b_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b} \qquad (13)$$

Last Condition of (4): $u(a,y)=f(y)$

$$u(a,y)=\sum_{n=1}^{\infty}\left(b_n \sinh\frac{n\pi a}{b}\right)\sin\frac{n\pi y}{b}=f(y) \quad (14)$$

by Fourier sine series of period $2b$

$$b_n \sinh\frac{n\pi a}{b}=\frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy \qquad (15)$$

Example 1: Let R be a rectangle $0<x<3, 0\le y\le 2$

$$f(y)=\begin{cases} y & 0\leqslant y\leqslant 1\\ 2-y & 1\leqslant y\leqslant 2\end{cases}$$

$$u(0,y)=0, u(3,y)=f(y),\ \ u(x,0)=0, u(x,2)=0$$

Sol: From (15)

$$b_n\sinh\frac{3n\pi}{2}=\int_0^2 f(y)\sin\frac{n\pi y}{2}\,dy$$

$$=\int_0^1 y\sin\frac{n\pi y}{2}\,dy+\int_1^2 (2-y)\sin\frac{n\pi y}{2}\,dy$$

$$u=y\,, dv=\sin\frac{n\pi y}{2}dy\,,\ \ du=dy,\ \ v=-\frac{2}{n\pi}\cos\frac{n\pi y}{2};$$

$$u=2-y,\ \ du=\sin\frac{n\pi y}{2}dy\,,\ \ du=-dy,\ \ v=-\frac{2}{n\pi}\cos\frac{n\pi y}{2}$$

$$b_n\sinh\frac{3n\pi}{2}=-\frac{2}{n\pi}y\cos\frac{n\pi y}{2}\bigg|_0^1+\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi y}{2}\bigg|_0^1-\frac{2}{n\pi}(2-y)\cos\frac{n\pi y}{2}\bigg|_1^2$$
$$-\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi y}{2}\bigg|_1^2$$

$$=-\frac{2}{n\pi}\cos\frac{n\pi}{2}+0+\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi}{2}-0-0+\frac{2}{n\pi}\cos\frac{n\pi}{2}-0+\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi}{2}$$

$$=\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{8\sin\frac{n\pi}{2}}{n^2\pi^2 \sinh\frac{3n\pi}{2}}$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{8\sin\frac{n\pi}{2}}{n^2\pi^2 \sinh\frac{3n\pi}{2}} \sinh\frac{n\pi x}{2} \sin\frac{n\pi y}{2}$$

Q1 page 611

Example 2: Find a solution $u(x,y)$ of Laplace equation in the rectangle $0 < x < a, 0 < y < b$, $u(0,y) = 0$, $u(a,y) = 0$, $u(x,0) = 0$, $u(x,b) = g(x)$

$$g(x) = \begin{cases} x & 0 \le x \le \frac{a}{2} \\ a - x & \frac{a}{2} \le x \le a \end{cases}$$

If $\lambda \ge 0$, $\lambda = \left(\frac{n\pi}{a}\right)^2$

$$X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0$$

$$X(0) = 0, X(a) = 0 \implies \text{Y(y)} = 0 \text{ contradiction no solution}$$
$$Y(0) = 0 \implies X(x) = 0 \text{ contradiction no solution}$$

if $\lambda < 0$. Let $\lambda = -\delta^2 \Rightarrow m^2 = \lambda \Longrightarrow m = \pm\delta i$

$$X(x) = c_1 \sin\delta x + c_2 \cos\delta x$$

$$X(0) = c_2 = 0, \quad X(x) = c_1 \sin\delta x$$

$$X(a) = c_1 \sin\delta a = 0; \; c_1 \ne 0 \Longrightarrow \sin\delta a = 0 \text{ iff } \delta a = n\pi$$

$$\Rightarrow \delta_n = \frac{n\pi}{a}, \quad \lambda_n = -\left(\frac{n\pi}{a}\right)^2, \quad n = 1,2,3,\cdots$$

$$X_n(x) = c_n \sin\frac{n\pi x}{a} \qquad (16)$$

$$Y''(y) - \frac{n^2\pi^2}{a^2} Y(y) = 0 \Longrightarrow m^2 = \frac{n^2\pi^2}{a^2} \Longrightarrow m = \pm\frac{n\pi}{a}$$

$$Y_n(y) = k_n e^{\frac{n\pi y}{a}} + j_n e^{-\frac{n\pi y}{a}}$$

$$Y_n(0) = 0 \Longrightarrow k_n + j_n = 0 \Longrightarrow j_n = -k_n$$

$$Y_n(y) = k_n\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right) = 2k_n \sinh\frac{n\pi y}{a} \qquad (17)$$

$$u_n(x,y) = c_n \sin\frac{n\pi x}{a} \cdot k_n \sinh\frac{n\pi y}{a}, \quad b_n = c_n k_n .$$

$$u_n(x,y) = b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \qquad (18)$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \qquad (19)$$

$$u(x,b) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi b}{a} = g(x)$$

$$b_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx \qquad (20)$$

Q2

Example 3: Find a solution u of Laplace equation in rectangle

$$u(0,y) = 0,\ \ u(a,y) = f(y),\ \ u(x,0) = h(x),\ \ u(x,b) = 0$$

$$X'' - \lambda X = 0, Y'' + \lambda Y = 0$$

(i) $u(0,y) = 0,\ u(a,y) = 0,\ u(x,0) = h(x),\ u(x,b) = 0$

(ii) $u(0,y) = h(y), u(a,y) = 0, u(x,0) = 0, u(x,b) = 0$

****

4.2. Dirichlet problem for a Circle

$\nabla^2 u = u_{xx} + u_{yy} = 0$ (1)   Cartesian coordinate of Two dimensions

$\nabla_u^2 = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ (2)   Polar coordinate

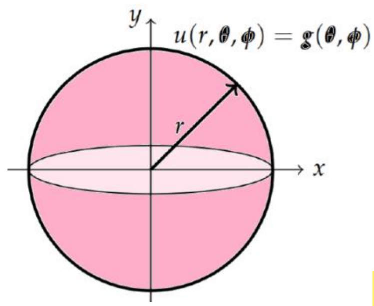
$$x = r\cos\theta\,,\ \ y = r\sin\theta$$

$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$ Cartesian of Three dimensions

$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$ (3)   Cylindrical $r(x,y), \theta(x,y)$

$$\nabla^2 u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\phi\phi} + \frac{\cot\phi}{r^2}u_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}\ \theta(x,y) = 0 \quad (4)$$

Spherical coordinate.



$$u_x(r,\theta) = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad (5)$$

$$r(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{\frac{1}{2}} \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\tan\theta = \frac{y}{x} \Longrightarrow \sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} \Longrightarrow \frac{1}{\cos^2\theta}\frac{\partial \theta}{\partial x} = \frac{-r\sin\theta}{r^2\cos^2\theta} = \frac{-\sin\theta}{r\cos^2\theta}$$

$$\frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin\theta$$

$$r_x = \cos\theta\,, \theta_x = -\frac{1}{r}\sin\theta \qquad (*)$$

$$u_x(r,\theta) = u_r\cos\theta - u_\theta\frac{1}{r}\sin\theta \qquad (6)$$

$$u_y(r,\theta) = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \qquad (7)$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta, \quad \tan\theta = \frac{y}{x} \Longrightarrow \sec^2\theta\frac{\partial \theta}{\partial y} = \frac{1}{x}$$

$$\Longrightarrow \frac{1}{\cos^2\theta}\frac{\partial \theta}{\partial y} = \frac{1}{r\cos\theta} \Longrightarrow \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta$$

$$r_y = \sin\theta\,, \theta_y = \frac{1}{r}\cos\theta \qquad (**)$$

$$u_y(r,\theta) = u_r\sin\theta + \frac{1}{r}u_\theta\cos\theta \qquad (8)$$

$$u_{xx}(r,\theta) = \frac{\partial}{\partial r}u_x\frac{\partial r}{\partial x} + \frac{\partial u_x}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial}{\partial r}\left(u_r\cos\theta - u_\theta\frac{1}{r}\sin\theta\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(u_r\cos\theta - \frac{1}{r}\sin\theta\, u_\theta\right)\frac{\partial \theta}{\partial x}$$

$$= \left(\cos\theta\, u_{rr} - \frac{1}{r}u_{\theta r}\sin\theta + \frac{1}{r^2}u_\theta\sin\theta\right)\cos\theta$$

$$+ \left(u_{r\theta}\cos\theta - u_r\sin\theta - \frac{1}{rx}u_{\theta\theta}\sin\theta - \frac{1}{r}u_\theta\cos\theta\right)\left(-\frac{1}{r}\sin\theta\right)$$

$$u_{xx} = \cos^2\theta u_{rr} - \frac{1}{r}u_{\theta r}\sin\theta\cos\theta + \frac{1}{r^2}u_\theta\sin\theta\cos\theta - \frac{1}{r}u_{r\theta}\sin\theta\cos\theta$$

$$+ \frac{1}{r}u_r\sin^2\theta + \frac{1}{r^2}u_{\theta\theta}\sin^2\theta + \frac{1}{r^2}u_\theta\cos\theta \qquad (9)$$

$$u_{yy}(r,\theta)=\frac{\partial}{\partial y}\left(u_r\sin\theta+\frac{1}{r}u_\theta\cos\theta\right)\frac{\partial\theta}{\partial y}$$

$$=\left(u_r\sin\theta\,\theta+\frac{1}{r}u_\theta\cos\theta\ \theta\right)r\frac{\partial r}{\partial y}+\left(u_r\sin\theta\ \theta+\frac{1}{r}u_\theta\cos\theta\ \theta\right)\theta$$

$$u_{yy}=u_{rr}\sin^2\theta-\frac{1}{r^2}u_\theta\sin\theta\cos\theta+\frac{1}{r}u_{\theta r}\sin\theta\cos\theta+\frac{1}{r}u_{r\theta}\sin\theta\cos\theta$$

$$+\frac{1}{r}u_r\cos^2\theta+\frac{1}{r^2}u_{\theta\theta}\cos^2\theta-\frac{1}{r^2}u_\theta\sin\theta\cos\theta\quad(10)$$

$$u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$$

Remark Let u be periodic in θ with period 2π and $r\leqslant a, u(a,\theta)=f(\theta)$

Apply the method of separation of variable

$$u(r,\theta)=R(r)\phi(\theta),\quad R(r)\neq 0,\phi(\theta)\neq 0,\qquad(11)$$

$$R''(r)\phi(\theta)+\frac{1}{r}R'(r)\phi(\theta)+\frac{1}{r^2}R(r)\phi''(\theta)=0]\div R(r)\phi(\theta)$$

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\phi''}{\phi}=\lambda$$

$$r^2R''+rR'-\lambda R=0\qquad(12)$$

$$\phi''+\lambda\phi=0\qquad(13)$$

1-if $\lambda<0$ let $\lambda=-\delta^2$ to solve (13)

$$\phi''(\theta)-\delta^2\phi(\theta)=0\Longrightarrow\ m^2=\delta^2\Longrightarrow m=\pm\delta$$

$$\phi(\theta)=c_1e^{\delta\theta}+c_2e^{-\delta\theta}\qquad(14)$$

since $\phi(\theta)$ is periodic of period 2π then $\phi(\theta)=\phi(\theta+2\pi)$

$$c_1e^{\delta\theta}+c_2e^{-\delta\theta}=c_1e^{\delta(\theta+2\pi)}+c_2e^{-\delta(\theta+2\pi)}$$

$$c_1e^{\delta\theta}\left(1-e^{2\delta\pi}\right)=c_2e^{-\delta(\theta+2\pi)}\left(1-e^{2\delta\pi}\right)$$

$$c_1e^{\delta\theta}=c_2e^{-\delta(\theta+2\pi)}$$

which possible only when $c_1=c_2=0\Longrightarrow\phi(\theta)=0$ contradiction no solution

2- $\lambda=0$: $\phi(\theta)=\phi(\theta+2\pi)\Longrightarrow\phi(0)=\phi(2\pi)$ from (13) we get

$$\phi(\theta)=c_1\theta+c_2\qquad(15)$$

$$c_2=2\pi c_1+c_2\Rightarrow 2\pi c_1=0\Rightarrow c_1=0$$

$$\phi(\theta)=c_2\qquad(16)$$

from (12) $r^2R'' + rR' = 0$

$$R'' + \frac{1}{r}R' = 0 \quad \text{let} \quad V = \frac{dR}{dr}$$

$$V' + \frac{1}{r}V = 0 \quad \Longrightarrow \quad I.f = e^{\int \frac{1}{r}dr} = r$$

$$[rV]' = 0 \Longrightarrow rV = k_1 \Longrightarrow V = \frac{k_1}{r}$$

$$\frac{dR}{dr} = \frac{k_1}{r} \Longrightarrow dR = \frac{k_1}{r}dr \Longrightarrow R(r) = k_1 \ln r + k_2$$

$\lim_{r\to 0} R(\theta) = k_1(-\infty) \Longrightarrow R \to \pm\infty$ is unbounded $\Rightarrow u(r,\theta)$ is unbounded unless $k_1 = 0$

$$\Longrightarrow R(r) = k_2 \qquad (17)$$

$$u(r,\theta) = c_2 \cdot k_2 \Rightarrow u_0(r,\theta) = b_0 \qquad (18)$$

3- If $\lambda > 0$ let $\lambda = \delta^2$ from (13) we get

$$\phi(\theta) + \delta^2\phi(\theta) = 0 \text{ from (13) we get}$$

$$m^2 + \delta^2 = 0 \Longrightarrow m^2 = -\delta^2 \Rightarrow m = \pm\delta i$$

$$\phi(\theta) = c_1 \sin\delta\theta + c_2\cos\delta\theta \qquad (19)$$

$$\phi(\theta) = \phi(\theta + 2\pi)$$

$$\theta = 0 \Longrightarrow \phi(0) = \phi(2\pi) \Longrightarrow c_2 = c_1 \sin 2\pi\delta + c_2 \cos 2\pi\delta \ \ (*)$$

$$\theta = -2\pi \Longrightarrow \phi(-2\pi) = \phi(0) \Longrightarrow -c_1 \sin 2\pi\delta + c_2\cos 2\pi\delta = c_2 \ (**)$$

$$c_1 \sin 2\pi\delta + c_2 \cos 2\pi\delta = -c_1 \sin 2\pi\delta + c_2 \cos 2\pi\delta$$

$$2c_1 \sin 2\pi\delta = 0 \Longrightarrow c_1 \sin 2\pi\delta = 0 \quad (***)$$

Put (***) in (*) $c_2 = c_2 \cos 2\pi\delta \Longrightarrow \cos 2\pi\delta = 1 \Longleftrightarrow 2\pi\delta = 2n\pi$

Or $\delta_n = n, \ \lambda_n = \delta_n^2 = n^2, \quad n = 1,2,3,\cdots$ so (19) will be

$$\phi_n(\theta) = c_n \sin n\theta + d_n \cos n\theta \qquad (20)$$

$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = 0,$ Euler equation

Let $R_n(r) = r^m, r \neq 0$

$$r^2(m(m-1)r^{m-2}) + rmr^{m-1} - n^2r^m = 0$$

$$r^m(m^2 - m + m - n^2) = 0 \Longrightarrow r^m(m^2 - n^2) = 0, r \neq 0$$

$$m = \pm n\pi$$

$$R_n(r) = k_n r^n + j_n r^{-n}$$

$\lim_{r\to 0} R_n(r) = k_n.0 + \frac{j_n}{0} = \pm\infty \Rightarrow u_n(r,\theta) = \pm\infty$ this is impossible unless $j_n = 0$

$$R_n(r) = k_n r^n, \quad \delta = n \qquad (21)$$

$$u_n(r,\theta) = k_n r^n (c_n \sin n\theta + d_n \cos n\theta)$$

$$u(r,\theta) = u_0(r,\theta) + \sum_{n=1}^{\infty} u_n(r,\theta), b_0 = \frac{d_0}{2}$$

$$u(r,\theta) = \frac{d_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \sin n\theta + d_n \cos n\theta) \qquad (22)$$

$$u(a,\theta) = \frac{d_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \sin n\theta + d_n \cos n\theta) = f(\theta)$$

$$\delta = n, \quad a^\delta c_n = \frac{2}{l}\int_0^l f(\theta) \sin \delta\theta \, d\theta \qquad (23)$$

$$a^\delta d_n = \frac{2}{l}\int_0^l f(\theta) \cos \delta\theta \, d\theta \qquad (24)$$

Example 1: Find the solution $u(r,\theta)$ of laplace equation in half circle $0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \pi, u(r,0) = 0, u(r,\pi) = 0, \ u(1,\theta) = \sin\theta$

Example 2: Find the solution $u(r,0)$ of laplace equation in semi circle $0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant \frac{\pi}{3}, u(r,0) = 0, u\left(r,\frac{\pi}{3}\right) = 0, \ u(1,\theta) = \sin 2\theta$

Sol. Example 2: $a = 1, l = \frac{\pi}{3}, \ \phi(0) = 0, \phi\left(\frac{\pi}{3}\right) = 0$ from (19)

$\phi(0) = c_2 = 0 \Rightarrow \phi(\theta) = c_1 \sin \delta\theta \Rightarrow \ \phi\left(\frac{\pi}{3}\right) = c_1 \sin\frac{\pi\delta}{3} = 0$

$c_1 \neq 0 \Rightarrow \sin\frac{\pi\delta}{3} = 0 \Leftrightarrow \frac{\pi\delta}{3} = n\pi \Rightarrow \delta = 3\pi, R(r) = k_n r^{3n}, \phi_n(\theta) = c_n \sin \delta\theta$. from (22) and (23) we get

$$u(r,\theta) = \sum_{n=1}^{\infty} u_n(r,\theta) = \sum_{n=1}^{\infty} b_n r^{3n} \sin 3n\theta$$

$$u(1,\theta) = \sum_{n=1}^{\infty} c_n. 1. \sin 3n\theta = \sin 2\theta$$

$$1^{3n} c_n = \frac{6}{\pi}\int_0^{\frac{\pi}{3}} \sin 2\theta \sin 3n\theta \, d\theta$$

$$=\frac{6}{\pi}\int_0^{\frac{\pi}{3}}\frac{1}{2}(\cos(2-3n)\theta-\cos(2+3n)\theta)d\theta$$

$$=\frac{3}{\pi}\left[\frac{\sin(2-3n)\theta}{(2-3n)}-\frac{\sin(2+3n)\theta}{(2+3n)}\right]_0^{\frac{\pi}{3}}$$

$$=\frac{3}{\pi}\left[\frac{\sin(2-3n)\frac{\pi}{3}}{2-3n}-\frac{\sin(2+3n)\frac{\pi}{3}}{2+3n}\right]$$

$$=\frac{3}{\pi}\left[\frac{\sin\frac{2\pi}{3}}{2-3n}-\frac{\sin\frac{2\pi}{3}}{2+3n}\right]=\frac{3}{\pi}\left[\frac{\frac{\sqrt{3}}{2}}{2-3n}-\frac{\frac{\sqrt{3}}{2}}{2+3n}\right]$$

$$=\frac{3\sqrt{3}}{2\pi}\left[\frac{1}{2-3n}-\frac{1}{2+3n}\right]=\frac{9\sqrt{3}}{4-9n^2}$$

Laplace equation

rectangle Laplace equation in 4.1

-1 - ملخص معادلة لابلاس

4,1. in rectangle

$$u_{xx}+u_{yy}=0,\qquad 0<x<a,0<y<b$$

$$x''-\lambda x=0$$

$$y''+\lambda y=0$$

$$u(x,0)=0,u(x,b)=0\Longrightarrow Y(0)=0,Y(b)=0$$

$$u(0,y)=0,u(a,y)=f(y)\Longrightarrow X(0)=0$$

if $\lambda\leqslant 0$ no solution

if $\lambda>0\quad Y_n(y)=c_n\sin\frac{n\pi}{b}y$

$$X_n(x) = k_n\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right) = 2k_n \sinh\frac{n\pi x}{b}$$

$$u_n(x,y) = 2k_n \sinh\frac{n\pi x}{b}\cdot c_n \sin\frac{n\pi y}{b} = b_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

$b_n \sinh\frac{an\pi}{b} = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}\,dy$

il-

$$u(x,0) = 0, u(x,b) = 0 \longrightarrow y(0) = 0, y(b) = 0$$
$$u(0,y) = g(y), u(a,y) = 0 \longrightarrow X(a) = 0$$

$\lambda \leqslant 0$ trivial sol.

$$\lambda > 0\ y_n(y) = c_n \sin\frac{n\pi y}{b}$$

$$x_n(x) = k_n \sinh\frac{(a-x)n\pi}{b}$$

$$u_y(r,\theta) = u_r \sin\theta + \frac{1}{r}u_\theta \cos^7\theta\,\frac{\partial r}{\partial y} = \sin\frac{\partial\theta}{\partial x} = -\frac{x^2 \sin\theta \cos^2\theta}{r^2\cos^\theta\theta(4)}$$

$$\theta y = \frac{1}{r}\cos\theta\,\sec^2\theta\,\frac{\partial\theta}{ry} = \frac{1}{r}\cos\theta$$

$$x'' - \lambda x = 0(6)\ y'' + \lambda y = 0(7)$$
$$u\ .1\ (x,0) = 0, u(x,b) = 0\ \longrightarrow y(0) = 0, y(b) = 0$$
$$u(0,y) = 0, u(a,y) = f(y) \longrightarrow X(0) = 0(9)$$

if $\lambda \leqslant 0$ trivial solution (ignore)

if $\lambda > 0$, $Y_n(y) = c_n \sin\frac{n\pi}{b}y$

$$X_n(x) = k_n\left(e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}\right) = 2k_n \sinh\frac{n\pi x}{b}$$

$$u_n(x,y) = 2k_n \sinh\frac{n\pi x}{b}\cdot c_n \sin\frac{n\pi y}{b} = b_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

$$u(x,y)=\sum_{n=1}^{\infty} b_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

$$b_n \sinh\frac{an\pi}{b}=\frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}dy$$

-ii $u(x,0)=0, u(x,b)=0 \longrightarrow y(0)=0, y(b)=0$

$$u(0,y)=g(y), u(a,y)=0 \longrightarrow X(a)=0$$

$\lambda \leqslant 0$.trivial sol

$\lambda > 0$, $Y_n(y)=c_n \sin\frac{n\pi y}{b}$

$$X_n(x)=k_n \sinh\frac{(a-x)n\pi}{b}$$

$u_n(x,y)=k_n \sinh\frac{(a-x)n\pi}{b}\cdot c_n \sin\frac{n\pi y}{b}=b_n \sinh\frac{(a-x)n\pi}{b}\sin\frac{n\pi y}{b}$.

$$u(x,y)=\sum_{1}^{\infty} b_n \sinh\frac{(a-x)n\pi}{b}\sin\frac{n\pi y}{b}$$

$$b_n \sinh\frac{na\pi}{b}=\frac{2}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}dy$$

ii- $u(x,0)=0, u(x,b)=0 \longrightarrow Y(0)=0, Y(b)=0$

$$u(0,y)=g(y), u(a,y)=0 \longrightarrow X(a)=0$$

$\lambda \leqslant 0$ trivial sol.

$\lambda > 0$ $Y_n(y)=c_n \sin\frac{n\pi y}{b}$

$X_n(x)=k_n \sinh\frac{(a-x)n\pi}{b}$

$$u_n(x,y)=k_n \sinh\frac{(a-x)n\pi}{b}\cdot c_n \sin\frac{n\pi y}{b}=b_n \sinh\frac{(a-x)n\pi}{b}\sin\frac{n\pi}{b}$$

$$u(x,y)=\sum_1^{\infty} b_n \sinh\frac{(a-x)n\pi}{b}\sin\frac{n\pi y}{b}$$

$b_n \sinh \frac{an\pi}{b} = \frac{2}{b}\int_0^b g(y) \sin \frac{n\pi y}{b} dy$

iii. $u(0,y) = 0, u(a,y) = 0, X(0) = 0, X(a) = 0$

$$u(x,0) = 0, u(x,b) = 0$$

$$X_n(x) = c_n \sin \frac{n\pi x}{a}, \qquad Y_n(y) = 2k_n \sinh \frac{n\pi y}{a}$$

$$u_n(x,y) = c_n \sin \frac{n\pi x}{a} \cdot 2k_n \sinh \frac{n\pi y}{a} = b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

$$u(x,y) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

$$b_n \sinh \frac{bn\pi}{a} = \frac{2}{a}\int_0^a h(x) \sin \frac{n\pi x}{a} dx$$

iv. $u(0,y) = 0,\ u(a,y) = 0 \longrightarrow X(0) = 0, X(a) = 0$

$$u(x,0) = k(x), u(x,b) = 0 \longrightarrow Y(b) = 0$$

$$X_n(x) = c_n \sin \frac{n\pi x}{a}$$

$$Y_n(y) = k_n \sinh \frac{(b-y)n\pi}{a}$$

$$u_n(x,y) = c_n \sin \frac{n\pi x}{a} \cdot k_n \sinh \frac{(b-y)n\pi}{a} = b_n \sin \frac{n\pi x}{a} \sinh \frac{(b-y)}{a}$$

$$u(x,y) = \sum_1^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{(b-y)n\pi}{a}$$

$$b_n \sinh \frac{bn\pi}{a} = \frac{2}{a}\int_0^a k(x) \sin \frac{n\pi x}{a} dx$$

2 - in circle $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

i. in Complete circle ($\phi(\theta)$ is periodic of period 2π
and

$$u(a,\theta) = f(\theta)\ \phi(\theta) = \phi(\theta \pm 2\pi))$$
$$r^2R'' + rR' - \lambda R = 0\ (12)$$
$$\phi'' + \lambda\phi = 0\ \ (13)$$

if $\lambda < 0$ trivial solution
if $\lambda = 0 \Rightarrow u(r,\theta) = C\ \bar{\omega}L^2 d\delta 1$
of $\lambda > 0$

$$\phi_n(\theta) = c_n \sin n\theta + d_n \cos n\theta (19)\ \delta = n$$
$$R_n(r) = k_n r^{\delta}(20)\ \delta = n$$
$$u_n(r,\theta) = k_n r^n (c_n \sin n\theta + d_n \cos n\theta)$$
$$u(r,\theta) = \sum_{n=0}^{\infty} k_n r^n (c_n \sin n\theta + d_n \cos n\theta)$$
$$= \frac{j_0}{2} + \sum_{n-1}^{\infty} r^n (e_n \sin n\theta + j_n \cos n\theta)$$

where

$$a^n e_n = \frac{2}{2\pi}\int_0^{n=1} f(\theta) \sin n\theta d\theta$$
$$u(a,\theta) = f(\theta) b, n, \bar{c}d$$
deb $0, f_5'$), wis aid $l = 2\pi$
$$\alpha^n j_n = \frac{2}{2\pi}\int_0^{0} f(\theta) \cos n\theta d\theta$$
$$0 \leqslant \theta \leqslant 2\pi$$

ii- (*) periodic po semicircle: chi θ ckaropt, n ar qi,

$$i, G, \frac{n}{i} i, 0 \leqslant \theta \leqslant l, l < 2\pi$$
$$u(r,0) = 0, u(r,l) = 0$$
$$u(a,\theta) = f(\theta)$$
$$\delta = \frac{n\pi}{l}$$

$$\phi(\theta) = c_n \sin\frac{n\pi\theta}{l}$$

$$R(r) = k_n r^{\frac{n\pi}{l}}$$

$$u_n(r,\theta) = k_n r^{\frac{n\pi}{l}} \cdot c_n \sin\frac{n\pi\theta}{l} = b_n r^{\frac{n\pi}{l}} \sin\frac{n\pi\theta}{l}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{\frac{n\pi}{l}} \sin\frac{n\pi\theta}{l} \quad u(a,\theta) = f(\theta)$$

$$b_n a^{\frac{n\pi}{l}} = \frac{2}{l}\int_0^l f(\theta)\sin\frac{n\pi\theta}{l}\,d\theta$$

$$l = \frac{\pi}{2}\dot{b}b, l_l^n \; 0 \leq \theta \leq \frac{\pi}{2}$$

$$\Rightarrow \phi(\theta) = c_n \sin 2n\theta \; u(2,\theta) =$$

$$R(r) = k_n r^{2n} \; 2\theta$$

$$u(r,\theta) = \sum_1^{\infty} b_n r^{2n} \sin 2n\theta$$

$$b_n 2^{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} 2\theta \sin 2n\theta d\theta = \frac{8}{\pi}\int_0^{\frac{\pi}{2}} \theta \sin 2n\theta d\theta$$

$$b_n 2^{2n} = \frac{8}{\pi}\left(-\frac{\pi}{4n}\cos n\pi\right) = \frac{2}{n}(-1)^{n+1}$$

$$b_n = \frac{2^{1-2n}}{n}(-1)^{n+1}$$

$$u(r,\theta) = \sum_1^{\infty} \frac{2^{1-2n}}{n}(-1)^{n+1} r^{2n} \sin 2n\theta$$

$$v(0,y)=0, u(a,y)=0 \longrightarrow X(0)=0, X(a)=0$$
$$u(x,0)=0, u(x,b)=h(x)$$
$$x_n(x)=c_n \sin\frac{n\pi x}{a}$$
$$Y_n(y)=2k_n \sinh\frac{n\pi y}{a}$$
$$u_n(x,y)=c_n \sin\frac{n\pi x}{a}\cdot 2k_n \sinh\frac{n\pi y}{a}=b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$
$$u(x,y)=\sum_{1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$$
$$b_n \sinh\frac{bn\pi}{a}=\frac{2}{a}\int_0^a h(x)\sin\frac{n\pi x}{a}dx$$

iv. $u(0,y)=0, u(a,y)=0 \longrightarrow X(0)=0, X(a)=0$

$$u(x,0)=k(x), u(x,b)=0 \longrightarrow y(b)=0$$
$$X_n(x)=c_n \sin\frac{n\pi x}{a}$$
$$V_n(y)=k_n \sinh\frac{(b-y)n\pi}{a}$$
$$u_n(x,y)=c_n \sin\frac{n\pi x}{a}\cdot k_n \sinh\frac{(b-y)n\pi}{a}=b_n \sin\frac{n\pi x}{a}\sinh\frac{(b-y)}{a\pi}$$
$$u(x,y)=\sum_{1}^{\infty} b_n \sin\frac{n\pi x}{a}\sinh\frac{(b-y)n\pi}{a}$$
$$b_n \sinh\frac{bn\pi}{a}=\frac{2}{a}\int_0^a k(x)\sin\frac{n\pi x}{a}$$

2 - in circle $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$

i. in Complete circle ($\phi(\theta)$ is periodic of period 2π and $u(a,\theta)=f(\theta)$ $\phi(\theta)=\phi(\theta\pm 2\pi)$)

$$r^2R''+rR'-\lambda R=0$$
$$\phi''+\lambda\phi=0 \quad (13)$$

if $\lambda < 0$ trivial solution

i $\lambda = 0 \Rightarrow u(r,\theta) = C\ \bar{\omega}L^2 d\delta 1$

if $\lambda > 0$

$$\phi_n(\theta) = c_n \sin n\theta + d_n \cos n\theta \ (19)\ \delta\delta = n$$

$$R_n(r) = k_n r^{\delta}\ (20)\ \delta = n$$

$$u_n(r,\theta) = k_n r^n (c_n \sin n\theta + d_n \cos n\theta)$$

$$u(r,\theta) = \sum_{n=0}^{\infty} k_n r^n (c_n \sin n\theta + d_n \cos n\theta)$$

$$= \frac{j_0}{2} + \sum_{n=1}^{\infty} r^n (e_n \sin n\theta + j_n \cos n\theta)$$

where

$$a^n e_n = \frac{2}{2\pi} \int_0^{n=1} f(\theta) \sin n\theta d\theta$$

$u(a,\theta) = f(\theta), b, n,$ úd.

$$\alpha^n j_n = \frac{2}{2\pi} \int_0^0 f(\theta) \cos n\theta d\theta$$

$$0 \le \theta \le 2\pi$$

ii- (*) periodic po ne semicircle : che θ cerrrpp, n dur ii, $\therefore Gl_0, l\dot{u}\dot{\omega} \leqslant \theta \leqslant l, l < 2\pi$

$$u(r,0) = 0, u(r,l) = 0$$

$$u(a,\theta) = f(\theta)$$

$$\delta = \frac{n\pi}{l}$$

$$\phi(\theta) = c_n \sin\frac{n\pi\theta}{l}$$

$$R(r) = k_n r^{\frac{n\pi}{l}}$$

$$R(r) = k_n r l$$

$$u_n(r,\theta) = k_n r\frac{n\pi}{l} \cdot c_n \sin\frac{n\pi\theta}{l} = b_n r^{\frac{n\pi}{l}} \sin\frac{n\pi\theta}{l}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{\frac{n\pi}{l}} \sin\frac{n\pi\theta}{l}$$

$$u(a,\theta) = f(\theta)$$

$$b_n a^{\frac{n\pi}{l}} = \frac{2}{l}\int_0^l f(\theta)\sin\frac{n\pi\theta}{l}\, d\theta$$

$l = \frac{\pi}{2}$ ن b, the $\longrightarrow 0 \leqslant \theta \leqslant \frac{\pi}{2}$

$$\Rightarrow \phi(\theta) = c_n \sin 2n\theta$$

$$u(2,\theta) =$$

$$R(r) = k_n r^{2n}$$

$$u(r,\theta) = \sum^{\infty} b_n r^{2n} \sin 2n\theta$$

$$b_n 2^{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} 2\theta \sin 2n\theta\, d\theta = \frac{8}{\pi}\int_0^{\frac{\pi}{2}} \theta \sin 2n\theta\, d\theta$$

$$b_n 2^{2n} = \frac{8}{\pi}\left(-\frac{\pi}{4n}\cos n\pi\right) = \frac{2}{n}(-1)^{n+1}$$

$$b_n = \frac{2^{1-2n}}{n}(-1)^{n+1}$$

$$u(r,\theta) = \sum_{1}^{\infty} \frac{2^{1-2n}}{n}(-1)^{n+1} r^{2n} \sin 2n\theta$$