.1 L'Hopital's Rule

THEOREM L'Hopital's Rule: Suppose that f(a) = g(a) = 0 or ∞, that f and g are differentiable on an open interval I containing a, and that g' (x) ≠0 on I if x ≠a. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

<u>Indeterminate Form 0/0</u>

EXAMPLE 1

(a) $\lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$

(b) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(c) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$

$$= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$

(d) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$

EXAMPLE 2

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0.$$

EXAMPLE 3

(a) $\lim_{x\to 0^+} \frac{\sin x}{x^2}$

$$= \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty$$

(b) $\lim_{x\to 0^-} \frac{\sin x}{x^2}$

$$= \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$$

<u>Indeterminate Forms ∞\ ∞ , ∞.0 and ∞-∞</u>

EXAMPLE 4: find the limit:

(a) $\lim_{x\to \pi/2} \frac{\sec x}{1 + \tan x}$ **(b)** $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}}$ **(c)** $\lim_{x\to\infty} \frac{e^x}{x^2}.$

Solution:

(a) $\lim_{x\to(\pi/2)^-} \frac{\sec x}{1 + \tan x}$ $\quad \frac{\infty}{\infty}$ from the left

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

(b) $\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ $\quad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$

(c) $\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$

EXAMPLE 5: Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$ **(b)** $\lim_{x \to 0^+} \sqrt{x} \ln x$

(a) $\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right) = \lim_{h \to 0^+} \left(\frac{1}{h} \sin h \right) = \lim_{h \to 0^+} \frac{\sin h}{h} = 1$ $\infty \cdot 0$; Let $h = 1/x$.

(b) $\lim_{x \to 0^+} \sqrt{x} \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}}$ $\infty \cdot 0$ converted to ∞/∞

$$= \lim_{x \to 0^+} \frac{1/x}{-1/2x^{3/2}}$$ l'Hôpital's Rule

$$= \lim_{x \to 0^+} \left(-2\sqrt{x} \right) = 0$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

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Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. \qquad ■$$

Indeterminate Powers

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^{∞}. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x} \ln (1 + x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln (1 + x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1 + x}}{1}$$

$$= \frac{1}{1} = 1.$$

If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

Therefore, $\lim_{x \to 0^+} (1 + x)^{1/x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^1 = e.$

EXAMPLE 8 Find $\lim_{x \to \infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x \to \infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x \to \infty} \frac{1/x}{1}$$

=0