3. $u(x) = 1 - \frac{1}{2}x^2 + \frac{1}{6}\int_0^x (x-t)^3 u(t)dt$

4. $u(x) = 1 + 3\int_0^x (x-t)u(t)dt$

5. $u(x) = \cos x - \sin x + 2\int_0^x \cos(x-t)\, u(t)dt$

2.5 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \qquad (2.17)$$

We will assume that the solution u(x) of the Volterra integral equation

$$u(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt\,, (2.18)$$

Substituting (2.17) into both sides of (2.18) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_0^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right), (2.19)$$

or for simplicity we use

$a_0 + a_1 x + a_2 x^2 + \cdots = T(f(x)) + \lambda \int_0^x K(x,t)\,(a_0 + a_1 t + a_2 t^2 + \cdots dt)$,

(2.20) where $T(f(x))$ is the Taylor series for $f(x)$. like powers of x in both sides of the resulting equation to obtain a recurrence relation in $a_j\,, j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j\,, j \geq 0$ by using the series solution method, If an exact solution is not obtainable, then the obtained series can be used for numerical purposes.

Example 2.27 Solve the Volterra IE by using the series solution method

$$u(x) = 1 + \int_0^x u(t)dt.$$

Substituting $u(x)$ by the series $u(x) = \sum_{n=0}^{\infty} a_n x^n$ into both sides of Eq. leads to

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \sum_{n=0}^{\infty} a_n t^n\, dt = 1 + \sum_{n=0}^{\infty} a_n \int_0^x t^n\, dt.$$

Evaluating the integral at the right side gives

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1},$$

that can be rewritten as

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \frac{a_{n-1} x^n}{n},$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + \cdots = 1 + a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \cdots$$

By comparing both sides gives the recurrence relation

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2} a_1 = \frac{1}{2}, a_3 = \frac{1}{3} a_2 = \frac{1}{3!}, \cdots a_n = \frac{1}{n} a_{n-1} = \frac{1}{n!}.$$

Substituting this result into (2.17) gives the series solution:

$$u(x) = \sum_{0}^{\infty} \frac{x^n}{n!} = e^x$$

,

Example 2.28 Solve the Volterra integral equation by using the series solution method

$$u(x) = x + \int_0^x (x-t) u(t) dt .$$

Substituting u(x) by the series $\sum_{n=0}^{\infty} a_n x^n$ into both sides of Eq. leads to

$$\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x (x-t) \sum_{n=0}^{\infty} a_n t^n \, dt.$$

$$\sum_{n=0}^{\infty} a_n x^n = x + x \sum_{n=0}^{\infty} a_n \int_0^x t^n \, dt - \sum_{n=0}^{\infty} a_n \int_0^x t^{n+1} \, dt.$$

$$a_0 + a_1 x + a_2 x^2 + \cdots = x + \sum_{n=0}^{\infty} \frac{a_n x^{n+2}}{n+1} - \sum_{n=0}^{\infty} \frac{a_n x^{n+2}}{n+2}$$

$$a_0 + a_1 x + a_2 x^2 + \cdots = x + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) a_n x^{n+2}$$

$$= x + \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} a_n \, x^{n+2} = x + \sum_{m=2}^{\infty} \frac{1}{m(m-1)} a_{m-2} \, x^m$$

By Letting $m = n+2, n = m-2$

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = x + \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + \cdots$$

Equating the coefficients of like powers of x in both sides gives

$a_0 = 0, a_1 = 1, a_2 = 0, a_3 = \frac{1}{6}a_1 = \frac{1}{3!}, a_4 = 0, a_5 = \frac{1}{20}a_3 = \frac{1}{120} = \frac{1}{5!}\cdots,\ a_{2n} = 0, a_{2n+1} = \frac{1}{(2n+1)!}.$

This result can be combined to obtain

$$u(x) = \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \sinh x$$

Example 2.29 Solve the Volterra integral equation using series solution method

$$u(x) = 1 - x\sin x + \int_0^x tu(t)dt.$$

For simplicity reasons, we will use few terms of the Taylor series for $\sin x$ and for the solution $u(x)$ to find

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$= 1 - x(x - \frac{x^3}{3!} + \cdots) + \int_0^x t(a_0 + a_1t + a_2t^2 + \cdots)dt$$

Integrating the right side and collecting the like terms of x we find

$$= 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{24} + \cdots + \frac{a_0x^2}{2} + \frac{a_1x^3}{3} + \frac{a_2x^4}{4} + \cdots$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = 1 + (\frac{a0}{2} - 1)x^2 + \frac{a1}{3}x^3 + (\frac{1}{6} + \frac{a2}{4})x^4 + \cdots.$

Equating the coefficients of like powers of x, yields

$a0 = 1, a1 = 0, a2 = \frac{a0}{2} - 1 = -\frac{1}{2!}, a3 = \frac{a1}{3} = 0, a4 = \frac{1}{6} + \frac{a2}{4} = \frac{1}{4!}, \cdots$

and generally

$$a_{2n+1} = 0, a_{2n} = \frac{(-1)^n}{(2n)!}\ , n \geq 0.$$

The solution in a series form is given by

$$u(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

that gives the exact solution by $u(x) = \cos x$.

Example 2.30 Solve the Volterra integral equation using the series solution method

$$u(x) = 2e^{x} - 2 - x + \int_0^x (x - t)u(t)dt.$$

Proceeding as before, we will use few terms of the Taylor series for e^x and for the solution $u(x)$ to find

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots + \int_0^x (x-t)(a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + \cdots)dt.$

Integrating the right side and collecting the like terms of x we find

$= x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots + \frac{1}{2}a_0x^2 + \frac{1}{6}a_1x^3 + \frac{1}{12}a_2x^4 + \cdots.$

$a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots = x + (1 + \frac{a_0}{2})x^2 + (\frac{1}{3} + \frac{a_1}{6})x^3 + (\frac{1}{12} + \frac{a_4}{12})\, x^4 + \cdots.$

Equating the coefficients of like powers of x in yields

$a0 = 0, a1 = 1, a2 = 1, a3 = \frac{1}{2!}, a4 = \frac{1}{3!}$,... and generally $a_n = \frac{1}{n!}$ $, n \geq 1, a0 = 0.$ The solution in a series form is given by

$u(x) = x(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots),$ that converges to the exact solution $u(x) = xe^x.$

Exercises 2.5

1. $u(x) = 1 - \int_0^x (x-t)u(t)dt$
2. $u(x) = 1 + \frac{x}{2} + \frac{1}{2}\int_0^x (x-t+1)u(t)dt$
3. $u(x) = 1 + xe^x - \int_0^x tu(t)dt.$
4. $u(x) = 3 + x^2 - \int_0^x (x-t)u(t)dt$
5. $u(x) = x\cos x + \int_0^x tu(t)dt$
6. $u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)dt$
7. $u(x) = x - x\ln(1+x) + \int_0^x u(t)dt.$

2.6 Volterra Integral Equations of the First Kind

The standard form of the Volterra integral equations of the first kind is given by

$$f(x) = \int_0^x K(x,t)u(t)dt, \qquad (2.20)$$

where the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $u(x)$ is the function to be determined. Recall that the unknown function u(x) appears inside and outside the integral sign for the Volterra integral equations of the second kind, whereas it occurs only inside the integral sign for the Volterra integral equations of the first kind. This equation of the first kind motivated mathematicians to develop reliable methods for solving it. In this section we will discuss three main methods that are commonly used for handling the Volterra integral equations of the first kind. Other methods are available in the literature but will not be presented in this text.

2.6.1 The Series Solution Method

As in the previous section, we will consider the solution $u(x)$ to be analytic

Example 2.31 Solve the Volterra integral equation by using the series solution method

$$\sin x - x\cos x = \int_0^x tu(t)dt\,.$$

Proceeding as before, only few terms of the Taylor series for $\sin x - x\cos x$ and for the solution u(x) will be used. Integrating the right side we obtain

$$\frac{x^3}{3} - \frac{x^5}{30} + \frac{x^7}{840} + \cdots = \int_0^x t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)dt\,,$$

$$= \tfrac{1}{2}a_0x^2 + \tfrac{1}{3}a_1x^3 + \tfrac{1}{4}a_2x^4 + \tfrac{1}{5}a_3x^5 + \tfrac{1}{6}a_4x^6 + \tfrac{1}{7}a_5x^7 + \cdots.$$

Equating the coefficients of like powers of x yields

$$a0 = 0, a1 = 1, a2 = 0, a3 = -\frac{1}{3!}, a4 = 0, a5 = \frac{1}{5!} \cdots$$

The solution in a series form is given by

$$u(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$$

that converges to the exact solution $u(x) = \sin x$.

Example 2.32 Solve the Volterra integral equation by using the series solution method

$$2 + x - 2e^x + xe^x = \int_0^x (x-t)u(t)dt.$$

Using few terms of the Taylor series for $2 + x - 2e^x + xe^x$ and for the solution $u(x)$, and by integrating the right side we find

$\frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{20} + \cdots = \int_0^x (x-t)(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)dt = \frac{1}{2}a_0 x^2 + \frac{1}{6}a_1 x^3 + \frac{1}{12}a_2 x^4 + \frac{1}{20}a_3 x^5 + \frac{1}{30}a_4 x^6 + \cdots$

Equating the coefficients of like powers of x yields

$a0 = 0,\ a1 = 1,\ a2 = 1,\ a3 = \frac{1}{2!},\ a4 = \frac{1}{3!}, \cdots$ The solution in a series form is given by $u(x) = x(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots)$, that converges to the exact solution $u(x) = xe^x$.

Example 2.33 Solve the Volterra integral equation by using the series solution method

$$x - \frac{1}{2}x^2 - \ln(1 + x) + x^2 \ln(1 + x) = \int_0^x 2tu(t)dt.$$

Using the Taylor series for $x - \frac{1}{2}x^2 - \ln(1 + x) + x^2 \ln(1 + x)$

and proceeding as before we find

$\frac{2}{3}x^3 - \frac{1}{4}x^4 + \frac{2}{15}x^5 - \frac{1}{12}x^6 + \cdots = \int_0^x 2t(a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots)dt =$

$a_0 x^2 + \frac{2}{3}a_1 x^3 + \frac{1}{2}a_2 x^4 + \frac{2}{5}a_3 x^5 + \frac{1}{3}a_4 x^6 + \cdots$

Equating the coefficients of like powers of x yields

$a0 = 0,\ a1 = 1,\ a2 = -\frac{1}{2},\ a3 = \frac{1}{3},\ a4 = -\frac{1}{4}, \ldots$

The solution in a series form is given by

$u(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

that converges to the exact solution $u(x) = \ln(1 + x)$.

Exercises 2.6.1 Use the series solution method to solve the Volterra integral equations of the first kind:

1. $e^x - 1 - x = \int_0^x (x - t + 1)u(t)dt.$
2. $x \cosh x - \sinh x = \int_0^x u(t)dt.$
3. $1 + xe^x - e^x = x\ 0\ tu(t)dt$
4. $-1 - x + 1\ 6\ x3 + e^x = \int_0^x (x - t)u(t)dt$
5. $-x + 2\sin x - x\cos x = \int_0^x (x - t)u(t)dt$
6. $-1 + \cosh x = \int_0^x (x - t)u(t)dt.$
7. $-x + \frac{1}{2}x^2 + \ln(1 + x) + x\ln(1 + x) = \int_0^x u(t)dt.$

2.6.2 The Laplace Transform Method

The Laplace transform method is a powerful technique that we used before for solving initial value problems and Volterra integral equations of the second kind. In the convolution theorem for the Laplace transform method, it was stated that if the kernel K(x, t) of the integral equation

$$f(x) = \int_0^x K(x,t)u(t)dt\ ,$$

$$f_1(x) * f_2(x)) = \int_0^x f_1(x - t)\, f_2(t)dt$$

$$\mathcal{L}\{(f_1(x) * f_2(x))\} = \mathcal{L}\left\{\int_0^x f_1(x - t)\, f_2(t)dt\right\} = \mathcal{L}\{(f_1(x)\}\, \mathcal{L}\{(f_2(x)\}$$

$$= F1(s)F2(s).$$

Example 2.34 Solve the Volterra integral equation of the first kind by using the Laplace transform method

$$e^x - \sin x - \cos x = \int_0^x 2e^{x-t}u(t)dt.$$

$$\frac{1}{s-1}-\frac{1}{s^2+1}-\frac{s}{s^2+1}=\mathcal{L}\left\{\int_0^x 2e^{x-t}u(t)dt\right\}=\mathcal{L}\{2e^x\}\mathcal{L}\{u(x)\}$$
$$=\frac{2}{s-1}\mathcal{L}\{u(x)\},$$

or equivalently

$$\frac{1}{s-1}-\frac{s+1}{s^2+1}=\frac{2}{(s-1)(s^2+1)}=\frac{2}{s-1}\mathcal{L}\{u(x)\}.$$

This in turn gives

$$\mathcal{L}\{u(x)\}=\frac{1}{(s^2+1)}.$$

Taking the inverse Laplace transform of both sides gives the exact solution $u(x)=\sin x$.

Example 2.35 Solve the Volterra integral equation of the first kind by using the Laplace transform method

$$1+x-e^x=\int_0^x (t-x)u(t)dt.$$

Notice that the kernel is $(t-x)=-(x-t)$. Taking the Laplace transform of both sides yields

$$\frac{1}{s}+\frac{1}{s^2}-\frac{1}{s-1}=-\frac{1}{s^2}\mathcal{L}\{u(x)\},$$
$$\frac{s(s-1)+s-1-s^2}{s^2(s-1)}=-\frac{1}{s^2}\mathcal{L}\{u(x)\}.$$

Solving for U(s) we find

$$\mathcal{L}\{u(x)\}=\frac{1}{(s-1)}.$$

Taking inverse Laplace transform of both sides gives the exact solution $u(x)=e^x$.

Example 2.36 Solve the Volterra integral equation of the first kind by using the Laplace transform method

$$-1+x^2+\frac{1}{6}x^3+2\sinh x+\cosh x=\int_0^x (x-t+2)u(t)dt.$$

Taking the Laplace transform of both sides of yields

$$-\frac{1}{s}+\frac{2}{s^3}+\frac{1}{s^4}+\frac{2}{s^2-1}+\frac{s}{s^2-1}=\frac{2s^4+3s^3+s^2-2s-1}{s^4(s^2-1)}$$

$$=\mathcal{L}\{x+2\}\mathcal{L}\{u(x)\}=\left(\frac{1}{s^2}+\frac{2}{s}\right)\mathcal{L}\{u(x)\}=(\frac{1+2s}{s^2})\mathcal{L}\{u(x)\},$$

$$\frac{2s^4+3s^3+s^2-2s-1}{s^2(s^2-1)}=(1+2s)\mathcal{L}\{u(x)\},$$

$$\frac{s^3+2s^4+s^2+2s^3-(1+2s)}{s^2(s^2-1)}=(1+2s)\mathcal{L}\{u(x)\},$$

$$\frac{s^3(1+2s)+s^2(1+2s)-(1+2s)}{s^2(s^2-1)}=(1+2s)\mathcal{L}\{u(x)\},$$

$$\mathcal{L}\{u(x)\}=\frac{s^3+s^2-1}{s^2(s^2-1)}=\frac{s^3}{s^2(s^2-1)}+\frac{s^2-1}{s^2(s^2-1)}=\frac{s}{s^2-1}+\frac{1}{s^2},$$

Taking the inverse Laplace transform of both sides gives the exact solution $u(x)=x+\cosh x$.

Exercises 2.6.2 Use the Laplace transform method to solve the Volterra integral equations of the first kind:

1. $x-\sin x=\int_0^x (x-t)u(t)dt$
2. $e^x+\sin x-\cos x=2\int_0^x e^{x-t}u(t)dt$
3. $1+x-\sin x-\cos x=\int_0^x (x-t)u(t)dt$
7. $x=\int_0^x (x-t+1)u(t)dt$
9. $1+x-\frac{1}{3!}x^3-e^x=\int_0^x (t-x)u(t)dt$
10. $1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3-\sin x-\cos x=\int_0^x (x-t+1)u(t)dt$
11. $3-7x+x^2+\sinh x-3\cosh x=\int_0^x (x-t-3)u(t)dt$
12. $1-\cos x=\int_0^x \cos(x-t)\,u(t)dt$

2.7. Conversion to a Volterra Equation of the Second Kind

In this section we will present a method that will convert Volterra integral equations of the first kind to Volterra integral equations of the second kind. The conversion technique works effectively only if $K(x,x) \neq 0$. Differentiating both sides of the Volterra integral equation of the first kind

$$f(x) = \int_0^x K(x,t)u(t)dt\ ,$$

with respect to x, and using Leibnitz rule, we find

$$f'(x) = K(x,x)u(x) + \int_0^x \frac{\partial K(x,t)}{\partial x} u(t)dt.$$

$$u(x) = \frac{f'(x)}{K(x,x)} - \frac{1}{K(x,x)}\int_0^x \frac{\partial K(x,t)}{\partial x} u(t)dt.$$

$$f_2(x) = \frac{f'(x)}{K(x,x)}\ ,\ K_2(x,t)) = -\frac{1}{K(x,x)}\frac{\partial K(x,t)}{\partial x}$$

$$u(x) = f_2(x) + \int_0^x K_2(x,t)\, u(t)dt.$$

Example 2.37 Solve by Laplace and Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$\sinh x = \int_0^x e^{x-t}\, u(t)dt.$$

$$\frac{1}{s^2-1} = \frac{1}{s-1}\mathcal{L}\{u(x)\}$$

$$\mathcal{L}\{u(x)\} = \frac{s-1}{s^2-1} = \frac{1}{s+1} \to u(x) = e^{-x}$$

Differentiating both sides using Leibnitz rule we obtain

$$\cosh x = u(x) + \int_0^x e^{x-t}\, u(t)dt.,$$

that gives the Volterra integral equation of the second kind

$$u(x) = \cosh x - \int_0^x e^{x-t}\, u(t)dt.$$

We select the Laplace transform method for solving this problem. Taking Laplace transform of both sides gives

$$\mathcal{L}\{u(x)\} = \frac{s}{s^2-1} - \mathcal{L}\left\{\int_0^x e^{x-t}\, u(t)dt\right\} = \frac{s}{s^2-1} - \mathcal{L}\{e^x * u(x)\} = \frac{s}{s^2-1} - \mathcal{L}\{e^x\}\mathcal{L}\{u(x)\}$$

$$= \frac{s}{s^2-1} - \frac{1}{s-1}\mathcal{L}\{u(x)\}$$

$$\mathcal{L}\{u(x)\}(\frac{s}{s-1}) = \frac{s}{s^2-1}$$

$$\mathcal{L}\{u(x)\} = \frac{1}{s+1}\ , u(x) = e^{-x}$$

Example 2.38 Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$1 + \sin x - \cos x = \int_0^x (x - t + 1)u(t)dt.$$

Differentiating both sides and using Leibnitz rule we obtain the Volterra integral equation of the second kind

$$u(x) = \cos x + \sin x - \int_0^x u(t)dt.$$

We select the modified decomposition method for solving this problem. Therefore we set the modified recurrence relation

$$u_0(x) = \cos x\,, u_1(x) = \sin x - \int_0^x u_0(t)dt = 0,$$

$u_{k+1}(x) = -\int_0^x u_k(t)dt = 0, k \geq 1.$

This gives the exact solution by $u(x) = \cos x$.

Remarks

1. It was stated before that if K(x, x) = 0, then the conversion of the first kind to the second kind fails. However, if $K(x,x) = 0$ and $\frac{\partial K(x,x)}{\partial x} = 0$, then by differentiating the Volterra integral equation of the first kind as many times as needed, provided that $K(x,t)$ is differentiable, then the equation will be reduced to the Volterra integral equation of the second kind.

2. The function $f(x)$ must satisfy specific conditions to guarantee a unique continuous solution for $u(x)$. The determination of these special conditions will be left as an exercise. However, for first remark, where $K(x,x) = 0$ but $\frac{\partial K(x,x)}{\partial x} = 0$.

we will differentiate twice, by using Leibnitz rule, as will be shown by the following illustrative example.

Example 2.40 Convert the Volterra integral equation of the first kind to the second kind and solve the resulting equation

$$x\sinh x = 2\int_0^x \sinh(x-t)\,u(t)dt$$

Differentiating both sides and using Leibnitz rule we obtain

$$x\cosh x + \sinh x = 2\int_0^x \cosh(x-t)\,u(t)dt,$$

which is still a Volterra integral equation of the first kind. However, because $\frac{\partial K(x,x)}{\partial x} = 0$, we differentiate again to obtain the Volterra integral equation of the second kind

$$u(x) = \cosh x + \frac{x}{2}\sinh x - \int_0^x \sinh(x-t)\,u(t)dt.$$

We select the modified decomposition method for solving this problem. Therefore we set the modified recurrence relation

$$u_0(x) = \cosh x\,, u_1(x) = \frac{x}{2}\sinh x - \int_0^x \sinh(x-t)\,u_0(t)dt = 0,$$

$$u_{k+1}(x) = -\int_0^x \sinh(x-t)\,u_k(t)dt = 0\,, k \ge 1.$$

The exact solution is given by $u(x) = \cosh x$.

Exercises 2.7 Convert the VIE of the first kind to a second kind and solve the resulting equation:

1. $e^x + \sin x - \cos x = \int_0^x 2e^{x-t}u(t)dt$
2. $x = \int_0^x (x-t+1)u(t)\,dt$
3. $e^x + \sin x - \cos x = \int_0^x 2\cos(x-t)\,u(t)dt$
4. $e^x - x - 1 = \int_0^x (x-t+1)u(t)dt$
5. $4 + x - 4e^x + 3xe^x = \int_0^x (x-t+2)u(t)dt$

6. $\tan x - \ln\cos x = \int_0^x (x - t + 1)u(t)dt\,, x < \frac{\pi}{2}$

7. $x\sin x = \int_0^x 2\sin(x - t)\,u(t)dt$

8. $e^x - \sin x - \cos x = \int_0^x 2\sin(x - t)u(t)dt$

9. $\sin x - x\cos x = \int_0^x 2\sinh(x - t)\,u(t)dt.$