13. $\frac{1}{f(D_x,D_y)}x^n = \frac{1}{f\left(1,\frac{D_y}{D_x}\right)}x^n = \frac{1}{a_0[1+g(\frac{D_y}{D_x})]}x^n$

$$= \frac{1}{a_0}\left[1 - g(\frac{D_y}{D_x}) + g^2(\frac{D_y}{D_x}) - g^3(\frac{D_y}{D_x}) + \cdots + g^n(\frac{D_y}{D_x}) + \cdots\right]x^n$$

Example 3.14. For a particular solution of the equation

$$\left(3D_x^2 + 4D_xD_y - D_y\right)u = e^{x-3y},$$

note that

$$u_p = \frac{1}{\left(3D_x^2 + 4D_xD_y - D_y\right)}e^{x-3y}$$

$$= \frac{1}{[3 + 4(-3) - (-3)]}e^{x-3y} = -\frac{1}{6}e^{x-3y}$$

Example 3.15. For a particular solution of partial differential equation

$$\left(3D_x^2 - D_y\right)u = \sin(ax + by)$$

we have

$$u_p = \frac{1}{\left(3D_x^2 - D_y\right)}\sin(ax + by) = \frac{1}{\left(-3a^2 - D_y\right)}\sin(ax + by)$$

$$= -\frac{D_y - 3a^2}{D_y^2 - 9a^4}\sin(ax + by) = \frac{b\cos(ax + by) - 3a^2\sin(ax + by)}{b^2 + 9a^4}$$

Example 3.16. To find a particular solution for the equation

$$\left(3D_x^2 - D_y\right)u = e^x\sin(x + y)$$

we have

$$u_p = \frac{1}{\left(3D_x^2 - D_y\right)}e^x\sin(x + y) = e^x\frac{1}{\left(3(D_x + 1)^2 - D_y\right)}\sin(x + y)$$

$$= e^x\frac{1}{\left(3D_x^2 + 6D_x + 3 - D_y\right)}\sin(x + y)$$

$$= e^x\frac{1}{\left(3(-1)_x^2 + 6D_x + 3 - D_y\right)}\sin(x + y)$$

$$= e^x \frac{1}{(6D_x - D_y)} \sin(x+y) = e^x \frac{(6D_x + D_y)}{(36D_x^2 - D_y^2)} \sin(x+y)$$

$$= e^x \frac{7\cos(x+y)}{-35}$$

$$= -\frac{1}{5} e^x \cos(x+y)$$

Example 3.17. To solve $u_{tt} - c^2 u_{xx} = 0$, such that $u(x,0) = e^{-x}$, $u_t(x,0) = 1 + x$, note that the partial differential equation can be written as $(D_t - cD_x)(D_t + cD_x)u = 0$, which gives the solution as $u = f(x+ct) + g(x-ct)$. This solution is known as the d'Alembert's solution (see Eq (5.24)). Applying the initial conditions, we get

$$f(x) + g(x) = e^{-x}$$

$$cf'(x) - cg'(x) = 1 + x$$

On integrating (3.16) with respect to x we get

$$f(x) - g(x) = \frac{1}{c}\left(x + \frac{x^2}{2}\right) + c_1$$

Eqs (3.15) and (3.17) yield

$$f(x) = \frac{1}{2} e^{-x} + \frac{1}{2c}\left(x + \frac{x^2}{2}\right) + \frac{c_1}{2}$$

$$g(x) = \frac{1}{2} e^{-x} - \frac{1}{2c}\left(x + \frac{x^2}{2}\right) - \frac{c_1}{2}$$

Hence

$$u = \frac{1}{2} e^{-x}(e^{ct} + e^{-ct}) + (x+1)t$$

A general scheme for initial value problems for the wave equation is as follows: Solve $u_{tt} = c^2 u_{xx}$, subject to the conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi'(x)$. Then as in the above example

$$f(x) + g(x) = \phi(x)$$
$$cf'(x) - cg'(x) = \psi'(x)$$

Consequently

$$f(x) = \frac{1}{2}\left[\phi(x) + \frac{1}{c}\psi(x) + c_1\right]$$
$$g(x) = \frac{1}{2}\left[\phi(x) - \frac{1}{c}\psi(x) - c_1\right]$$

which yields

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(x-ct)] + \frac{1}{2c}[\psi(x+ct) - \psi(x-ct)]$$

Example 3.18. It is interesting to note that we can solve the Laplace equation by the above method. We will solve

$$u_{xx} + u_{yy} = 0$$

such that $u(x,0) = \phi(x)$ and $u_y(x,0) = \psi'(x)$. We can express $u_{xx} + u_{yy} = 0$ as

$$(D_x + iD_y)(D_x - iD_y)u = 0$$

and, therefore, its general solution is

$$u = f(x+iy) + g(x-iy)$$

Applying the initial conditions, we get $f(x) + g(x) = \phi(x)$, and $if'(x) - ig'(x) = \psi'(x)$. Consequently

$$f(x) = \frac{1}{2}[\phi(x) - i\psi(x) + c]$$
$$g(x) = \frac{1}{2}[\phi(x) + i\psi(x) - c]$$

Thus,

$$u(x,y) = \frac{1}{2}[\phi(x+iy) + \phi(x-iy)] + \frac{i}{2}[\psi(x-iy) - \psi(x+iy)]$$

The final value of u is real. If $\phi(x) = e^{-x}$, and $\psi' = \frac{1}{1+x^2}$, the solution is given by

$$\begin{aligned} u(x,y) &= \frac{1}{2}\left[e^{(x+iy)} + e^{(x-iy)}\right] + \frac{i}{2}[\tan^{-1}(x-iy) - \tan^{-1}(x+iy)] \\ &= e^{-x}\cos y - \frac{1}{4}\ln\left[\frac{x^2+(1-y)^2}{x^2+(1+y)^2}\right] \end{aligned}$$

where we have used the formula

$$\tan\alpha = iz, \text{ or } \alpha = \frac{i}{2}\ln\frac{(1+z)}{1-z}$$

with $\alpha = \tan^{-1}(x-iy) - \tan^{-1}(x+iy)$.

Example 3.19. Consider

$$u_{tt} = c^2 u_{xx},\ 0 < x < 1$$

subject to the conditions $u(0,t) = u(l,t) = 0$, for $t \geq 0$, and $u(x,0) = x$, $u_t(x,0) = 0$. The general solution is

$$u = f(x+ct) + g(x-ct)$$

From the boundary conditions we find that

$$f(ct) + g(-ct) = 0, \text{ or } f(z) + g(-z) = 0$$

which yields $f(z) = -g(-z)$. Also $f(l+ct) + g(l-ct) = 0$ is equivalent to

$$f(ct+l) - f(ct-l) = 0$$

which in turn gives $f(z) = f(z+2l)$. This last equation implies that the function $f(x)$ is a periodic function of period $2l$. The solution, thus, reduces to

$$u = f(ct+x) - f(ct-x)$$

Applying the initial conditions, we get

$$f(x) - f(-x) = x, \text{ and } f'(x) - f'(-x) = 0$$

i.e., $f'(x)$ is an even function, which means that $f(x)$ is an odd function, i.e., $f(x) = -f(-x)$. Hence $2f(x) = x$. Since $f(x)$ is an odd periodic function of period $2l$, it can be expressed as a Fourier sine series. Thus

$$f(x) = \frac{x}{2} = \frac{l}{\pi}\sum_{0}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$

which yields

$$u(x,t) = \frac{l}{\pi}\sum_{0}^{\infty}\frac{(-1)^{n+1}}{n}\left[\sin\frac{n\pi}{l}(ct+l) - \sin\frac{n\pi}{l}(ct-l)\right]$$

$$= \frac{2l}{\pi}\sum_{0}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}$$

Other techniques from ordinary differential equations such as the method of undetermined coefficients and the variation of parameters

3.4. Exercises

Evaluate (use the inverse operator method of §3.1):

3.1. $(D-3)^{-1}(x^3+3x-5)$.

Ans. $-\frac{1}{27}(9x^3+9x^2+33x-34)$.

3.2. $(D-1)^{-1}(2x)$.

Ans. $-2x$.

3.3. $(D-1)^{-1}(x^2)$.

Ans. $-(x^2+2x+2)$.

3.4. $(4D^2-5D)^{-1}(x^2e^{-x})$.

Ans. $-\frac{e^{-x}}{9}\left(x^2+\frac{26}{9}x+\frac{266}{81}\right)$.

3.5. $(D^2-3D+2)^{-1}\sin 2x$.

Ans. $\frac{3}{20}\cos 2x-\frac{1}{20}\sin 2x$.

3.6. $D^{-2}(2\sin 2x)$.

Ans. $-\frac{1}{2}\sin 2x$..

3.7. $D^{-3}x$.

Ans. $\frac{x^4}{24}$.

3.8. $D^{-2}(3e^{3x})$.

Ans. $\frac{e^{3x}}{3}$.

3.9. $D^{-1}(2x+3)$.

Ans. x^2+3x.

3.10. $(D^3-D^2)^{-1}(2x^3)$.

Ans. $-2\left(\frac{x^5}{20}+\frac{x^4}{4}+x^3+3x^2\right)$.

3.11. $(D^2+3D+2)^{-1}(e^{ix})$.

Ans. $\frac{1-3i}{10}e^{ix}$.

3.12. $(D^2-3D+2)^{-1}(3\sin x)$.

Ans. $\frac{3}{10}(\sin x+3\cos x)$.

3.13. $(D^2+3D+2)^{-1}(8+6e^x+2\sin x)$.

Ans. $4+e^x+\frac{1}{5}(\sin x-3\cos x)$.

3.14. $(D^5+2D^3+D)^{-1}(2x+\sin x+\cos x)$

Ans. $x^2+\frac{x^2}{8}(\cos x-\sin x)$.

Find the general solution of the following partial differential equations:

3.15. $(3D_x^2-2D_xD_y-5D_y^2)u=3x+y+e^{x-y}$.

ANS. $u=f(5x+3y)+g(x-y)+\frac{11}{54}x^3+\frac{1}{6}x^2y+\frac{1}{8}xe^{x-y}$.

3.16. $(D_x^4-10D_x^2D_y^2+9D_y^4)u=135\sin(3x+2y)$.

ANS. $f_1(3x+y)+f_2(x-3y)+g_1(x+y)+g_2(x-y)-\sin(3x+2y)$.

3.17. $(D_x-2D_y)^3u=125e^x\sin y$.

ANs. $f_1(2x+y)+xf_2(2x+y)+x^2f_3(2x+y)-e^x(2\cos y+11\sin y)$.

3.18. Find the particular solution for the following partial differential equations:

(a) $(D_x^2-D_y)u=17e^{x+y}\sin(x-2y)$.

Ans. $-e^{x+y}\{\sin(x-2y)+4\cos(x-2y)\}$.

(b) $(D_x^2+D_y^2)u=6xy+25e^{3x+4y}$.

Ans. x^3y+e^{3x+4y}.

(c) $(D_x^2+D_y^2-D_x)u=37e^{5y}\cos(3x+4y)$.

ANs. $e^{5y}\sin(3x+4y)$.

3.19. Show that $u=f(ay-bx)e^{-cy/b}$ is also a solution of

$$(aD_x+bD_y+c)u=0$$

3.20. Find the general solution of $3u_x + 4u_y - 2u = 1$, subject to the initial condition $u(x,0) = x^2$.

Solution. Here $\tan\theta = 4/3$, thus

$$\frac{\partial w}{\partial \xi} - \frac{2}{5}w = \frac{1}{5}$$

whose general solution is

$$w(\xi,\eta) = -\frac{1}{2} + g(\eta)e^{2\xi/5}$$

or

$$u(x,y) = -\frac{1}{2} + g\left(\frac{3}{5}y - \frac{4}{5}x\right)e^{6x/25+8y/25}$$

3.21. Find the general solution of $u_x - u_y + u = 1$, such that $u(x,0) = \sin x$.

SOLUTION. $\tan\theta = -1$, thus $\theta = 4\pi/4$, and

$$\frac{\partial w}{\partial \xi} - \frac{1}{\sqrt{2}}w = -\frac{1}{\sqrt{2}}$$

whose general solution is $w = 1 + g(\eta)e^{\xi/\sqrt{2}}$, or

$$u(x,y) = 1 + g\left(1 - \frac{x+y}{\sqrt{2}}\right)e^{(y-x)/2}$$

Using the initial condition, we get $\sin x = 1 + G(-x/\sqrt{2})e^{-x/2}$, so that

$$g(\eta) = -(\sin\sqrt{2}\eta + 1)e^{-\eta/\sqrt{2}}$$

Then

$$u(x,y) = 1 - (\sin\sqrt{2}\eta + 1)e^{-\eta/\sqrt{2}}e^{\xi/\sqrt{2}} = 1 + [1 - \sin(x+y)]e^{y}$$

3.22. Solve $u_x + u_y - u = 0$, subject to the initial condition $u(x,0) = h(x)$.

SOLUTION. Here $\tan\theta = 1$, thus $\theta = \pi/4$, and $\sqrt{2}\frac{\partial w}{\partial \xi} = w$, whose general solution is $w = g(\eta)e^{\xi/\sqrt{2}}$, or

$$u(x,y) = g(\eta)e^{\xi/\sqrt{2}}$$

The initial condition yields

$$h(x) = g(-x/\sqrt{2})e^{x/2} = g(\eta)e^{\eta/\sqrt{2}}$$

or $g(\eta) = h(-\sqrt{2}\eta)e^{\eta/\sqrt{2}}$. Hence

$$u(x,y) = h(-\sqrt{2}\eta)e^{\xi/\sqrt{2}} = h(x-y)e^{y}$$

3.23. Solve $u_{tt} - c^2 u_{xx} = 0$, subject to the conditions $u(x,0) = \ln(1+x^2)$ and $u_t(x,0) = e^{-x}$.

Ans.

$$u(x,t) = \frac{1}{2}[\ln\{1+(x+ct)^2\} + \ln\{1+(x-ct)^2\}] + \frac{1}{c}e^{-x}\cosh ct$$