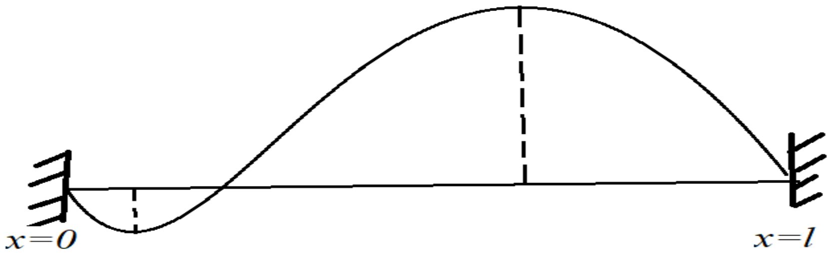
Chapter 3
Wave equation

2.0 Introduction examples of waves equations: a caustic wave, electromagnetic wave, seismic Wave, violin Wave equation
Vibrations of an elastic string violin string or one dimension wave eq.



$$\alpha^2 u_{xx} = u_{tt} \qquad (2.1)$$

$$u_{tt} = \alpha^2 u_{xx} + F(x,t), \qquad x = l$$

$$u_{tt} = \alpha^2 u_{xx} - \beta u_t - \lambda u + F(x,t)$$

$u(x,t)$ is displacement of the string from equilibrium

$$\alpha^2 = \frac{T}{\rho} \neq 0, \qquad (2.2)$$

T : is the tension (force) in the string
ρ: is the mass per unit length of the string martial

Boundary Conditions

$$u(0,t) = 0, \;\; u(l,t) = 0, \;\; 0 \leqslant t < \infty. \qquad (2.3)$$

Initial conditions

$$u(x,0) = f(x), \qquad 0 \leqslant x \leqslant l. \qquad (2.4)$$

Initial velocity

$$u_t(x,0) = g(x), \qquad 0 \leq x \leq l \qquad (2.5)$$

f, g are given function

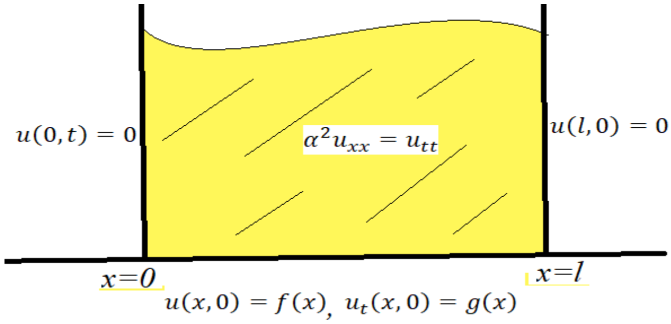
اذا كانت الازاحة و السرعة بالاطراف ثابتة و متجانسة فتكون

$$f(0) = f(l) = 0, \qquad g(0) = g(l) = 0 \qquad (2.6)$$

حل معادله الموجه كانت من المسائل الرياضيه الرئيسيه في منتصف القرن الثامن عشر واول من درس معادله الموجه عام 1746 دي الامبرت وبعده 1748 اويلر ثم 1753 برنولي و 1759 لاكرانج

Euler 17480 m, D' Alembert 1746of, L, S, m, O, N lagrange 1759 . Bernoulli 1753

This problem is initial value problem of t and Boundary value problem of x.



$$\alpha^2(u_{xx} + u_{yy}) = u_{tt}$$

Is Two dimensions wave eq.

$$\alpha^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}$$

Three dimensions wave eq.

2.1: Case 1

B.C. $u(0,t) = 0, u(l,0) = 0,\ 0 < x < l,\ 0 < t < \infty.$ (2.7)

I.C. $u(x,0) = f(x),\ u_t(x,0) = 0.$ (2.8)

$$u(x,t) = X(x)T(t) \qquad (2.9)$$

$$u_{xx} = X''T, \qquad u_{tt} = XT''$$

$$\frac{X''}{X} = \frac{T''}{\alpha^2 T} = -\lambda$$

$$X''(x) + \lambda X(x) = 0. \qquad (2.10)$$

$$T''(t) + \lambda T(t) = 0, \qquad (2.11)$$

$$X(x) \neq 0,\ T(t) \neq 0.$$

By (2.7),(2.8) $\Rightarrow X(0) = 0,\ X(l) = 0.$ (2.12)

$$u_t(x,0) = 0 \Rightarrow X(x)T'(0)) = 0,\ X(x) \neq 0$$

$$T'(0) = 0, \qquad (2.13)$$

to solve (2.10), (2.12)

- for $\lambda = 0, \&\ \lambda < 0$ $\Rightarrow$trivial (no solution)
- $\lambda > 0 \Rightarrow \lambda = \delta^2,\ X'' + \delta^2 X = 0$

$$m^2 = -\delta^2 \Rightarrow m = \pm\delta i$$

$$X_1 = \sin\delta x,\ X_2 = \cos\delta x$$

$$X(x) = c_1 \sin\delta x + c_2 \cos\delta x$$

$$X(0) = c_2 = 0$$

$$X(x) = c_1 \sin \delta x \Longrightarrow X(l) = c_1 \sin \delta l = 0$$

$$\sin \delta l = 0 \;\text{ iff }\; \delta l = n\pi, \qquad n = 0,1,2,\cdots$$

$$\delta_n = \frac{n\pi}{l} \;\Rightarrow\; \lambda_n = \frac{n^2\pi^2}{l^2}, \qquad (2.14)$$

$$X_n(t) = c_n \sin \frac{n\pi}{l} x, \qquad (2.15)$$

Use (2.14) into eq.(2.11) we get

$$T_n''(t) + \frac{\alpha^2 n^2 \pi^2}{l^2} T_n = 0, \qquad (2.16)$$

$$m^2 = -\frac{\alpha^2 n^2 \pi^2}{l^2} \Rightarrow m = \pm \frac{\alpha n \pi}{l} i$$

$$T_n(t) = k_n \sin \frac{n\pi\alpha}{l} t + J_n \cos \frac{n\pi\alpha}{l} t \quad (2.16-\text{a})$$

by Cond. (2.13) $T'(0) = 0$

$$T_n'(t) = k_n \frac{n\pi\alpha}{l} \cos \frac{n\pi\alpha}{l} t - j_n \frac{n\pi\alpha}{l} \sin \frac{n\pi\alpha}{l} t$$

$$T_n'(0) = k_n \frac{n\pi\alpha}{l} = 0 \Longrightarrow k_n = 0$$

$$\therefore T_n(t) = j_n \cos \frac{n\pi\alpha}{l} t \qquad (2.17)$$

$$\therefore u_n(x,t) = c_n \sin \frac{n\pi x}{l} j_n \cos \frac{n\pi\alpha t}{l}$$

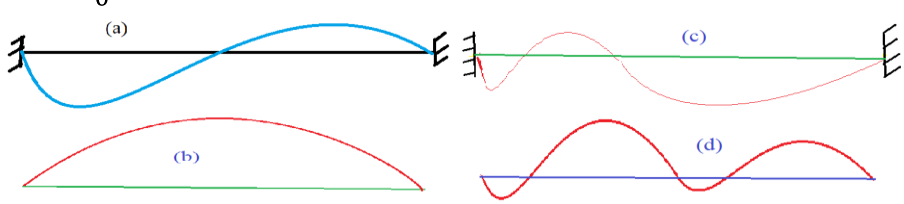
$$= b_n \sin \frac{n\pi}{l} x \cos \frac{n\pi\alpha}{l} t, \quad b_n = c_n j_n \qquad (218)$$

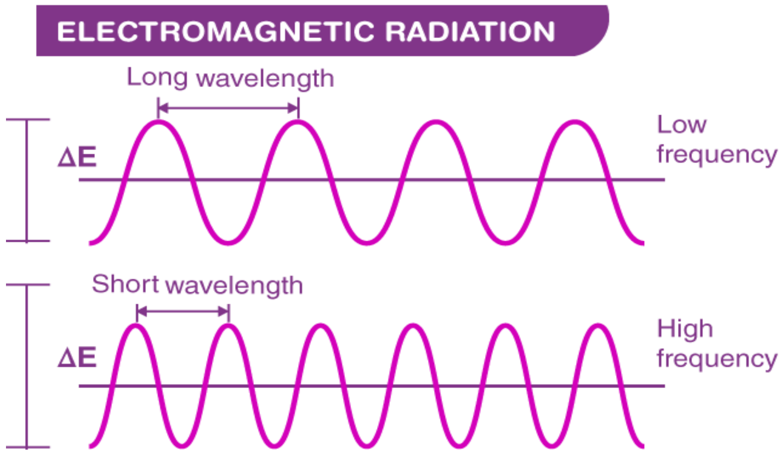
The general solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi\alpha t}{l} \qquad (2.19)$$

$$u(x,0) = f(x) \Longrightarrow u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = f(x)$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \qquad n = 1,2,\ldots \qquad (2.20)$$





Frequency≡ $\delta_n \alpha = \frac{n\pi\alpha}{l}$, Wave length $= \frac{2l}{n}$

(a) Freq.: $\frac{2\pi\alpha}{l}$, Wave length: $\frac{2l}{2} = l$. (b) Freq.: $\frac{\pi\alpha}{l}$, Wave length: $2l$.

(c) Freq.: $\frac{3\pi\alpha}{l}$, Wave length: $\frac{2l}{3}$.

Example: A vibrating string of length 30 cm satisfies the wave equation $4u_{xx} = u_{tt}$Assume the end of the string are fixed and initial displacement is given by

سلك مهتز طوله 30 سم يحقق معادلة الموجة $4u_{xx} = u_{tt},\ 0 < x < 30,\ t > 0$ افرض ان طرفي السلك ثابتة وان الازاحة الابتدائية معطى بالدالة

$$u(x,0) = \begin{cases} \frac{x}{10} & 0 \leqslant x \leqslant 10 \\ \frac{30-x}{20} & 10 < x \leqslant 30 \end{cases} ,u_t(x,0) = 0 \qquad (2.21)$$

Find the displacement $u(x,t)$?

Sol: $\alpha^2 = 4 \rightarrow \alpha = 2,\ l = 30$, from (2.19) and (2.20) we get

$$u(x,t) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{30}\cos\frac{2n\pi t}{30}$$

$$b_n = \frac{2}{30}\int_0^{30} f(x)\sin\frac{n\pi x}{30}dx$$

$$= \frac{1}{150}\int_0^{10} x\sin\frac{n\pi x}{30}dx + \frac{1}{300}\int_{10}^{30}(30-x)\sin\frac{n\pi x}{30}dx$$

$$u = x,\ du = dx,\ dv = \sin\frac{n\pi x}{30}dx\ , v = -\frac{30}{n\pi}\cos\frac{n\pi x}{30}$$

$$= \frac{1}{150}\left[\left[\frac{-30}{n\pi} x\cos\frac{n\pi}{30}x + \left(\frac{30}{n\pi}\right)^2 \sin\frac{n\pi x}{30}\right]_0^{10}\right]$$

$$+ \frac{1}{300}\left[\left[-(30-x)\frac{30}{n\pi}\cos\frac{n\pi}{30}x - \left(\frac{30}{n\pi}\right)^2 \sin\frac{n\pi x}{30}\right]_{10}^{30}\right]$$

$$= \frac{-2}{n\pi}\cos\frac{n\pi}{3} + \frac{6}{(n\pi)^2}\sin\frac{n\pi}{3} + \frac{2}{n\pi}\cos\frac{n\pi}{3} + \frac{3}{(n\pi)^2}\sin\frac{n\pi}{3}$$

$$= \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$u(x,t) = \sum_1^\infty \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{30}\cos\frac{2n\pi t}{30}$$

$$\text{Freq.: } \frac{2n\pi}{30} = \frac{n\pi}{15}\ , \ \text{Wave length } = \frac{60}{n}$$

Justification of the Solution: to insure that

$$u(x,t) = \sum_1^\infty b_n \sin\frac{n\pi}{l}x\cos\frac{n\pi\alpha}{l}t \qquad (2.22)$$

is a solution of wave equation

$$u_x(x,t) = \sum_{n=1}^\infty b_n \frac{n\pi}{l}\cos\frac{n\pi}{l}x\cos\frac{n\pi\alpha}{l}t$$

$$u_{xx}(x,t) = -\sum_{n=1}^\infty b_n \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi}{l}x\cos\frac{n\pi\alpha}{l}t$$

$$u_{tt} = -\alpha^2\sum_{n=1}^\infty \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}\cos\frac{n\pi\alpha t}{l}$$

First: we will show that (2.22) equivalent to

$$u(x,t) = \frac{h(x-\alpha t) + h(x+\alpha t)}{2}$$

where h is periodic of period $2l$ s.t.

■ $h(x) = \begin{cases} f(x) & 0 \leqslant x \leqslant l \\ -f(-x) & -l < x < 0 \end{cases} \qquad (2.23)$

■ $h(x+2l) = h(x)$

And h has a Fourier series

$$h(x) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad (2.24)$$

$$h(x-\alpha t) = \sum_{1}^{\infty} b_n \left(\sin\frac{n\pi}{l}(x-\alpha t)\right)$$

$$= \sum_{1}^{\infty} b_n \left(\sin\frac{n\pi}{l}x\cos\frac{n\pi\alpha}{l}t - \cos\frac{n\pi}{l}x\sin\frac{n\pi\alpha}{l}t\right)$$

$$h(x+\alpha t) = \sum_{1}^{\infty} b_n \sin\frac{n\pi}{l}(x+\alpha t)$$

$$= \sum_{1}^{\infty} b_n \left(\sin\frac{n\pi}{l}x\cos\frac{n\pi\alpha}{l}t + \cos\frac{n\pi}{l}x\sin\frac{n\pi\alpha}{l}t\right)$$

$$\frac{h(x-\alpha t)+h(x+\alpha t)}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos\frac{n\pi\alpha}{l}t$$

2.2 Case 2: General problem for elastic string:

$$\alpha^2 u_{xx} = u_{tt}, \quad 0 < x < l, t > 0$$

B.C. $u(0,t) = 0, u(l,0) = 0, \ 0 < x < l, \ 0 < t < \infty.$ (2.7)

I.C. $u(x,0) = 0, \ u_t(x,0) = g(x), \ 0 \leqslant x \leqslant l$ (2.25)

$$u(x,t) = X(x)T(t),$$

$$X_n(x) = c_n \sin\frac{n\pi x}{l}$$

$$T_n(t) = k_n \sin\frac{n\pi\alpha}{l}t + j_n \cos\frac{n\pi\alpha}{l}t \qquad (2.16-a)$$

now from first part (2.25) we have $T_n(0) = 0$

$$\Longrightarrow J_n = 0, \quad n = 1,2,\cdots$$

$$T_n(t) = k_n \sin\frac{n\pi\alpha}{l}t, \qquad (2.26)$$

$$u_n(x,t) = c_n \sin\frac{n\pi x}{l} k_n \sin\frac{n\pi\alpha t}{l}$$

$$u_n(x,t) = b_n \sin\frac{n\pi}{l}x \sin\frac{n\pi\alpha}{l}t, \quad n = 1,2,3,\dots \qquad (2.27)$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi \alpha t}{l} \qquad (2.28)$$

second part of (2.25) $u_t(x,0) = g(x)$

$$u_t(x,t) = \sum_{n=1}^{\infty} b_n \frac{n\pi\alpha}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi\alpha t}{l}$$

$$u_t(x,0) = \sum_{n} b_n \frac{n\pi\alpha}{l} \sin\frac{n\pi x}{l} = g(x)$$

$b_n \frac{n\pi\alpha}{l}$ are the coefficients in the Fourier Series then:

$$b_n \frac{n\pi\alpha}{l} = \frac{2}{l}\int_0^l g(x) \sin\frac{n\pi}{l} x dx \; n = 1,2,\dots \qquad (2.29)$$

So the solution is by (2.28), (2.29)

Example A vibrating string of length 40 cm satisfy the equation $16u_{xx} = u_{tt}$,

Fixed at the ends with condition $u(x,0) = 0, u_t(x,0) = \begin{cases} 2 & 0 \leqslant x < 10 \\ 4x & 10 \leqslant x \leqslant 40 \end{cases}$

Find the displacement of the string

Ans: $l = 40, \alpha = 4, g(x) = \begin{cases} 2 & 0 \leqslant x < 10 \\ 4x & 10 \leqslant x \leqslant 40 \end{cases}$ from (Case 2)

The general solution (2.28)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{40} \sin\frac{n\pi t}{10}$$

from (2.29)

$$b_n \frac{n\pi}{10} = \frac{2}{40}\int_0^{40} g(x) \sin\frac{n\pi x}{40} dx$$

$$= \frac{1}{20}\int_0^{10} 2\sin\frac{n\pi x}{40} dx + \frac{1}{20}\int_{10}^{40} 4x\sin\frac{n\pi x}{40} dx$$

$$= \frac{1}{10}\left[-\frac{40}{n\pi}\cos\frac{n\pi x}{40}\right]_0^{10} + \frac{1}{5}\left[-\frac{40}{n\pi}\cos\frac{n\pi x}{40} + \left(\frac{40}{n\pi}\right)^2 \sin\frac{n\pi x}{40}\right]^{10}$$

$$= \frac{1}{10}\left[-\frac{40}{n\pi}\cos\frac{n\pi}{4} + \frac{40}{n\pi}\right] + \frac{1}{5}\left[-\frac{40}{n\pi}\cos n\pi + \frac{40}{n\pi}\cos\frac{n\pi}{4} - \left(\frac{40}{n\pi}\right)^2\right.$$

$$= -\frac{4}{n\pi}\cos\frac{n\pi}{4} + \frac{4}{n\pi} - \frac{8}{n\pi}(-1)^n + \frac{40}{n\pi}\cos\frac{n\pi}{4} - \left(\frac{40}{n\pi}\right)^2 \sin\frac{n\pi}{4}\frac{n\pi}{4}\Big]$$

$$= \frac{36}{n\pi}\cos\frac{n\pi}{4} + \frac{4}{n\pi} - \frac{8}{n\pi}(-1)^n - \left(\frac{40}{n\pi}\right)^2 \sin\frac{n\pi}{4}$$

$$b_n = \frac{360}{(n\pi)^2}\cos\frac{n\pi}{4} + \frac{40}{(n\pi)^2} - \frac{80}{(n\pi)^2}(-1)^n - \frac{16000}{(n\pi)^3}\sin\frac{n\pi}{4}$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left(\frac{360}{(n\pi)^2}\cos\frac{n\pi}{4} + \frac{40}{(n\pi)^2} - \frac{80}{(n\pi)^2}(-1)^n - \frac{16000}{(n\pi)^3}\sin\frac{n\pi}{4}\right)\sin\frac{n\pi x}{40}\sin\frac{n\pi t}{10}$$

2.3 Case 3: More general problem for elastic string:

$$\alpha^2 u_{xx} = u_{tt} \quad 0 < x < l, t > 0$$

B.C. $u(0,t) = 0, u(l,0) = 0,\ 0 < x < l,\ 0 < t < \infty.$ (2.7)

I.C. $u(x,0) = f(x),\ u_t(x,0) = g(x),\ 0 \leqslant x \leqslant l$ (2.30)

where $f(x)$ initial position $g(x)$ initial velocity

Case 1:	$\alpha^2 V_{xx} = V_{tt}$	Case 2:	$\alpha^2 w_{xx} = w_{tt}$
$V(0,t) = 0$	$V(x,0) = f(x)$	$W(0,t) = 0$	$W(x,0) = 0$
$V(l,t) = 0$	$V_t(x,0) = 0$	$W(l,t) = 0$	$W_t(x,0) = g(x)$
(2.7)	(2.8)	(2.7)	(2.25)

Case 3:	$\alpha^2 u_{xx} = u_{tt}$
$u(0,t) = 0$	$u(x,0) = f(x)$
$u(l,t) = 0$	$u_t(x,0) = g(x)$
(2.7)	(2.30)

$$V(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\, cos\frac{n\pi\alpha t}{l} \qquad (2.31)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx \qquad (2.32)$$

$$W(x,t) = \sum_{n=1}^{\infty} d_n \frac{n\pi\alpha}{l} \sin\frac{n\pi x}{l} \sin\frac{n\pi\alpha t}{l} \qquad (2.33)$$

$$d_n \frac{n\pi\alpha}{l} = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \qquad (2.34)$$

$$\text{let} \quad u(x,t) = V(x,t) + W(x,t) \qquad (*)$$

$$\therefore u_{xx} = V_{xx} + W_{xx}, \quad u_{tt} = V_{tt} + W_{tt}$$

$$\alpha^2 u_{xx}^{-u_{tt}} = \alpha^2 V_{xx} + \alpha^2 W_{xx} - V_{tt} - W_{tt} = \alpha^2 V_{xx} - V_{tt} + \alpha^2 W_{xx} - W_{tt}$$

$$= 0 + 0 = 0 \text{ so } u(x,t) \text{ satisfy } (1)$$

$$u(0,t) = V(0,t) + W(0,t) = 0 + 0 = 0$$

$$u(l,t) = V(l,t) + W(l,t) = 0$$

then $u(x,t)$ satisfy boundary. Condition

$$u(x,0) = V(x,0) + W(x,0) = f(x) + 0 = f(x)$$

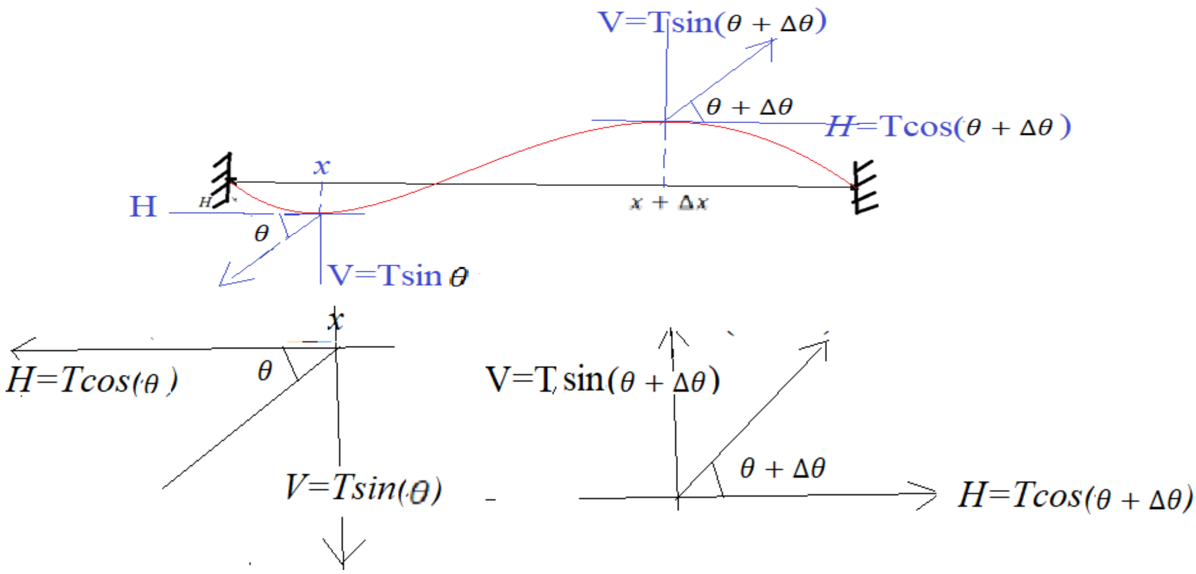
$$u_t(x,0) = V_t(x,0) + W_t(x,0) = 0 + g(x) = g(x)$$

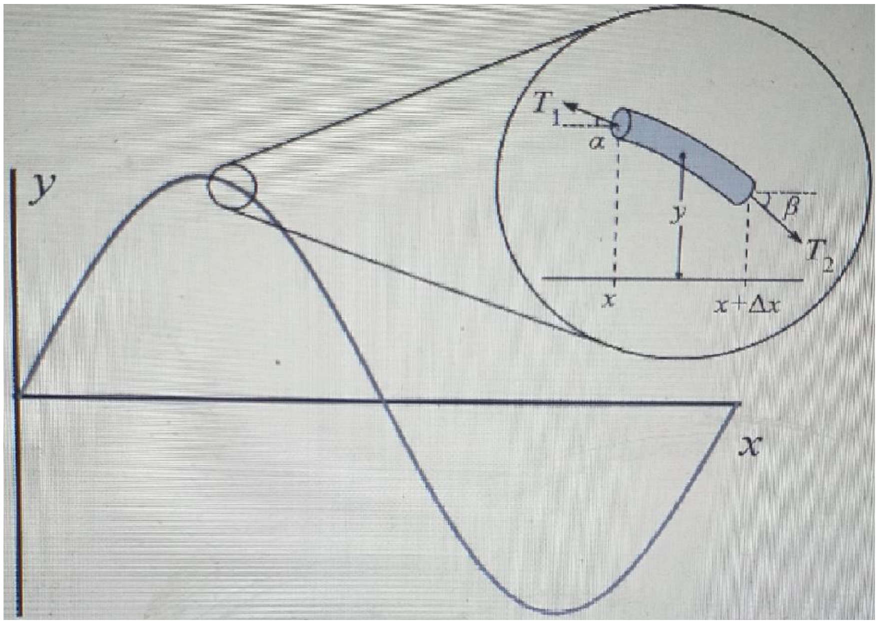
Thus $u(x,t)$ satisfy Condition (2.31)

Conclusion To solve case 3 we first solve case1 and case 2 and using (*) to get the solution

2.4 Derivation of the wave equation

Consider a perfectly flexible elastic string





القوه الخارجيه الصافيه والناتجه من الشد في نهايات عنصر تساوي حاصل ضرب كتله العنصر في تعجيل كتله مركزه

Newton's law as it applies to the element Δx of the string ρ mass per unit Force difference $\Delta F = \rho \Delta x u_{tt}$.

There is no horizontal acceleration then

Horizontal Component لايوجد فرق في القوة الافقية

$$T(x+\Delta x,t)\cos(\theta+\Delta\theta) - T(x,t)\cos\theta = 0 \qquad (1)$$

اي ان المركبة الافقية $H(x,t)$ ثابتة بالنسبة الى x ولهذا فان $H_x(x,t) = 0$

Vertical Components satisfy من ناحية اخرى

$$T(x+\Delta x,t)\sin(\theta+\Delta\theta) - T(x,t)\sin\theta = \Delta F = \rho\Delta x u_{tt}(\bar{x},t) \qquad (2)$$

$\bar{x}$: is the coordinate of the Center of mass of the element of the string

$$x < \bar{x} < x + \Delta x$$

$$V(x,t) = T(x,t)\sin\theta$$

$$V(x+\Delta x,t) = T(x+\Delta x,t)\sin(\theta+\Delta\theta)$$

$$x < \bar{x} < x + \Delta x$$

$$H(x,t) = T(x,t)\cos\theta \implies T(x,t) = \frac{H(x,t)}{\cos\theta}$$

from (2) we get

$$\frac{V(x+\Delta x,t) - V(x,t)}{\Delta x} = \rho u_{tt}(\bar{x},t)$$

$$\lim_{\Delta x \to 0} \frac{V(x+\Delta x,t) - V(x,t)}{\Delta x} = \rho u_{tt}(\bar{x},t), \quad \Delta x \to 0 \Rightarrow \bar{x} \to x$$

$$V_x(x,t) = \rho u_{tt}(x,t) \qquad (3)$$

$$V(x,t) = T(x,t)\sin\theta = H(x,t)\frac{\sin\theta}{\cos\theta} = H(x,t)\tan\theta = H(x,t)u_x(x,t)$$

$H_x(x,t) = 0$, then (3) will be

$$\therefore V_x(x,t) = (H(x,t)u_x(x,t))_x = \rho u_{tt}(x,t)$$

$$\Longrightarrow H(x,t)u_{xx} = \rho u_{tt}. \qquad (4)$$

If $\theta \approx 0$, $H(x,t) = T(x,t)\cos\theta = T(x,t)$, $H = T\cos\theta = T$

$$\Longrightarrow u_{tt} = \frac{T}{\rho}u_{xx} = \alpha^2 u_{xx}\,, \qquad \alpha^2 = \frac{T}{\rho} \qquad (5)$$

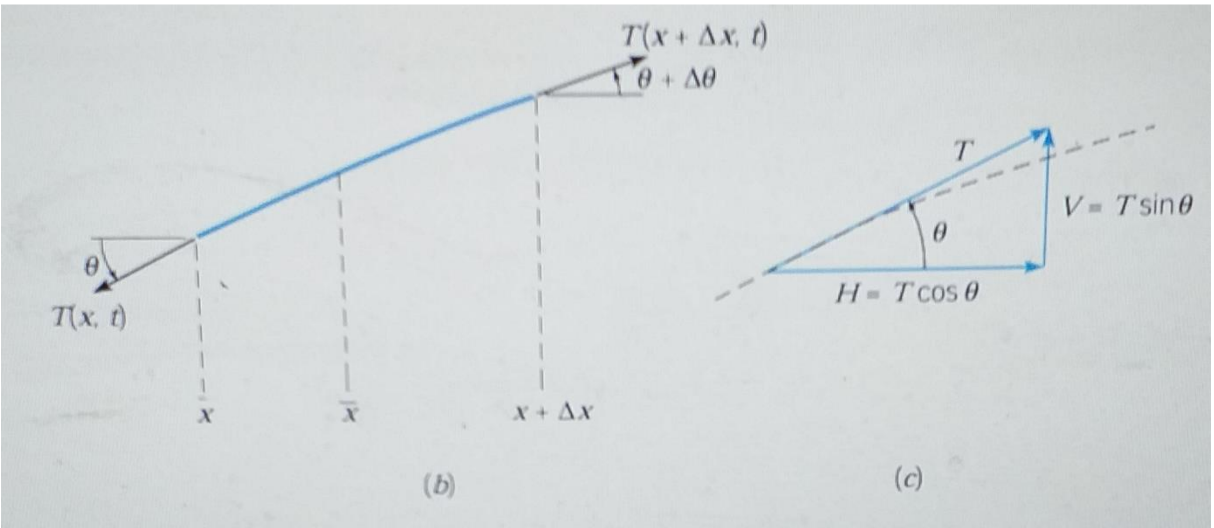


FIGURE 10.B.1 (*a*) An elastic string under tension. (*b*) An element of the displaced string. (*c*) Resolution of the tension T into components.