Chapter 2

Separation of variable: Heat conduction in a Rod

Three type of distinct physical phenomena

1- diffusive processes chive

2- oscillatory processes

3- time independent (steady processes)

linear of second order best form Classify into Three Categories:

1- heat conduction equation

2 - Wave equation

3-potential equation bid

let u be aheat conduction for a straight bar of uniform cross section

2.1 Heat Conduction equation

The equation

$$\alpha^2 u_{xx} = u_t \,, \quad 0 < x < l, t > 0$$

is called heat conduction equation where

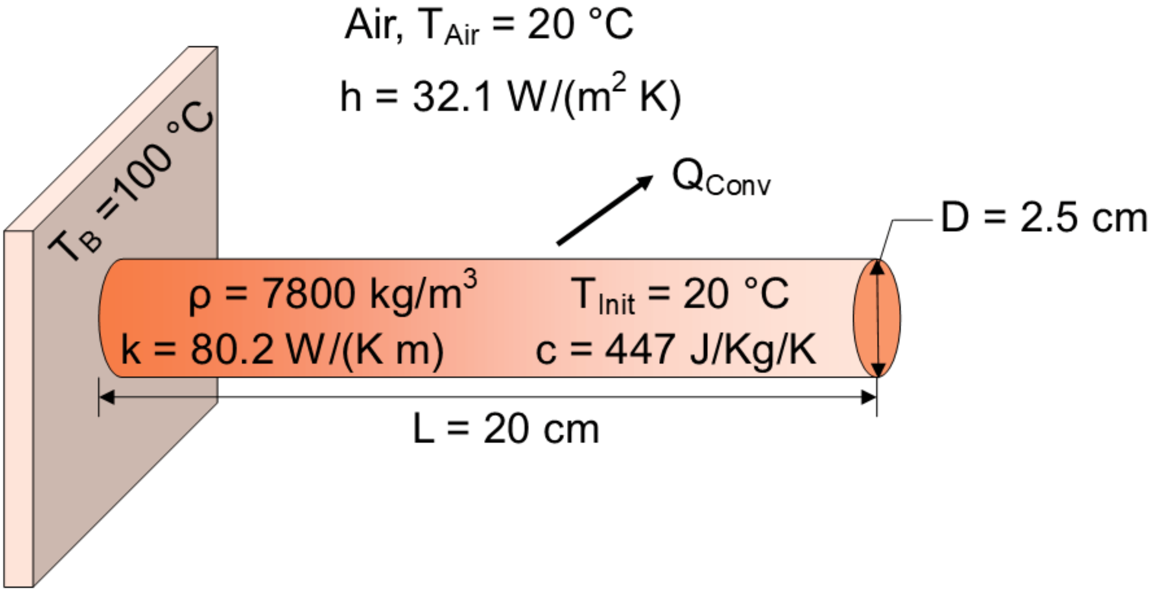
α^2: thermal diffusivity constant depend on the material from which the bar is made and. $\alpha^2 = \frac{k}{\rho s}$

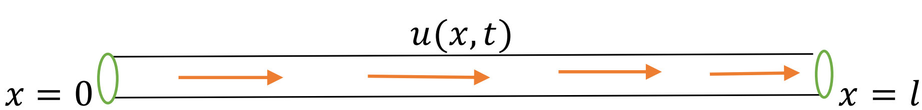
p: density

s: specific heat of the material in the bar

initial temperature distribution in the bar

α^2	$\mathrm{cm}^2/\mathrm{sec}$
Silver	1.71
plumper	0.86
Alumina	0.14
Gastiron	0.12
Granite	0.011
water	0.000144





$$\alpha^2 u_{xx} = u_t\ , \quad 0 < x < l, \quad t > 0 \qquad (1)$$

First The ends of the bar are held at fixed temperature, at $x = 0$ is T_1 and $x = l$ is T_2, when $x < 0$ & $x > l$, $T_1 = T_2 = 0$, so the **Boundary conditions**:

$$u(0,t) = T_1, \quad u(l,t) = T_2 \qquad (2)$$

Initial condition $u(x,0) = f(x), \quad 0 \leqslant x \leqslant l \qquad (3)$

$f(x)$ Given function, we will assume homogeneous boundary conditions

$$u(0,t) = 0, u(l,t) = 0, \quad t > 0 \qquad (4)$$

$(1), (3), (4)$ is said linear BVP

Separation of variable: The assumption is that $u(x,t)$ is product of two separated functions

$$u(x,t) = X(x)T(t) \qquad (5)$$

$$u_{xx} = x''(x)T(t), \quad u_t = x(t)T'(t) \quad (6)$$

Substituting (6) in (1) we get

$$\alpha^2 X''(x)T(t) = X(x)T'(t)] \div \alpha^2 X(x)T(t) \qquad (7)$$

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} \implies$$

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} = -\lambda, \qquad (8)$$

$$X''(x) + \lambda X(x) = 0 \qquad (9)$$

$$T'(t) + \alpha^2 \lambda T(t) = 0 \qquad (10)$$

Using boundary conditions (4)

$$u(0,t) = 0 \implies u(0,t) = x(0)T(t) = 0 \quad (11)$$

if $T(t) = 0 \to u(x,t) = 0$ is trivial. So $T(t) \neq 0$ for some t

$$\implies x(0) = 0 \ (12)$$

Similarly $u(l,t) = 0 \implies u(l,t) = x(l)T(t) = 0 \implies \quad x(l) = 0 \quad (13)$

eq. (9) with (12) & (13) which have solution

$$x_n(x) = \sin\frac{n\pi x}{l} \ ? \quad n = 1,2,3,\ldots \quad (14)$$

لتوضيح ذلك لاحظي من المعادلة (9) نحصل على

$$m^2 + \lambda = 0 \implies m = \pm\sqrt{-\lambda}$$

وحل هذه لها عدة حالات هي:

1- $\lambda = 0 \Longrightarrow x_1 = e^0 = 1, x_2 = xe^0 = x$ The general solution

$$X(x) = c_1 x_1 + c_2 x_2 = c_1 + c_2 x$$

$X(0) = 0 \rightarrow c_1 = 0, X(l) = 0 \rightarrow c_2 = 0$

$\Longrightarrow X = 0$ trivial solution (neglect) تهمل

2. if $\lambda < 0$ let $\lambda = -\delta^2, -\lambda = \delta^2, \delta \neq 0 \Longrightarrow m_1 = \sqrt{-\lambda} = \delta,\ m_2 = -\delta$

$$\Rightarrow x_1 = e^{\delta x}, x_2 = e^{-\delta x} \Longrightarrow X(x) = c_1 e^{\delta x} + c_2 e^{-\delta x}$$

$$X(0) = c_1 + c_2 = 0,\ x(l) = c_1 e^{\delta l} + c_2 e^{-\delta l} = 0$$

$$\begin{matrix} c_1 + c_2 = 0 \\ c_1 e^{\delta l} + c_2 e^{-\delta l} = 0 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ e^{\delta l} & e^{-\delta l} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{vmatrix} 1 & 1 \\ e^{\delta l} & e^{-\delta l} \end{vmatrix} = e^{-\delta l} - e^{\delta l} \neq 0$$

$\Rightarrow c_1 = 0,\ \ c_2 = 0$ *is the only solution*

$\Longrightarrow X = 0$ trivial solution (neglect)

3- If $\lambda > 0$, let $\lambda = \delta^2, m_1 = \delta i, m_2 = -\delta i, \quad \delta = \sqrt{\lambda}$

$$X_1 = \sin \delta x \quad X_2 = \cos \delta x$$ الحلول المستقلة

$$X(x) = c_1 \sin \delta x + c_2 \cos \delta x$$ الحل العام

$X(0) = c_2 = 0,\ \ X(l) = c_1 \sin \delta l = 0$ (12),(13) باستخدام الشرط

$\sin \theta = 0$ iff $\theta = n\pi\ n = 0, \pm 1, \pm 2, \dots$

$$\sin \delta l = 0 \quad \text{iff} \quad \delta_n = \frac{n\pi}{l}$$

$$X_n(x) = c_n \sin \frac{n\pi x}{l}, \qquad n = 1,2,3, \dots \quad (14)$$

$$\lambda_n = \delta_n^2 = \frac{n^2\pi^2}{l^2}\ n = 1,2,3, \dots \qquad (15)$$

Substitute (15) in (10) we get

$$T_n'(t) + \frac{n^2\pi^2}{l^2}\alpha^2 T_n(t) = 0 \Longrightarrow \ln T_n(t) = -\frac{n^2\pi^2}{l^2}\alpha^2 t + a_n$$

$$T_n(t) = k_n e^{-\frac{n^2\pi^2}{l^2}\alpha^2 t}, \qquad k_n = e^{a_n}. \quad (16)$$

$$u_n(x,t) = c_n k_n \sin \frac{n\pi x}{l} e^{-\frac{n^2\pi^2\alpha^2 t}{l^2}},\ \ n = 1,2,3, \dots \quad (17)$$

is a homogenous solution of eq. (1) and (4) $\forall n$.

The function u_n is called fundamental Solution of the heat Conduction problem (1), (3), (4), let $b_n = j_n c_n k_n$ then The general solution

$$u(x,t) = \sum_{n=1}^{\infty} j_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{n^2\pi^2\alpha^2}{l^2}t}, \quad (19)$$

to see Condition (3) $u(x,0) = f(x)$

$f(x)$ كميه الحراره المؤثره على الموصل في اي موقع x فيه من بدايه التجربه

$$u(x,0) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l} = f(x), \quad (20)$$

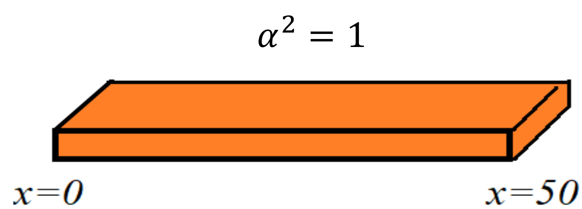
which is a sine series so

$$a_n = 0,\ n = 0,1,2,\dots$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad (21)$$

Hence the solution of the heat conduction problem (1), (3), (4) is given by (19) and (21). of granite and marble a piece of tiles made

Example 1: Find the temperature $u(x,t)$ at any time of a 50 cm long piece of tile made of granite and marble ($\alpha^2 = 1$) insulated on the sides, which initially has uniform temperature of 20°C. Whose ends are maintained at 0°C for all $t > 0$.



Ans: $l = 50$ cm, $\alpha^2 = 1$, $f(x) = 20$, $0 < x < 50$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{50} e^{-\left(\frac{n\pi}{50}\right)^2 t}, \quad (22)$$

where

$$b_n = \frac{40}{50}\int_0^{50} \sin\frac{n\pi x}{50}dx = \frac{40}{n\pi}\left[-\cos\frac{n\pi}{50}x\right]_0^{50}$$

$$= \frac{40}{n\pi}(1-\cos n\pi) = \frac{40}{n\pi}(1-(-1)^n) = \begin{cases} \frac{80}{n\pi} & if \;\; n \;\; odd \\ 0 & if \;\; n \;\; even \end{cases} \quad (23)$$

Then

$$u(x,t) = \sum_{n=1,3,5,7,\ldots}^{\infty} \frac{80}{n\pi} \sin\frac{n\pi x}{50} e^{-\left(\frac{n\pi}{50}\right)^2 t}, \qquad (24)$$

Ex. Find the equ. of Separation of variable of 1. $u_{xx} + te^{-x}u_t = 0$

$u = X(x)T(t) \implies u_{xx} = X''T,\; u_t = XT'$

$$X''T(t) = -te^{-x}XT' \implies e^x\frac{X''}{X} = -t\frac{T'}{T} = -\lambda$$

$$e^x X''(x) + \lambda X(x) = 0,\; tT' - \lambda T(t) = 0$$

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2.2 Nonhomogeneous Boundary conditions

$$\alpha^2 u_{xx} = u_t \qquad (1), \quad u(x,0) = f(x) \quad (3)$$

$$u(0,t) = T_1,\; u(l,t) = T_2 \; t > 0 \qquad (8)$$

steady state temperature distribution will be $v(x)$

$u(x,t) = v(x), \quad t \to \infty$ for large t

$$\alpha^2 u_{xx}(x,t) = u_t \to \alpha^2 v''(x) = 0 \to v''(x) = 0 \quad (9)$$

$$v(0) = T_1,\; v(l) = T_2 \quad (10)$$

The solution of (9) with (10) yields

$$v(x) = \frac{(T_2 - T_1)}{l}x + T_1 \qquad (11)$$

Let

$$u(x,t) = v(x) + w(x,t) \qquad (12)$$

w is transient distribution

from eq. (1) $\alpha^2(v+w)_{xx} = (v+w)_t$

$$\alpha^2 w_{xx} = w_t \qquad (13)$$

$$w(x,t) = u(x,t) - v(x)$$

$$\begin{aligned} w(0,t) &= u(0,t) - v(0) = T_1 - T_1 = 0 \\ w(l,t) &= u(l,t) - v(l) = T_2 - T_2 = 0 \end{aligned} \qquad (14)$$

from eq.(3) $u(x,0) = f(x)$

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x) \quad (15)$$

eq. $(13), (14)$ and (15) is similar to $(1), (3), (4)$

$$u(x,t) = \sum_{1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} + \frac{(T_2 - T_1)}{l} x + T_1 \quad (16)$$

where

$$b_n = \frac{2}{l}\int_0^l \left(f(x) - \frac{T_2 - T_1}{l} x - T_1\right) \sin\frac{n\pi x}{l} dx, \quad (17)$$

Example $u_{xx} = u_t$, $0 < x < 30$, $u(0,t) = 20$, $u(30,t) = 50$

$$u(x,0) = 60 - 2x,\ 0 < x < 30$$

1- Find the steady state temperature, 2. Transient distribution, 3. $u(x,t)$

Ans.:1. $V(0) = 20, V(30) = 50$

$$V(x) = x + 20,$$

2. Transient distribution $w_{xx} = w_t$, $\alpha^2 = 1, f(x) = 40 - 3x, l = 30$

$$w(0,t) = 0,\ \ w(30,t) = 0,$$

$$w(x,0) = f_1(x) = f(x) - v(x) = 60 - 2x - x - 20 = 40 - 3x$$

$$w_n = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{30} e^{-\left(\frac{n\pi\alpha}{30}\right)^2 t}$$

$$b_n = \frac{2}{l}\int_0^l f_1(x) \sin\frac{n\pi x}{l} dx = \frac{1}{15}\int_0^{30} (40 - 3x) \sin\frac{n\pi x}{30} dx$$

$$= \frac{20}{n\pi}(5\cos n\pi + 4)$$

$$w_n = \sum_{n=1}^{\infty} \frac{20}{n\pi}(5\cos n\pi + 4)\sin\frac{n\pi x}{30} e^{-\left(\frac{n\pi}{30}\right)^2 t}.$$

3. $u(x,t) = v(x) + w(x,t)$

$$= 20 + x + \sum_{n=1}^{\infty} \frac{20}{n\pi}(5\cos n\pi + 4)\sin\frac{n\pi x}{30} e^{-\left(\frac{n\pi}{30}\right)^2 t}.$$

Example $u_{xx} = u_t$, $0 < x < 30, u(0,t) = 30$, $u(30,t) = 0$,

$$u(x,0) = \frac{x(60 - x)}{30}$$

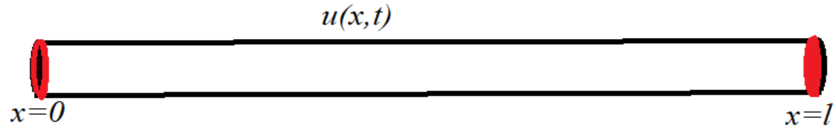
1. Find the steady state
2. Transient temperature
3. $u(x,t)$

2.3 Bar with insulated ends

If the bar are insulated at ends then the rate of flow of heat across section (is proportional) the rate of change of temperature in the x direction.

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$u_x(x,t) = \lim_{\Delta x\to 0}\frac{u(x+\Delta x,t)-u(x,t)}{\Delta x}$$



$$u_x(0,t)=0,\ \ u_x(l,t)=0, \qquad (18)$$

$$u(x,t)=X(x)T(t) \qquad (19)$$

$$\frac{X''}{X}=\frac{1}{\alpha^2}\frac{T}{T}=-\lambda$$

$X''+\lambda X=0,\Longrightarrow\ m^2+\lambda=0\Longrightarrow\ m=\pm\sqrt{-\lambda}$ (20) Characteristic equation

$$T'+\alpha^2\lambda T=0, \qquad (21)$$

From (18)

$$\left.\begin{aligned} u_x(0,t)=X'(0)T(t)=0\Longrightarrow X'(0)=0\\ u_x(l,t)=X'(l)T(t)=0\Longrightarrow X'(l)=0\end{aligned}\right\} \qquad (22)$$

To solve (20)

1- if $\lambda<0$ let $\lambda=-\delta^2,\ m_1=\delta, m_2=-\delta,\ \delta\neq 0$

$$X_1=e^{\delta x}, X_2=e^{-\delta x}\ \Longrightarrow\ X(x)=c_1e^{\delta x}+c_2e^{-\delta x}$$

$X'(x)=c_1\delta e^{\delta x}-c_2\delta e^{-\delta x}=\ c_1\sinh\delta x+c_2\cos\delta x$

$$X'(0)=c_1\delta-c_2\delta=0\longrightarrow c_1-c_2=0, c_1=c_2.$$

$$X'(l)=c_1\delta e^{\delta l}-c_2\delta e^{-\delta l}=0\Longrightarrow\ c_1(e^{\delta l}-e^{-\delta l})=0$$

$$e^{\delta l}\neq e^{-\delta l},\ \ e^{\delta l}-e^{-\delta l}\neq 0$$

$$\longrightarrow c_1=0,\ \ c_2=0$$

u is trivial. (neglect)

2- If $\lambda = 0 \Rightarrow X''(x) = 0 \Rightarrow x_1 = 1, x_2 = x$

$$X = c_1 + c_2 x \text{ from } (22)$$

$$X'(0) = c_2 \Rightarrow c_2 = 0 \Rightarrow X(x) = c_1$$

$$T'(t) = 0 \Rightarrow T(t) = c_3$$

$$\Rightarrow u(x,t) = c_1 c_3 = c_4 \quad (23) \quad \text{Constant.}$$

3- If $\lambda > 0$, let $\lambda = \delta^2, \quad \delta \neq 0$

$$X''(x) + \delta^2 X(x) = 0, \quad m = \pm\delta i, \quad X_1 = \sin \delta x, \quad X_2 = \cos \delta x$$

$$X(x) = c_1 \sin \delta x + c_2 \cos \delta x$$

$$X'(x) = c_1 \delta \cos \delta x - c_2 \delta \sin \delta x$$

$$X'(0) = c_1 \delta = 0 \Rightarrow c_1 = 0$$

$$X'(l) = -c_2 \delta \sin \delta l = 0, \quad c_2 \delta \neq 0 \quad (24)$$

From (24) we get $\sin\theta = 0$ iff $\theta_n = n\pi, \quad n = 0, \pm1, \pm2, \pm3, \cdots$

$\delta_n l = n\pi \Longrightarrow \delta_n = \frac{n\pi}{l} \Longrightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2$ use it in (21)

$$T'(t) + \left(\frac{n\pi\alpha}{l}\right)^2 T(t) = 0$$

$$T_n(t) = k_n e^{-\left(\frac{n\pi}{l}\alpha\right)^2 t}$$

$$X_n(x) = c_n \cos\frac{n\pi}{l} x$$

$$\therefore u_n(x,t) = c_n \cos\frac{n\pi}{l} x \cdot k_n e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t}, \quad n = 1,2,3,\cdots$$

The fundamental Solution

$$u_n(x,t) = c_n k_n \cos\frac{n\pi}{l} x \, e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \quad (25)$$

Let $a_n = j_n c_n k_n$ The general Solution is

$$u(x,t) = \sum_{n=1}^{\infty} j_n u_n(x,t) = \frac{a_0}{2} + \sum_{1} a_n \cos\frac{n\pi x}{l} e^{-\left(\frac{n\pi\alpha}{l}\right)^2 t} \quad (26)$$

By using $u(x,0) = f(x)$ in (26)

$$u(x,0)=\frac{a_0}{2}+\sum_{i}^{\infty} a_n \cos\frac{n\pi x}{l}=f(x)$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \qquad (27)$$

Example: Find the temperature $u(x,t)$ in ametal rod of length 25 cm that is insulated on the ends whose initial temperature distribution is
$u(x,0)=x,\;\; 0<x<25$

Sol:

from (18), (26), (27)

$$u_x(0,t)=0, u_x(25,t)=0$$

$$u(x,t)=\frac{a_0}{2}+\sum_{1}^{\infty} a_n\cos\frac{n\pi x}{25}e^{-\left(\frac{n\pi x}{25}\right)^2 t}$$

$$a_0=\frac{2}{25}\int_0^{25} f(x)dx=\frac{2}{25}\int_0^{25} xdx=\frac{2}{25}\frac{(25)^2}{2}=25$$

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx=\frac{2}{25}\int_0^{25} x\cos\frac{n\pi x}{25}dx$$

$$=\frac{50}{(n\pi)^2}(\cos n\pi-1)=\frac{50}{(n\pi)^2}((-1)^n-1)=\begin{cases}0 & \text{if } n \text{ even}\\ \dfrac{-100}{(n\pi)^2} & \text{if } n \text{ odd}\end{cases}$$

The general sol.

$$u(x,t)=\frac{25}{2}-\sum_{1,3,5,\cdots}^{\infty}\frac{100}{(n\pi)^2}\cos\frac{n\pi x}{25}e^{-\left(\frac{n\pi\alpha}{25}\right)^2 t}$$

Let $n=2m-1$

$$u(x,t)=\frac{25}{2}-\sum_{m=1}^{\infty}\frac{100}{((2m-1)\pi)^2}\cos\frac{(2m-1)\pi x}{25}e^{-\left(\frac{(2m-1)\pi\alpha}{25}\right)^2 t} \qquad (28)$$

(nonhomogeneous)

2.4. More general Case

one end of the bar be at a fixed temperature while the other insulted so the boundary Condition have some cases:

$$u(0,t) = T\,,\;\; u_x(\ell,t) = 0 \quad (29)$$

$$u_x(0,t) = 0,\;\; u(\ell,t) = T \quad (30)$$

$$u_x(0,t) - \delta_1 u(0,t) = 0,\;\; u_x(l,t) + \delta_2\, u(l,t) = 0 \qquad (29-\text{a})$$

$$u_x(0,t) + \delta_1\, u(0,t) = 0,\;\; u_x(l,t) - \delta_2\, u(l,t) = 0 \quad (30-\text{a}))$$

δ_1, δ_2 are positive constant.

Ex. $\alpha^2 u_{xx} = u_t,\; u(0,t) = 10,\; u_x(50,t) = 0$

The derivation of (29) leads to The derivation of (29-a) Leads

$$u(x,t) = X(x)T(t)$$

$$u_x(x,t) = X'(x)T(t)$$

$$u_x(0,t) - \delta_1 u(0,t) = X'(0)T(t) - \delta_1 X(0)T(t) = 0.$$

$$\to T(t)[X'(0) - \delta_1 X(0)] = 0$$

$$\to X'(0) - \delta_1 X(0) = 0$$

$$u_x(l,t) + \delta_i u(l,t) = 0$$

$$0 = X'(l)T(t) + \delta_2 X(l)T(t) \Longrightarrow T(t)[X'(l) + \delta_2 X(l)] = 0$$

$$X'(l) + \delta_2 X(l) = 0$$

$$X'' + \lambda X(x) = 0$$

$$T'(t) + \lambda\alpha^2 T(t) = 0$$

Example Special Case problem 20 page 590

$$\alpha^2 u_{xx} = u_t \;\; 0 < x < l,\; 0 < t < \infty$$

Boundary conditions:

$$u(0,t) = 0$$

$$u_x(l,t) + \delta_2 u(l,t) = 0$$

Initial Cond. $u(x,0) = f(x) \quad 0 < x < l$

From first condition $\Longrightarrow x(0) = 0$

From second condition $\Longrightarrow X'(l) + \delta_2 X(l) = 0$

$$X'' + \lambda X = 0 \;\text{ if } \lambda < 0, \text{let } \;\lambda = -h^2$$

$$T' + \alpha^2 \lambda T = 0$$

$$\frac{dT}{T} = \alpha^2 h^2 dt$$

$$T(t) = ke^{\alpha^2 h^2 t}, \text{ as } t \to \infty \Longrightarrow T(t) \to \infty \Longrightarrow u(x,t) \longrightarrow \infty$$

which is impossible (Neglect)

If $\lambda = 0 \Longrightarrow X'' = 0 \Longrightarrow X(x) = c_1 + c_2 x$

$$X(0) = 0 \Longrightarrow c_1 = 0$$

$$X(x) = c_2 x \Longrightarrow X'(l) + \delta_2 X(l) = 0$$

$$c_2 + \delta_2 c_2 l = 0 \Rightarrow c_2(1 + \delta_2 l) = 0$$

$$1 + \delta_2 l \neq 0 \Rightarrow c_2 = 0$$

$$\therefore X = 0 \Rightarrow u = 0 \text{ trivial (Neglect)}$$

If $\lambda > 0$ let $\lambda = \delta^2$

$$X'' + \delta^2 X = 0$$

$$T' + \alpha^2 \delta^2 T = 0$$

$$X = c_1 \sin(\delta x) + c_2 \cos(\delta x), \quad X(0) = 0, \Rightarrow c_2 = 0$$

$$X(x) = c_1 \sin(\delta x)$$

$$T = c_3 e^{-\alpha^2 \delta t}$$

$$c_1 \delta \cos(\delta l) + \delta_2 c_1 \sin(\delta l) = 0$$

$$c_1 \delta + \delta_2 c_1 \tan(\delta x) = 0$$

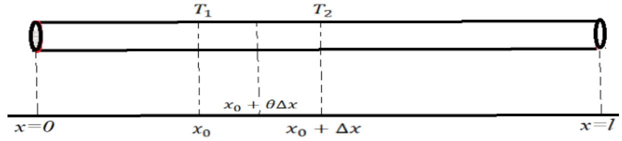
$$\tan(\delta x) = -\frac{\delta}{\delta_2}$$

$$u_n(x,t) = b_n \sin(\delta_n x) e^{-\delta_n^2 \alpha^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \delta_n x \, e^{-\delta_n^2 \alpha^2 t}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \delta_n x \, dx$$

2.5 Derivation of the heat Conduction equation



كمية الحرارة لكل وحدة زمنية Amount of heat per unit time $= \frac{kA|T_2 - T_1|}{d}$ (1)

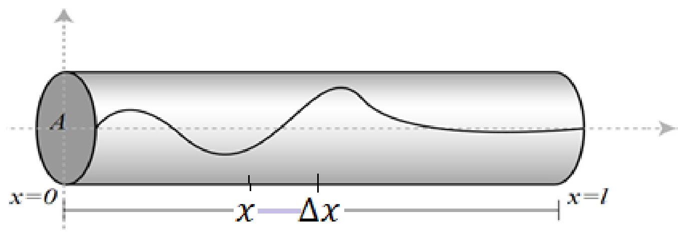
k : Thermal conductivity, $u(x_0,t)=T_1, u(x_0+\Delta x,t)=T_2$

A: Area of cross section

rod uniform thin and homogenous one material

(مركبة الحرارة الافقية H عند x_0)=المعدل الفوري لنقل الحرارة $u(x_0,t)$، من اليسار إلى اليمين

Horizontal heat $H(x_0,t)$ =The instantaneous rate of heat transfer $u(x_0,t)$ from left to right is



$$H(x_0,t)=-\lim_{d\to 0} kA\frac{u\left(x_0+\frac{d}{2},t\right)-u\left(x_0-\frac{d}{2},t\right)}{d}$$

$$=-kA\lim_{d\to 0}\frac{u\left(x_0+\frac{d}{2},t\right)-u\left(x_0-\frac{d}{2},t\right)}{d}=-kAu_x(x_0,t) \quad (2)$$

Then

$$H(x_0+\Delta x,t)=-kAu_x(x_0+\Delta x,t) \quad (3)$$

The net rate of heat flow between x_0 and $x_0+\Delta x$ is

صافي معدل تدفق الحرارة بين x_0 و $x_0+\Delta x$ هو

$$Q=H(x_0,t)-H(x_0+\Delta x,t)=kA\big(u_x(x_0+\Delta x,t)-u_x(x_0,t)\big) \quad (4)$$

The amount of heat entering the bar element in time

كمية الحرارة التي تدخل عنصر الموصل في الوقت المناسب

$$Q\Delta t=kA\big(u_x(x_0+\Delta x,t)-u_x(x_0,t)\big)\Delta t \qquad (5)$$

The average change in temperature u, in the time interval t, is proportional to the amount of heat Q t introduced and inversely proportional to the mass m of the element

معدل تغيرفي الحراره Δu لتغير الزمن Δt في الموقع Δx يتناسب طرديا مع كميه الحراره الداخله $Q\Delta t$ وعكسيا مع التغير في الكتله النوعيه للعنصر $s\Delta m$

$$\Delta u=\frac{Q\Delta t}{s\Delta m}=\frac{Q\Delta t}{s\rho A\Delta x} \qquad \text{الكتلة} = \text{الحجم} \times \text{الكثافة} \quad (6)$$

تغير الحرارة لتغير الزمن للموقع x بين $x_0 < x < x_0 + \Delta x$

$\Delta u \simeq u(x, t + \Delta t) - u(x, t)$

$$\text{let } x = x_0 + \theta\Delta x, \quad 0 < \theta < 1, \quad x_0 < x < x_0 + \Delta x$$

$$\Delta u = u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t) = \frac{Q\Delta t}{s\rho A\Delta x} \qquad (7)$$

$$Q\Delta t = [u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)]s\rho A\Delta x, \qquad (8)$$

from (5) & (8) we get

$$kA[u_x(x_0 + \Delta x, t) - u_x(x_0, t)]\Delta t$$
$$= s\rho A[u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)]\Delta x. \ (9)$$

dividing (9) by $\Delta x\Delta t$ and let $\Delta x \to 0, \Delta t \to 0$

$$kA \lim_{\Delta x \to 0} \frac{u_x(x_0 + \Delta x, t) - u_x(x_0, t)}{\Delta x}$$
$$= s\rho A \lim_{\Delta t \to 0} \frac{u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)}{\Delta t}$$
$$\frac{kA}{s\rho A} u_{xx} = u_t, \quad \alpha^2 = \frac{k}{s\rho} \Longrightarrow \alpha^2 u_{xx} = u_t$$

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Example 1. Let $\alpha^2 u_{xx} = u_t, \ 0 < x < l$

$$u(0, t) = 0, \ u_x(l, t) + \gamma u(l, t) = 0, \ \ u(x, 0) = f(x)$$

(a) Show that $X'' + \lambda X = 0,$

$$X(0) = 0, \ X'(l) + \gamma X(l) = 0, \qquad T'(t) + \lambda\alpha^2 T = 0$$

(b) if $\lambda \le 0$, there is no solution

(c) if $\lambda > 0$. has a solution satisfy $\quad \delta \cos \delta l + \gamma \sin \delta l = 0$

Example 2. (10 page (601)) Elastic string of length l, Fixed at $x = 0$ while the end $x = l$ Free, $u(0, t) = 0, u_x(l, t) = 0, u(x, 0) = f(x) \ u_t(x, 0) = 0,$

$$f(x) = \begin{cases} 1 & \frac{l}{2} - 1 < x < \frac{l}{2} + 1, l > 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $u(x,t)$?

Ans: $u(x,t) = \sum_{n=odd}^{\infty} b_n \sin\frac{n\pi x}{2l}\cos\frac{n\pi\alpha}{2l}t$

Q1 Find the steady state of $\alpha^2 u_{xx} = u_t$ where:

a. $u(0,t) = 10, u(50,t) = 40,$ b. $u(0,t) = 30, u(40,t) = -20$

c. $u_x(0,t) = 0, u(l,t) = T,$ d. $u(0,t) = \mathrm{T},\ u_x(l,t) = 0$

e. $u_x(0,t) - u(0,t) = 0, u(l,t) = T$

f. $u(0,t) = T,\ u_x(l,t) + u(l,t) = 0$

Sol: (a) $V(x) = \frac{T_2 - T_1}{l}x + T_1\ T_1 = 10, T_2 = 40, l = 50$

$$= \frac{3}{5}x + 10$$

(c) $V(x) = \dfrac{T_2 - T_1}{l}x + T_1 \Rightarrow V'(x) = \dfrac{T_2 - T_1}{l} \Rightarrow V'(0) = 0$

$\dfrac{T_2 - T_1}{l} = 0 \to T_2 = T_1\ V(l) = T_2 = T$

$\therefore V(x) = T_1 = T$

e. $V(x) = \dfrac{T_2 - T_1}{l}x + T_1,\ V(l) = T_2 = T$

$V'(x) = \dfrac{T_2 - T_1}{l} \Rightarrow \dfrac{T_2 - T_1}{l} - T_1 = 0 \to \dfrac{T_2 - T_1 - lT_1}{l} = 0$

$T_2 - T_1 = lT_1 \Rightarrow T = T_2 = (1 + l)T_1 \Rightarrow T_1 = \dfrac{T}{1 + l}$

$$\Rightarrow V(x) = \frac{T - \frac{T}{1+l}}{l}x + \frac{T}{1+l} = \frac{T}{1+l}x + \frac{T}{1+l} = \frac{T}{1+l}(x+1)$$

Q2 Can we use the separation of variable for each PDE:

a. $xu_{xx} + u_t = 0,$ b. $u_{xx} + u_{xt} + u_t = 0,$ c. $u_{xx} + (x + y)u_{yy} = 0$

d. $(f(x)u_x)_x - g(x)u_{tt} = 0$ e. $u_{xx} + u_{yy} + xu = 0$

(a) $u(x,t) = X(x)T(t) \Longrightarrow xX''(x)T(t) + X(x)T'(t) = 0$

$$xX''(x)T(t) = -X(x)T'(t) \div X(x)T(t) \Rightarrow \frac{xX''(x)}{X(x)} = -\frac{T'(t)}{T(t)} = -\lambda$$

$$xX''(x) = -\lambda X(x) \Rightarrow X''(x) + \frac{\lambda}{x}X(x) = 0$$

$$\frac{T'(t)}{T(t)} = \lambda \Longrightarrow T'(t) - \lambda T(t) = 0$$

$$(b)\ X''(x)T(t) + X'(x)T'(t) + X(x)T'(t) = 0$$

$$\Rightarrow \frac{X''(x)}{X(x)} + \frac{X'(x)}{X(x)}\frac{T'(t)}{T(t)} + \frac{T'(t)}{T(t)} = 0$$

$$\Longrightarrow \frac{X''(x)}{X(x)} = -\frac{T'(t)}{T(t)}\left(\frac{X'(x)}{X(x)} + 1\right)$$

$$\Rightarrow \frac{X''(x)}{X'(x) + X(x)} = -\frac{T'(t)}{T(t)} = -\lambda$$

$$\Rightarrow X''(x) + \lambda X'(x) + \lambda X(x) = 0$$

$$T'(t) - \lambda T(t) = 0$$

$$f:\ X''(x)Y(y) + (x+y)X(x)Y''(y) = 0$$

$$X''(x)Y(y) = -(x+y)X(x)Y''(y)] \div X(x)Y(y)$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -(x+y)\frac{Y''(y)}{Y(y)} \quad \text{not separable}$$

يمكن تعميم معادله الحراره الى مستوى ذات بعدين

Two dimension $\alpha^2(u_{xx} + u_{yy}) = u_t$

or Three dimension او الى مستوي ذات ثلاثه ابعاد

$$\alpha^2(u_{xx} + u_{yy} + u_{zz}) = u_t$$